

# Global phase-space approach to rate-induced tipping in nonautonomous dynamical systems

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In nonautonomous dynamical systems, rate-induced tipping (R-tipping) is a critical transition triggered by the rate of change of a time-varying parameter, rather than its absolute value. In recent years, there is a growing interest in R-tipping due to its relevance to significant problems of current interest, such as potential, catastrophic collapse of various ecosystems induced by climate change. This Perspective Article provides a brief overview of the basic concept, theory, and real-world implications of R-tipping from a global phase-space point of view. The key quantity underlying the global approach is the probability of R-tipping defined with respect to initial conditions in the phase space. A recently discovered scaling law governing this probability and the rate of parameter change is introduced, with numerical support from a class of high-dimensional, complex, and empirical ecological networks: pollinator-plant mutualistic networks. Issues such as prediction of tipping and protection of ecosystems from R-tipping are discussed.

Once upon a time, there was a donkey grazing near a river. At some distance from the river there is a high ground. A flash flood had occurred and the donkey began to run towards the high ground. If the flood front did not move so fast, the donkey would be able to get to the high ground in time and survive. However, if the flood moved sufficiently fast, the donkey would be swept into the water and would not survive. In addition to the moving speed of the water front, whether the donkey would survive also depends on its initial position: near or far away from the river. For any specific position, there is a critical speed of the flood, below which there is survival but above which death or extinction occurs. This is an intuitive picture of the phenomenon of rate-induced tipping, or R-tipping. In nonlinear dynamics, if the system is autonomous and bistable with two coexisting steady-state attractors, one corresponding to survival and another to extinction, each with its own basin of attraction, the basin boundary is stationary, i.e., it does not move in time. When a parameter changes with time so that the system becomes nonautonomous, the basin boundary will move with time. Whether a specific initial condition will lead to system survival or extinction depends on its location in the phase space as well as the “speed” at which the basin boundary moves. For an initial condition originally in the basin of the survival attractor, if it moves more “slowly” than the basin boundary, it will be swept into the basin of the extinction attractor, leading to a tipping of its fate. For any

such initial condition, there exists a critical speed of the movement of the basin boundary, or a critical rate of the parameter change, above which R-tipping will occur. For a given rate of parameter change, some initial conditions in the phase space will experience R-tipping, while some others will not by remaining in the basin of the survival attractor, in spite of the time variation of the parameter. The probability of R-tipping can then be defined for any given rate of parameter change. How does this probability depend on the rate? Characteristically, there are two possible scenarios. If, as the rate becomes nonzero, this probability increases from zero but slowly, a safe interval will arise in the rate in which the probability remains at some near-zero value - a desired situation. However, if the probability grows immediately and drastically as soon as the rate of parameter change increases from zero, no such safe interval or region will exist. In this case, in order to keep the probability of R-tipping diminishingly small, practically the rate of parameter change must be reduced to zero. Unfortunately, in nonautonomous dynamical systems, the dependence of the probability of R-tipping on the rate of parameter change tends to follow the second scenario. This article presents mathematical reasoning establishing such a dependence, or the scaling law, and describe numerical support from a class of complex, high-dimensional empirical ecological networks: plant-pollinator mutualistic networks. One implication is dire: in order to protect ecosystems from climate change, the rate of adverse parameter change, such as the rate of emissions of chemicals into the atmosphere, must be kept at near zero values.

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## I. INTRODUCTION

A tipping point is a critical threshold within a complex dynamical system where a small perturbation can lead to significant and often irreversible system changes<sup>1–38</sup>. Due to its implications for system stability and resilience, the phenomenon of tipping has garnered substantial attention across various scientific disciplines including climate science<sup>1,14,35</sup>, neuroscience<sup>8,18</sup>, and engineering<sup>9</sup>. In autonomous dynamical systems, a tipping point is the result of parameters reaching a critical level or of the influence of noise, which are referred to as bifurcation-induced tipping or noise-induced tipping<sup>14,37</sup>, respectively. A common dynamical mechanism for tipping is a backward saddle-node bifurcation where, as the bifurcation parameter increases through a critical point, a stable steady state and an unstable one coalesce and disappear together. Assume that the stable steady state corresponds to a “healthy” or “survival” state of the system. Before the bifurcation, there is bistability where the “healthy” steady state and another steady state associated with the “collapse” or “extinction” of the system coexist. After the bifurcation, the “healthy” steady state no longer exists, leaving the “collapse” state as the only attractor of the system. The backward saddle-node bifurcation thus leads to a tipping, after which the system functions are destroyed.

Dynamical systems in the real world are often of the nonautonomous type, due to the ubiquitous variations of the systems parameters with time. For example, influenced by the global climate change, the parameters of many ecosystems can drift with time, making them nonautonomous. When a parameter of the system changes with time, its rate of change is of particular concern - one example is the rate of carbon-dioxide emission into the atmosphere, which has been increasing steadily in recent decades. From the point of view of tipping, one might tempt to think that the time rate of change of a parameter may not be important, insofar as its value has not reached the threshold for tipping. More specifically, consider the two cases where a bifurcation parameter of the system changes slowly or fast, respectively, and suppose that the value of the parameter is still far away from a tipping point from the bifurcation point of view. In which case is a tipping event more likely? This question was addressed in a series of seminal works<sup>3,8,14</sup>, where the rate of the parameter change acts as a “super parameter” of the system and can induce a tipping. This is known as the phenomenon of rate-induced tipping, or R-tipping.

The history of research on R-tipping and the current status are briefly described, as follows. In 2008, Scheffer et al. proposed the critical-rate hypothesis suggesting that the rate at which environmental changes occur can be more significant than the magnitude of those changes in determining the long-term ecosystem states<sup>3</sup>. The concepts of critical ramping rate and rate-dependent tipping were introduced by Wiczorek et al. in 2010<sup>8</sup>.

Subsequently, Ashwin et al.<sup>14,23</sup> categorized three primary mechanisms that can lead to tipping in climate systems: bifurcation-induced tipping, noise-induced tipping, and R-tipping. A method to analyze R-tipping is examining the bifurcation diagrams of the corresponding autonomous system<sup>23,39</sup>. Another approach is constructing asymptotic series expansions to characterize R-tipping<sup>40</sup>. The interplay between noise and the rate of parameter change in triggering a tipping event was studied<sup>41,42</sup>. R-tipping has also been investigated in cases where the quasi-static attractor is not necessarily an equilibrium state but can be periodic<sup>43</sup>, chaotic<sup>28</sup> in multi-dimensional<sup>44–46</sup>, discrete-time<sup>47</sup>, and spatiotemporal dynamical systems<sup>48</sup>. Furthermore, early warning signals, which are a critical indicator for detecting tipping in dynamical systems, have been adapted to R-tipping<sup>49–51</sup>.

Recent years have witnessed a steady growth of interest in R-tipping due to its ubiquity in various natural and man-made systems<sup>35,38,52,53</sup>. For example, R-tipping is particularly relevant to phenomena induced by climate change<sup>54,55</sup>. Rapid environmental changes, such as accelerated global warming, can induce tipping events in various components of the Earth’s climate system, including ice sheets, ocean currents, and ecosystems<sup>35</sup>. For example, the rapid melting of Arctic sea ice due to increasing temperatures can trigger feedback mechanisms that further accelerate ice loss and contribute to global climate instability<sup>56,57</sup>. Ecosystems, characterized by their complex interactions and nonlinear dynamics, are also susceptible to R-tipping<sup>37,58</sup>. Rapid changes in environmental conditions, such as deforestation, pollution, or climate change, can push ecosystems past critical thresholds, leading to regime shifts<sup>37</sup>. These shifts can result in significant biodiversity loss and altered ecosystem services<sup>59</sup>. For instance, coral reefs are highly vulnerable to rapid increases in sea temperature and acidification, which can cause widespread coral bleaching and degradation<sup>60</sup>. Engineering systems, including power grids, transportation networks, and industrial processes, are not immune to R-tipping<sup>61</sup>. These systems often operate under dynamic conditions where parameters such as the load, demand, or operational settings can change rapidly<sup>62</sup>. If the rates of these changes surpasses their corresponding critical thresholds, system failures, cascading outages, or catastrophic breakdowns can occur.

Most previous studies on R-tipping were from a local perspective, focusing on initial conditions that are proximate to the system’s steady state. The local approach is particularly effective if the system is always near an equilibrium point. However, real-world phenomena frequently occur far from some steady state due to noise, disturbances, and inherent variabilities within the system<sup>63–66</sup>. The nonequilibrium initial conditions can significantly influence a system’s response to rapid parameter changes, making it essential to consider initial conditions from a larger region of the phase space than the vicinity of some equilibrium state<sup>36,59</sup>.



FIG. 1. Probability of R-tipping versus the time rate of parameter change: two possible scenarios. The first one, as indicated by the black dashed curve, is that the R-tipping probability increases from zero slowly with the rate, where there is a “safe region” or “safe interval” of the rate in which the probability remains at some near zero value, thereby providing opportunities of control or intervention to reduce the rate of parameter change. The second scenario, as illustrated by the solid blue curve, is that the R-tipping probability grows dramatically and immediately as the rate of parameter change increases from zero. In this case, there is no safe region: in order to keep the probability near zero, the rate of parameter change must practically be reduced to zero. Mathematical reasoning and numerical evidence from empirical complex ecological networks tend to support the second scenario for nonautonomous dynamical systems in the real world.

In this Perspective Article, we provide an overview of the R-tipping phenomenon from a global perspective. The key quantity of interest is the probability of R-tipping. In particular, for a given rate of the parameter change, one chooses a large number of initial conditions from a relevant region of the phase space. The probability of R-tipping is essentially the fraction of initial conditions that lead to R-tipping for this particular rate value. How does this probability depend on the rate of parameter change? Heuristically, there are two scenarios of possible dependence at the opposite extremes, as illustrated schematically in Fig. 1. One is that the probability increases slowly from zero as the rate increases. Such a “benign” dependence might be our hope, e.g., for a climate system, as this means there is certain tolerance of the system against even fast parameter changes, leaving rooms and opportunities for control and mitigation. The opposite scenario is that the probability of R-tipping grows drastically and immediately as the rate of parameter change increases from zero. In this case, the system has no tolerance against any rate value, even if it is small. The implication is dire: in order to avoid R-tipping, the rate of parameter change must be reduced to practically near zero values. Which scenario of R-tipping do real-world dynamical systems tend to follow? The answer, unfortunately, is likely to be the second scenario<sup>36</sup>. Quantitatively, this scenario can be characterized by a general scaling law between the probability of R-tipping

and the rate of parameter change. We shall describe a mathematical theory to understand the scaling law and numerical evidence from a class of complex real-world ecological networks: the plant-pollinator networks from different geographical regions of the world. The overall message is that, to prevent R-tipping from occurring in ecosystems and climate systems, the time rate of parameter change, such as the rate of emissions of harmful chemicals into the environment, must be significantly suppressed and kept at near zero.

## II. NECESSITY OF A GLOBAL PHASE-SPACE APPROACH TO R-TIPPING

In real-world dynamical systems, parameters are not stationary but constantly change with time. For example, in climate systems, factors such as temperature and CO<sub>2</sub> emissions are not static but increase over time, influencing the system’s stability with potentially severe consequences that can lead to the possible collapse of the system. To gain a qualitative understanding of the phenomenon of R-tipping in nonautonomous dynamical systems from a global perspective, we compare it with bifurcation-induced tipping in autonomous dynamical systems.

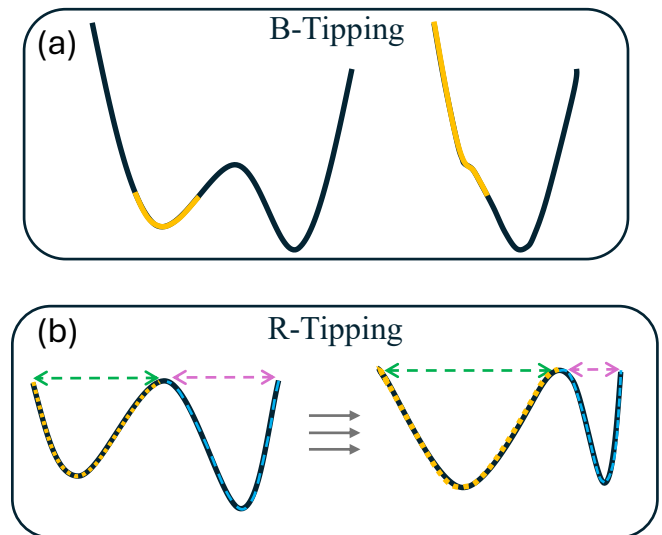


FIG. 2. Schematic comparison of bifurcation-induced tipping and R-tipping in a simple potential model. (a) Bifurcation-induced tipping: static changes in the control parameter lead to the disappearance of one steady state of the system. (b) R-tipping: the state of the system is determined by both the initial condition and the speed at which the parameter changes.

For bifurcation-induced tipping, we consider the scenario where, in the parameter regime of interest, two stable steady states (or attractors) coexist: a “left” state corresponding to “normal” functioning, as highlighted by yellow in Fig. 2(a), and a “right” or “abnormal” state.

Each state has its own basin of attraction. If external factors cause an increase in a bifurcation parameter of the system, a tipping point can occur through a backward saddle-node bifurcation, which is reached when the “normal” fixed point disappears, leaving the “abnormal” state as the sole attractor. As a result, the basin of attraction for the original “normal” state is absorbed into that of the “abnormal” state. This means that, once the system crosses this tipping point, it inevitably transitions to the “abnormal” state, with no possibility of returning to the “normal” state through small perturbations. This type of bifurcation highlights the critical threshold beyond which a system undergoes an irreversible transition to a drastically different state.

The R-tipping scenario is illustrated in Fig. 2(b). Rapid variations of a parameter can trigger a critical transition even if the parameter values remain within some safe bounds. Consider the setting where, in the range of parameter variations, the system exhibits bistability with two coexisting stable steady-state attractors, represented by yellow and blue, respectively, in Fig. 2(b). Each state has its own basin of attraction, as indicated by green and purple arrows, respectively. During the transition, the basin of attraction of the left state (yellow) expands, while the basin of attraction of the right state (blue) shrinks. In this case, the final state of the system is determined by both the initial condition and the speed at which the parameter changes. In particular, rapid parameter changes can push the system past some critical threshold, causing it to settle into a different stable state than it would under slower, more gradual parameter changes.

Most previous studies of R-tipping focused on low-dimensional dynamical systems from a “local” point of view, emphasizing the behavior of specific initial conditions in the vicinity of a stable equilibrium point and trajectories to address issues such as the critical rate for tipping. Commonly, R-tipping was conceived as an abrupt change in the system behavior occurring at a specific rate of change of a bifurcation parameter<sup>23</sup>. However, real-world systems are inherently dynamic and often operate under nonideal conditions<sup>66,67</sup>: they are constantly influenced by external disturbances, fluctuating environmental conditions, and inherent variability where disturbances and stochastic variations are the norm rather than the exception<sup>63,68</sup>. For example, ecological systems are affected by changes in species interactions and environmental factors all the time, which in turn influence population dynamics and stability<sup>64–66</sup>. In these contexts, species may face varying levels of predation, competition, and resource availability, leading to complex and unpredictable behaviors. The dynamic nature of these systems means that they often drift away from the equilibrium point, making them susceptible to tipping events triggered by rapid changes in external conditions. This variability necessitates the study of R-tipping from a global perspective, taking into account the entire relevant phase space rather than focusing solely on the

local dynamics surrounding the equilibrium point.

More generally, initial conditions away from an equilibrium point can influence the system’s response to rapid parameter changes, making it essential to consider a broader range of initial conditions in R-tipping studies. For example, in ecological systems, the natural variability in species populations and environmental conditions invalidate the assumption of initial conditions being near some equilibrium. Similarly, in engineering systems, operational parameters can drift due to external shocks or internal variability, necessitating a comprehensive analysis that accounts for a wide range of initial states. In fact, the integration of diverse initial conditions into R-tipping research is not merely an academic exercise, but a practical necessity for accurately assessing and managing the risks associated with critical transitions in natural and engineered systems. A early example of the dependency of R-tipping on the initial condition was reported in a thermoacoustic system<sup>69</sup>.

### III. SCALING LAW OF R-TIPPING PROBABILITY

In our recent work<sup>36</sup>, a scaling law governing the dependence of the probability of R-tipping on the rate of parameter change was uncovered. To explain the scaling law, we assume that a parameter of the system, denoted as  $p(t)$ , increases at the linear rate  $r$  from an initial value  $p_1$  at time  $t = 0$  to a final value  $p_2 > p_1$  during the time interval  $[0, T]$  with  $T = (p_2 - p_1)/r$ . We assume a qualitative correspondence between the environmental condition and  $p(t)$ : a smaller value of  $p$  indicates a relatively more health state of the system. In ecosystems, such a parameter could be, e.g., the decay rate of some species, which tends to increase as the environment deteriorates with time. Let  $\Phi(r)$  be the probability of R-tipping. The scaling law is

$$\Phi(r) \sim \exp \left[ -C \frac{(p_2 - p_1)}{r} \right], \quad (1)$$

where  $C > 0$  is a constant. This scaling law has the following features: (1) the probability  $\Phi(r)$  is an increasing function of  $r$ , (2) the derivative  $\Phi'(r)$  is a decreasing function of  $r$ , and (3) the maximum rate of increase in  $\Phi(r)$  occurs for  $r \gtrsim 0$ . The second feature indicates that the rate at which  $\Phi(r)$  increases slows down as  $r$  increases, eventually approaching zero as  $r \rightarrow \infty$ . The third feature emphasizes the fact that, insofar as the parameter varies with time, even with a very small rate of change, the chance of R-tipping can be nonzero and large. This has alarming real-world implications: a slow change in the parameter could still precipitate a system collapse with catastrophic consequences, so simply slowing down the rate of parameter change might not be sufficient to prevent tipping. Instead, the rate of change must be reduced to a nearly zero value to prevent R-tipping.

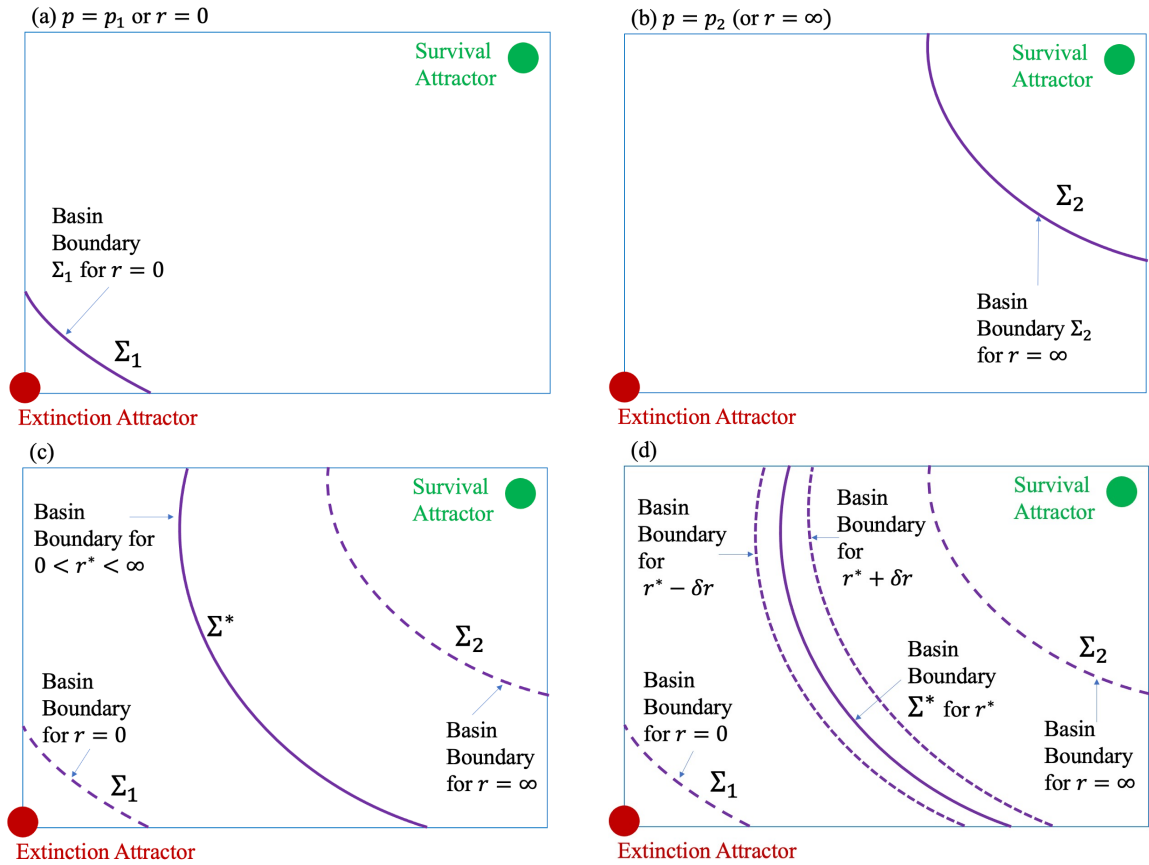


FIG. 3. Phase-space structure for different values of the rate parameter and dependence of the critical rate on initial condition. The nonautonomous bistable system has a time-varying parameter  $p(t)$  that increases at the linear rate  $r$  from an initial value  $p_1$  at time  $t = 0$  to a final value  $p_2 > p_1$  during the time interval  $[0, T]$ , where  $p_1$  ( $p_2$ ) corresponds to a health (deteriorated) environment. There are two attractors throughout the parameter variation: a survival and an extinction attractor. (a) The extreme case  $r = 0$  [ $p(t) = p_1$ ], where the basin boundary  $\Sigma_1$  is located closer to the extinction attractor. (b) The opposite extreme case  $r \rightarrow \infty$ , where basin boundary  $\Sigma_2$  is located closer to the survival attractor. (c) The location of the basin boundary  $\Sigma^*$  for any finite rate value  $r^*$ , which is in between  $\Sigma_1$  and  $\Sigma_2$ . (d) The basin boundaries for rate values  $r^* \pm \delta r$  with infinitesimal  $\delta r$ , revealing that all initial conditions on  $\Sigma^*$  share  $r^*$  as the critical rate value for R-tipping.

### A. Dependence of critical rate for R-tipping on initial condition

To prepare for the derivation of the scaling law (1), we analyze the dependence of the critical rate for R-tipping on the initial condition by using the general setting of bistability in a two-dimensional phase space in which two steady-state attractors coexist: survival and extinction, as illustrated by the filled green and red circles, respectively, in Fig. 3. To gain insights, we begin by considering two extreme cases:  $r = 0$  and  $r \rightarrow \infty$ . For  $r = 0$ , there is no change in the parameter  $p$  with time: it stays at the initial value  $p_1$ . For this relatively small parameter value, the system is mostly “healthy” so the basin of attraction of the survival attractor is “larger.” In this case, the majority of the initial conditions in the phase space will land the system in the survival attractor, so the the basin boundary  $\Sigma_1$  between the two basins is located closer to the extinction attractor, as shown in

Fig. 3(a). In the opposite extreme  $r \rightarrow \infty$ , for any  $t > 0$  the system is already at the larger parameter value  $p_2$  so the system is in a decayed environment that is hostile to system’s normal functioning. In this case, the majority of the initial conditions in the phase space will lead to extinction and the basin boundary  $\Sigma_2$  is closer to the survival attractor, as shown in Fig. 3(b). The phase-space structures for these two extreme cases, as illustrated in Figs. 3(a) and 3(b) for  $r = 0$  and  $r \rightarrow \infty$  respectively, implies that for any finite value of  $r$ , say  $r^*$ , the basin boundary  $\Sigma^*$  must lie in between  $\Sigma_1$  and  $\Sigma_2$ , as shown in Fig. 3(c).

We now provide a mathematical reasoning for the dependence of the critical rate for R-tipping on the initial condition, which provides a geometric principle to determine the critical rate for any given initial condition. In particular, in Fig. 3(c), all initial conditions on the basin boundary  $\Sigma^*$  have the critical rate  $r^*$ . To see this, consider two parameter values in the vicinity of  $r^*$ :  $r^* \pm \delta r$ , where  $\delta r$  is infinitesimal. For  $r = r^* - \delta r < r^*$ , the the

basin boundary is located slightly closer to the extinction attractor than  $\Sigma^*$ , as shown in Fig. 3(d). In this case, all initial conditions on  $\Sigma^*$  belong to the basin of the survival attractor. For  $r = r^* + \delta r > r^*$ , the basin boundary is located slightly closer to the survival attractor than  $\Sigma^*$ , so all initial conditions on  $\Sigma^*$  now belong to the basin of the extinction attractor. As a result, an infinitesimal increment in the rate from  $r^*$  will cause all initial conditions on  $\Sigma^*$  to switch their fate: from survival to extinction, indicating that  $r^*$  is the critical rate of R-tipping for all the initial conditions on the basin boundary  $\Sigma^*$ . For a different value of the rate, say  $r^\dagger$ , the location of the basin boundary in the phase space is different, so the initial conditions on this boundary will have the critical rate  $r^\dagger$ , which is different from  $r^*$ . It is therefore apparent that, from the point of view of the whole phase space, the concept of critical rate may not be meaningful as there is an uncountably infinite number of critical rate values for R-tipping, depending on the initial condition.

## B. Derivation of scaling law (1)

To derive the scaling law (1), we consider two different rates of parameter change:  $r_1$  and  $r_2 > r_1$ . In an autonomous system, the boundary is the stable manifold of an unstable fixed point, denoted as  $\mathbf{f}$ , as indicated by the filled green circle in Fig. 4. When the system becomes nonautonomous with a time-varying parameter  $p(t)$ , the unstable fixed point and the basin boundary become time dependent, so we write  $\mathbf{f}(t)$  or  $\mathbf{f}(p)$ . We focus on an infinitesimal neighborhood of  $\mathbf{f}(t)$ , where the basin boundary is approximately straight, as shown in Figs. 4(a) and 4(b) for  $r = r_1$  and  $r = r_2$ , respectively. For the two cases, the parameter variation occurs within the interval  $[p_1, p_2]$  in the time interval  $[0, T_1]$  and  $[0, T_2]$ , respectively. Since  $r_2 > r_1$ , we have  $T_2 < T_1$ . In each case, the arrowed dashed line through  $\mathbf{f}(t)$  indicates the direction along which it moves in the phase space as the parameter changes over time. The solid green (at  $t = 0$ ) and blue (for  $t = T_1$  or  $T_2$ ) line segments through  $\mathbf{f}(t)$  represent the boundaries separating the extinction from the survival basin. Before the parameter variation starts ( $p = p_1$ ), initial conditions below the solid green lines belong to the basin of the extinction attractor, while those above belong to the survival attractor. After the parameter variation ends ( $p = p_2$ ) the initial conditions below the solid blue lines belong to the extinction basin, and those above belong to the survival basin.

During the parameter variation,  $\mathbf{f}(t)$  shifts from the green circle's position to the blue circle's position, causing its stable manifold (the basin boundary) to move accordingly. Initial conditions in the light-green shaded area, which initially belong to the survival attractor's basin for  $p = p_1$ , will be exponentially stretched along  $\mathbf{f}(t)$ 's unstable direction and compressed in the stable direction, forming a long blue rectangle. Since  $T_1 > T_2$ ,

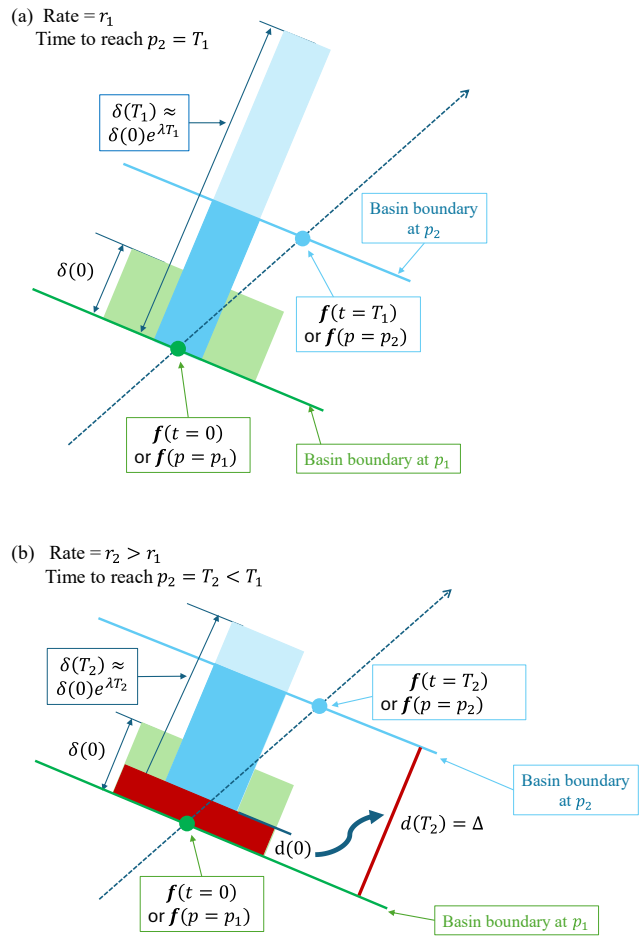


FIG. 4. A geometrical argument leading to the scaling law (1). See text for details.

the blue rectangle for  $r = r_1$  is longer and thinner compared to that for  $r = r_2$ . Due to  $\mathbf{f}(t)$ 's movement and the changing basin boundary as the parameter varies, a portion of the long rectangle (dark shaded blue) now falls within the basin of the extinction attractor. The initial conditions in the original green rectangle that evolve into this dark shaded blue region are those that switch from the survival to the extinction attractor due to the time-dependent parameter change, experiencing R-tipping, as indicated by the red rectangle within the green area in Fig. 4(b). The fraction of such initial conditions for any given rate  $r$  determines the R-tipping probability. Denoting the fraction of R-tipping initial conditions by  $d(0)$  and the distance between the basin boundaries along  $\mathbf{f}(t)$ 's unstable direction by  $\Delta$ , we have

$$d(T) = \Delta = d(0) \exp(\lambda T), \quad (2)$$

where  $\lambda$  is the unstable eigenvalue of  $\mathbf{f}(t)$ , which is assumed to remain approximately constant in the course of time variation of the parameter. Substituting  $T = (p_2 - p_1)/r$  into Eq. (2) leads to the scaling law (1).

#### IV. SCALING LAW OF R-TIPPING IN COMPLEX ECOLOGICAL NETWORKS

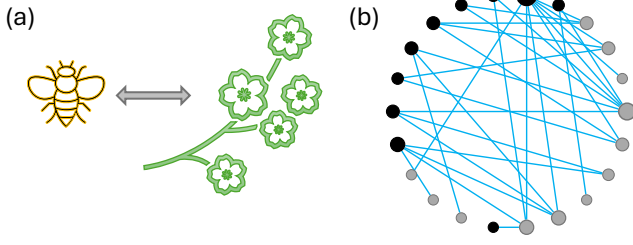


FIG. 5. Illustration of mutualistic networks. (a) Mutualistic interaction between a plant and a pollinator species. (b) Network structure of an empirical mutualistic network from the Web of Life database,

The derivation of the scaling law (1) is based on a two-dimensional phase-space structure. Real-world systems are high-dimensional. Can the scaling law (1) be expected to hold in high-dimensional nonautonomous dynamical systems? One class of such systems that was used<sup>36</sup> to test the scaling law (1) is complex plant-pollinator mutualistic networks<sup>20,24,26,30,70–77</sup>, where a species in the plant group benefits from interacting with some species in the pollinator group, and vice versa, as illustrated in Fig. 5(a). The structures of over 100 empirical mutualistic pollinator–plant networks from many geographical regions in the world have been well documented (Web of Life database: [www.Web-of-Life.es](http://www.Web-of-Life.es)). One example of such a network is shown in Fig. 5(b). At the present, a detailed computational model for these empirical mutualistic networks is available<sup>70</sup>. In spite of their high dimensionality, due to the nature of steady-state dynamics associated with tipping, a mutualistic network can be reduced effective to a two-dimensional system<sup>24</sup> through some conventional dimension-reduction methods, providing a feasible paradigm for numerically testing the scaling law (1).

The dynamics of a plant-pollinator mutualistic network, comprising  $N_A$  pollinator and  $N_P$  plant species, are described by a system of  $N = N_A + N_P$  nonlinear differential equations of the Holling type in terms of the species abundances<sup>70</sup>:

$$\begin{aligned} \dot{P}_i &= P_i \left( \alpha_i^P - \sum_{l=1}^{N_P} \beta_{il}^P P_l + \frac{\sum_{j=1}^{N_A} \gamma_{ij}^P A_j}{1 + h \sum_{j=1}^{N_A} \gamma_{ij}^P A_j} \right), \\ \dot{A}_j &= A_j \left( \alpha_j^A - \kappa_j - \sum_{l=1}^{N_A} \beta_{jl}^A A_l + \frac{\sum_{i=1}^{N_P} \gamma_{ji}^A P_i}{1 + h \sum_{i=1}^{N_P} \gamma_{ji}^A P_i} \right), \end{aligned} \quad (3)$$

$$(4)$$

where  $P_i$  and  $A_j$  are the abundances of the  $i^{\text{th}}$  and  $j^{\text{th}}$  plant and pollinator species, respectively,  $i = 1, \dots, N_P$  and  $j = 1, \dots, N_A$ . The equations account for intrinsic growth rates, intraspecific and interspecific competi-

tions, and mutualistic interactions, with specific parameters characterizing the dynamics, such as the pollinator decay rate  $\kappa$ , intrinsic growth rate  $\alpha^{P(A)}$ , and the half-saturation constant  $h$ . The mutualistic interactions are further quantified through parameters  $\gamma_{ij}^{P(A)} = \xi_{ij} \gamma_0 / K_i^T$  that depend on whether a mutualistic interaction exists ( $\xi_{ij}$ ), the general interaction parameter  $\gamma_0$ , and the degree of the plant species  $K_i$ .

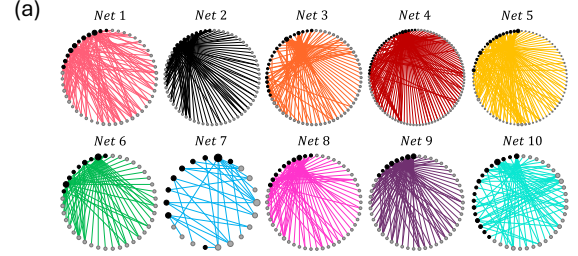


Table I. Ten empirical mutualistic networks from the Web of Life database

Network	Country	$N_P$	$N_A$	$\kappa$ -interval
Net 1	Canary Islands	11	38	[0.90, 0.93]
Net 2	South Africa	9	56	[0.70, 0.99]
Net 3	Argentina	21	45	[0.75, 0.93]
Net 4	Argentina	23	72	[0.87, 0.96]
Net 5	New Zealand	18	60	[0.90, 0.95]
Net 6	USA	7	33	[0.70, 0.99]
Net 7	Acores	10	12	[0.74, 0.88]
Net 8	Denmark	10	40	[0.80, 0.93]
Net 9	Denmark	8	42	[0.70, 0.95]
Net 10	Greenland	17	26	[0.77, 0.98]

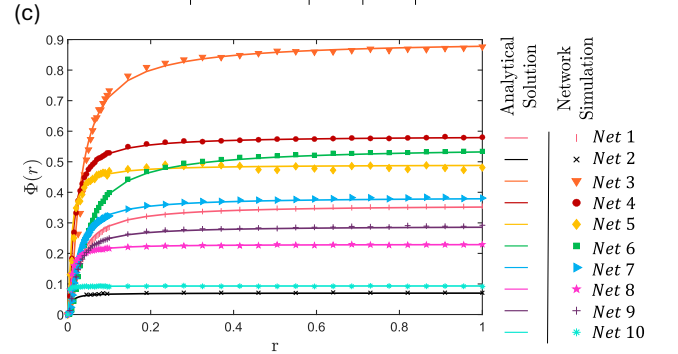


FIG. 6. Scaling law of R-tipping for ten empirical mutualistic networks. The time-varying parameter is the pollinator decay rate  $\kappa$ . (a) The network structures. (b) Structural parameters and origins of the ten empirical networks. (c) R-tipping probability  $\Phi(r)$  versus the rate  $r$  of parameter change. The dots are the results from direct numerical simulations of the high-dimensional networks with an ensemble of random initial conditions from a substantial volume of the high-dimensional phase space. The solid curves are the theoretical fits of the scaling law (1). Other parameter values in the computational model are fixed:  $\alpha = 0.3$ ,  $\beta = 1$ ,  $h = 0.4$ ,  $\gamma^P = 1.93$ , and  $\gamma^A = 1.77$ .

To introduce the rate change of a parameter, three different scenarios were considered<sup>36</sup> where negative environmental impacts can lead to: (1) a linear increase in the species decay rate, (2) a linear decrease in the mutualistic interaction strength, and (3) simultaneous

linear changes in both parameters over time. Simulations were conducted using ten empirical mutualistic pollinator–plant networks, where the intervals of parameter variations were selected to ensure the occurrence of bistability. The structures of the ten networks are illustrated in Fig. 6(a), and their structural parameters and origin are listed in Fig. 6(b). Figure 6(c) presents examples of the scaling law of the probability of R-tipping  $\Phi(r)$  with the rate  $r$  where, for each network, the species decay rate  $\kappa$  varies with time linearly across the interval specified in Fig. 6(b). It can be seen that the R-tipping probabilities for all ten networks exhibit qualitatively similar behavior: as the rate of parameter change increases from zero, the probability initially rises rapidly and then plateaus at an approximately constant value, as predicted by the scaling law (1). The final value of the R-tipping probability depends on the specific structural characteristics of the network.

## V. DISCUSSION

Investigating R-tipping is important for understanding how complex systems respond to constant changes in their parameters due to environmental effects. There are two approaches to R-tipping in nonautonomous dynamical systems: local and global. The local approach focuses on the behavior near some equilibrium point, taking advantage of the corresponding autonomous system to determine the conditions under which the system transitions from one state to another. This approach has provided significant insights into the R-tipping mechanism. The global approach offers a comprehensive and holistic picture, particularly for real-world systems that are typically open, dynamic, and under the influences of stochastic disturbances. Due to these effects, in the phase space the dynamical trajectory of the system can hardly be confined near some equilibrium point but rather, it can wander in an extended region of the phase space. Under such circumstances, a meaningful way to investigate R-tipping is through a probability defined with respect to initial conditions from the relevant phase-space region. As a result, the notion of critical rate for R-tipping loses its applicability, as it depends on the initial condition and there is an uncountably infinite number of such critical values. The scaling law (1), which can be derived through a geometric argument and has been validated in a number of complex empirical plant-pollinator mutualistic networks, suggests a dire consequence of the rate of parameter changes: insofar as it is not zero, the probability of R-tipping can be substantial. This can have implications, e.g., in environmental policy making, rendering necessary testing the scaling law in the broad spectrum of nonautonomous dynamical systems arising from various disciplines.

A topic of current interest is data-driven prediction of tipping in nonlinear and complex dynamical systems. Traditional methods often rely on detailed knowledge of

the system’s dynamics, which may not always be available. A promising approach is machine learning, especially recurrent neural-network architectures capable of identifying dynamical patterns and predicting critical transitions by learning from data<sup>78–86</sup>. To develop effective machine-learning models for predicting tipping, two considerations are essential. First, the availability of training data is crucial. While traditional classification problems require data from both sides of the critical transition, this is impractical since post-transition data are not available. Training must be based on data from the pre-critical regime, necessitating the development of models that can extrapolate from this information. Second, the machine-learning architecture must be capable of Self-organizing evolution, mimicking the inherent dynamics of the target system. The recently developed adaptable reservoir-computing<sup>79–83</sup> provides a possibility.

A significant problem to which R-tipping is relevant is the possible collapse of the Atlantic Meridional Overturning Circulation (AMOC)<sup>87–89</sup> that plays a crucial role in maintaining moderate temperature conditions in Western Europe by transporting warmer, upper waters northward and returning colder, deeper waters southward<sup>89,90</sup>. While studies have indicated a tendency for AMOCs to weaken in the last 30 years<sup>91,92</sup>, at the present the AMOC is still in a stable state that ensures the continuation of these critical ocean flows. However, due to the increasing human influence on the climate change, a potential halt of this circulation signifying a collapse of the AMOC represents a shift to another stable steady state within the underlying dynamical system, which is characteristic of R-tipping. Recent research based on simplified stochastic dynamical system models suggests that the AMOC may be on the course towards collapse, with a tipping point potentially occurring as early as 2025<sup>57</sup>. Estimating the probability of the collapse due to R-tipping based on measurement data is an urgent but open problem.

## DATA AVAILABILITY

All relevant data are available from the authors upon request.

## CODE AVAILABILITY

All relevant computer codes are available from the authors upon request.

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