Supplementary Information for

Closed-loop control of complex nonlinear networks: A tradeoff between time and energy

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CONTENTS

I. UPPER BOUNDS FOR CONTROL TIME AND ENERGY COST

We provide mathematical estimates of the upper bounds for control time and the associated energy cost with the proposed closed-loop controller $\boldsymbol{u}^S$.

A. Preliminaries

We list two Lemmas that will be used in our analysis.

Lemma S1.1 ([1]). *Let $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $0 < p < 1$. The following inequality holds:*

$$\sum_{i=1}^{n} \xi_i^p \geq \left(\sum_{i=1}^{n} \xi_i\right)^p .$$

Lemma S1.2 ([2]). *For any $0 < q \leq p$, there exist two positive numbers $\zeta_{1,2}$ such that*

$$\zeta_1 \|\cdot\|_p \leq \|\cdot\|_q \leq \zeta_2 \|\cdot\|_p,$$

where $\|\cdot\|_h$ $(h = p, q)$ is the L_h-norm for the n-dimensional space $\mathbb{R}^n$. Specifically, $\zeta_1 = 1$ and $\zeta_2 = n^{\frac{1}{q}-\frac{1}{p}}$.

B. Estimate of control time

For the general closed-loop controlled network dynamics in the main text, we introduce the following Lyapunov function:

$$V(\boldsymbol{x}) = \sum_{i=1}^{N} x_i^\top x_i = \sum_{i=1}^{N} \|x_i\|^2 = \|\boldsymbol{x}\|^2, \tag{S1.1}$$

where $\boldsymbol{x} = \left[x_1^\top, \cdots, x_N^\top\right]^\top \in \mathbb{R}^{Nd}$ and $\|\cdot\|$ represents the L_2-norm of the given vector. We assume $\boldsymbol{x}(0) \notin \mathcal{U} = \left\{\|\boldsymbol{x}(0)\| < 1\right\}$. Differentiating the function V along a typical trajectory of the system, we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= 2\sum_{i=1}^{N} x_i^\top f(x_i) + 2\sum_{i=1}^{N} x_i^\top \sum_{j=1}^{N} c_{ij}\Gamma x_j - 2k\sum_{i=1}^{N} x_i^\top x_i \\ &\leq 2(l-k)\sum_{i=1}^{N} x_i^\top x_i - 2\boldsymbol{x}^\top \boldsymbol{H}\boldsymbol{x} \leq -2(k - l - \eta_{\max})V(t),\end{aligned} \tag{S1.2}$$

where $\boldsymbol{H} \equiv \frac{1}{2}\left[(\boldsymbol{C}\otimes\boldsymbol{\Gamma})^\top + \boldsymbol{C}\otimes\boldsymbol{\Gamma}\right]$ is a matrix and $\eta_{\max}$ is its maximum eigenvalue. The global Lipschitz condition on f can be relaxed to the one-sided uniform Lipschitz condition (a function f is said to be one-sided uniformly Lipschitzian if for some $l > 0$, we have $|x^\top f(x)| \leq l\|x\|^2$ for all $x \in \mathbb{R}^n$). Choosing $k > l + \eta_{\max}$ and integrating the differential inequality (S1.2) from 0 to t, we get $V[(\boldsymbol{x}(t)] = \|\boldsymbol{x}(t)\|^2$, which is circumscribed by an exponentially decreasing quantity. We thus have $V(\boldsymbol{x}(t^*)) = 1$ and

$$\|\boldsymbol{x}(t^*)\| = 1 \;\text{ with }\; t^* \leq \frac{\ln\|\boldsymbol{x}(0)\|}{\rho} > 0, \tag{S1.3}$$

where $\rho = k - l - \eta_{\max}$ (as defined in the main text).

We next prove that $\|\boldsymbol{x}(t)\| < 1$ for all $t \in (t^*, +\infty)$. Intuitively, this is a result of *system dissipation*. The proof is carried out by contradiction. Specifically, assume this is not the case. We can then obtain the first time instant at which the trajectory $\boldsymbol{x}(t)$, after entering the unit ball $\mathcal{U}$, hits the ball again. Denote this time by

$$t' = \inf \left\{ t \in [\hat{t}, t_1) \,\middle|\, \|\boldsymbol{x}(t)\| = 1 \right\},$$

where the time instants $\hat{t}$ and t_1 satisfy $\|\boldsymbol{x}(t)\| < 1$ with $t^* < t < \hat{t} < t' < t_1 < +\infty$. All the time instants can be found because of the continuity of the trajectory $\boldsymbol{x}(t)$ and the assumption that $\boldsymbol{x}(t)$ can hit the unit ball. For $t \in [\hat{t}, t')$, taking the derivative of $V(t)$ with respect to t yields

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= 2\sum_{i=1}^{N} x_i^\top f(x_i) + 2\sum_{i=1}^{N} x_i^\top \sum_{j=1}^{N} c_{ij}\Gamma x_j - 2k\sum_{i=1}^{N} x_i^\top \operatorname{sig}(x_i)^\alpha \\ &\leq 2(l + \eta_{\max})\boldsymbol{x}^\top \boldsymbol{x} - 2k \sum_{i=1}^{N} x_i^\top \operatorname{sig}(x_i)^\alpha. \end{aligned} \tag{S1.4}$$

From Lemma S1.1, we have

$$\sum_{i=1}^{N} x_i^\top \operatorname{sig}(x_i)^\alpha = \sum_{i=1}^{N}\sum_{j=1}^{d} |x_{ij}|^{\alpha+1} \geq \left(\sum_{i=1}^{N}\sum_{j=1}^{d} |x_{ij}|^2\right)^{\frac{\alpha+1}{2}},$$

which gives a further estimation for dV/dt:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq 2(l + \eta_{\max})V(t) - 2k\,[V(t)]^{\frac{\alpha+1}{2}}. \tag{S1.5}$$

Since $V(t) = \|\boldsymbol{x}(t)\|^2 \leq 1$ for all $t \in [\hat{t}, t')$, we have $V(t) \leq V^{\frac{\alpha+1}{2}}(t)$ for all $t \in [\hat{t}, t')$. Hence, the estimation in (S1.5) can be refined as:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -2\rho V^{\frac{\alpha+1}{2}}(t), \quad \text{for all } \ t \in [\hat{t}, t'). \tag{S1.6}$$

This implies $\mathrm{d}V/\mathrm{d}t \leq 0$ for all $t \in [\hat{t}, t')$, so we have

$$1 > \|\boldsymbol{x}(\hat{t})\|^2 = V[\boldsymbol{x}(\hat{t})] \geq V[\boldsymbol{x}(t)]$$

for all $t \in [\hat{t}, t')$. In the limit $t \to t'$, we have $1 > V(\boldsymbol{x}(\hat{t})) \geq V(\boldsymbol{x}(t')) = 1$. This is a contradiction, which implies that for all $t \in (t^*, t_1)$, $\boldsymbol{x}(t) \in \mathcal{U}$ holds, where t_1 can be extended to $+\infty$.

We can now prove that the trajectory $\boldsymbol{x}(t)$ of the general nonlinear network system in the main text approaches the desired target within a finite-time duration in $(t^*, +\infty)$. In particular, from the estimation in (S1.6) and the theory of differential inequalities [3], we have $V(t) \leq W(t)$, where $t \in (t^*, +\infty)$ and $W(t)$ satisfies the following equation:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -2\rho W^{\frac{\alpha+1}{2}}(t), \quad \text{for all } \ t > t^*, \tag{S1.7}$$

with the initial condition $W(t^*) = V(t^*) = 1$. From (S1.7), we have

$$\frac{1}{1-\alpha} W^{\frac{1-\alpha}{2}}(t) = -\rho t + c_0, \quad \text{for all } \; t > t^*, \tag{S1.8}$$

where $c_0 = \rho t^* + \frac{1}{1-\alpha} V^{\frac{1-\alpha}{2}}(t^*)$ and t^* is defined in (S1.3). From (S1.8), we have

$$V(t) \leq W(t) = [(1-\alpha)(-\rho t + c_0)]^{\frac{2}{1-\alpha}}. \tag{S1.9}$$

Letting $W(t) = 0$, we obtain the upper bound for the time T_f^S to achieve control:

$$T_f^S \leq t^* + \frac{\|\boldsymbol{x}(t^*)\|^{1-\alpha}}{\rho(1-\alpha)} = t^* + \frac{1}{\rho(1-\alpha)}.$$

For the case of $\boldsymbol{x}(0) \in \mathcal{U}$, a similar argument leads to the upper bound for T_f^S as

$$T_f^S \leq \frac{\|\boldsymbol{x}(0)\|^{1-\alpha}}{\rho(1-\alpha)}.$$

The estimated upper bound for T_f^S can thus be summarized as

$$T_f^{S_{\text{up}}} = \begin{cases} \frac{1}{\rho} \ln \|\boldsymbol{x}(0)\| + \frac{1}{\rho(1-\alpha)}, & \boldsymbol{x}(0) \notin \mathcal{U}, \\ \frac{1}{\rho(1-\alpha)} \|\boldsymbol{x}(0)\|^{1-\alpha}, & \boldsymbol{x}(0) \in \mathcal{U}. \end{cases} \tag{S1.10}$$

For the special case of controlled linear network dynamics $\dot{\boldsymbol{x}} = \boldsymbol{C}\boldsymbol{x} + \left[\boldsymbol{u}^S\right]^\top$, we set $l = 0$, $\boldsymbol{\Gamma} = 1$, $b_{ii} = 1$, and all other $b_{im} = 0$. The upper bound of the required control time can be estimated as

$$T_f^{S_{\text{up}}} = \begin{cases} \frac{1}{\rho} \ln \|\boldsymbol{x}(0)\| + \frac{1}{(k-\mu_{\max})(1-\alpha)}, & \boldsymbol{x}(0) \notin \mathcal{U}, \\ \frac{1}{(k-\mu_{\max})(1-\alpha)} \|\boldsymbol{x}(0)\|^{1-\alpha}, & \boldsymbol{x}(0) \in \mathcal{U}, \end{cases}$$

where $\mu_{\max}$ is the maximal eigenvalue of the matrix $\frac{1}{2}\left[\boldsymbol{C} + \boldsymbol{C}^\top\right]$.

C. Estimate of control energy cost

Case 1: $\boldsymbol{x}(0) \notin \mathcal{U}$. From the definition in the main text, the energy cost is given by

$$\mathcal{E}_c^S = \int_0^{T_f} \sum_{i=1}^N \left\|u_i^S(t)\right\|^2 \mathrm{d}t = \int_0^{t^*} \sum_{i=1}^N \left\|u_i^L(t)\right\|^2 \mathrm{d}t + \int_{t^*}^{T_f} \sum_{i=1}^N \left\|u_i^F(t)\right\|^2 \mathrm{d}t.$$

Outside the unit ball $\mathcal{U}$, the energy cost can be estimated as

$$\int_0^{t^*} \sum_{i=1}^N \left\|u_i^L(t)\right\|^2 \mathrm{d}t = k^2 \int_0^{t^*} \|\boldsymbol{x}(t)\|^2 \mathrm{d}t = k^2 \int_0^{t^*} V(t)\mathrm{d}t.$$

From the estimate (S1.2), we get

$$\begin{aligned} k^2 \int_0^{t^*} V(t)\mathrm{d}t \;&\leq\; k^2 V(0) \int_0^{t^*} \mathrm{e}^{-2\rho t}\mathrm{d}t \\ &=\; k^2 V(0) \left(-\frac{1}{2\rho}\mathrm{e}^{-2\rho t^*} + \frac{1}{2\rho}\right) k^2 \left[\frac{1}{2\rho} - \frac{1}{2\rho\|\boldsymbol{x}(0)\|^2}\right]. \end{aligned} \tag{S1.11}$$

Note that

$$\sum_{i=1}^{N}\left\|u_i^F(t)\right\|^2 = k^2\sum_{i=1}^{N}\sum_{j=1}^{d}|x_{ij}(t)|^{2\alpha} = k^2\left\|\boldsymbol{x}(t)\right\|_{2\alpha}^{2\alpha} \le \zeta k^2\left\|\boldsymbol{x}(t)\right\|^{2\alpha} = \zeta k^2 V^{\alpha}(t),$$

where the inequality follows from Lemma S1.2 and $\zeta = (\zeta_2)^{2\alpha} = \left[(Nd)^{\frac{1}{2\alpha}-\frac{1}{2}}\right]^{2\alpha} = (Nd)^{1-\alpha}$. This, with (S1.9), gives an estimate of the energy cost inside $\mathcal{U}$:

$$\begin{aligned}&\int_{t^*}^{T_f}\sum_{i=1}^{N}\left\|u_i^F(t)\right\|^2\mathrm{d}t \le \zeta k^2\int_{t^*}^{T_f}V^{\alpha}(t)\mathrm{d}t \le \zeta k^2\int_{t^*}^{T_f}(1-\alpha)^{\frac{2}{1-\alpha}}(-\rho t+c_0)^{\frac{2\alpha}{1-\alpha}}\mathrm{d}t\\ =\ &\zeta k^2\frac{1}{\rho(1+\alpha)}(1-\alpha)^{\frac{1+\alpha}{1-\alpha}}\left[(-\rho t^*+c_0)^{\frac{1+\alpha}{1-\alpha}}-(-\rho T_f+c_0)^{\frac{1+\alpha}{1-\alpha}}\right],\end{aligned} \tag{S1.12}$$

where $c_0 = 1/(1-\alpha)$. Substituting the estimation of T_f into (S1.12), we get

$$\int_{t^*}^{T_f}\sum_{i=1}^{N}\left\|u_i^F(t)\right\|^2\mathrm{d}t \le \zeta k^2\frac{1}{\rho(1+\alpha)}. \tag{S1.13}$$

Finally, from (S1.11) and (S1.13), we obtain the upper bound estimate of the energy-cost as

$$\mathcal{E}_c^{S_{\mathrm{up}}} = k^2\frac{1}{2\rho}\left[1-\|\boldsymbol{x}(0)\|^{-2}+\frac{2\zeta}{1+\alpha}\right].$$

Case 2: $\boldsymbol{x}(0)\in\mathcal{U}$. The energy cost is

$$\mathcal{E}_c = \int_0^{T_f}\sum_{i=1}^{N}\left\|u_i^F(t)\right\|^2\mathrm{d}t \le \zeta k^2\int_0^{T_f}V^{\alpha}(t)\mathrm{d}t.$$

Following the argument for Case 1, we get

$$\begin{aligned}\mathcal{E}_c\ &\le\ \zeta k^2\int_0^{T_f}(1-\alpha)^{\frac{2}{1-\alpha}}(-\rho t+\tilde{c}_0)^{\frac{2\alpha}{1-\alpha}}\mathrm{d}t\\ &=\ \zeta k^2\frac{1}{\rho(1+\alpha)}(1-\alpha)^{\frac{1+\alpha}{1-\alpha}}\left[(\tilde{c}_0)^{\frac{1+\alpha}{1-\alpha}}-(-\rho T_f+\tilde{c}_0)^{\frac{1+\alpha}{1-\alpha}}\right],\end{aligned}$$

where $\tilde{c}_0 = \frac{1}{1-\alpha}\|\boldsymbol{x}(0)\|^{1-\alpha}$. From the estimated T_f in (S1.10), we get

$$\mathcal{E}_c^{S_{\mathrm{up}}} = \frac{\zeta k^2}{\rho(1+\alpha)}\|\boldsymbol{x}(0)\|^{1+\alpha}.$$

To summarize, the analytical estimate for the upper bound of the energy cost is given by

$$\mathcal{E}_c^{S_{\mathrm{up}}} = \begin{cases} k^2\frac{1}{2\rho}\left[1-\|\boldsymbol{x}(0)\|^{-2}+\frac{2\zeta}{1+\alpha}\right], & \boldsymbol{x}(0)\notin\mathcal{U};\\ k^2\frac{\zeta}{\rho(1+\alpha)}\|\boldsymbol{x}(0)\|^{1+\alpha}, & \boldsymbol{x}(0)\in\mathcal{U},\end{cases}$$

where $\zeta = (Nd)^{1-\alpha}$.

TABLE S1. Results of controlling 22 nonlinear food-web networks with the controllers $\boldsymbol{u}^{S,F,L}$, where $K_i = 5$, $A_i = 1$, $k = 2$, and $\alpha = \frac{1}{2}$. The dynamical variables in the initial state are chosen randomly from the interval $[0, 5]$. Each data point is the result of averaging 100 control realizations.

Food-web name	# of nodes	# of edges	T_f^S	T_f^F	T_f^L
Chesapeake	39	177	2.88	5.45	7.32
ChesLower	37	166	2.84	5.37	7.05
ChesMiddle	37	203	2.85	5.34	6.96
ChesUpper	37	206	2.90	5.43	7.30
CrystalC	24	125	2.91	5.49	7.34
CrystalD	24	100	2.91	5.48	7.15
Everglades	69	916	2.92	5.50	7.35
Florida	128	2106	2.92	5.50	7.25
Maspalomas	24	82	2.80	5.27	7.48
Michigan	39	221	2.91	5.49	7.18
Mondego	46	400	2.90	5.44	7.18
Narragan	35	220	2.94	5.52	7.50
Rhode	20	53	2.90	5.46	7.22
St. Marks	54	356	2.87	5.37	7.28
baydry	128	2137	2.92	5.50	7.36
baywet	128	2106	2.92	5.49	7.06
cypdry	71	640	2.90	5.47	7.16
cypwet	71	631	2.90	5.48	7.05
gramdry	69	915	2.92	5.50	7.28
gramwet	69	916	2.93	5.52	7.30
Mangrove Dry	97	1491	2.92	5.50	7.28
Mangrove Wet	97	1492	2.93	5.52	7.31

II. CONTROLLING FOOD-WEB NETWORKS: DATA AND ANALYSES

All the results on control time for controlling the 22 food-web networks are shown in Tab. S1. The food-web data are from the website:
http://vlado.fmf.uni-lj.si/pub/networks/data/bio/foodweb/foodweb.htm

As shown in Fig. S1, the required control time and energy cost for controlling the Florida food-web network exhibit exactly the opposite trends with increasing k and α. This, together with Fig. 2 in the main text, reveals a control trade-off between the time and the energy cost inherent to the controller $\boldsymbol{u}^S$.

III. EIGENVALUE DISTRIBUTIONS OF ECOLOGICAL NETWORKS

Here we prove that, for May's classic ecosystem, $\boldsymbol{H}$'s eigenvalues are distributed in the interval $\left[-r - \sqrt{2NP}\sigma_0, -r + \sqrt{2NP}\sigma_0\right]$ in a probabilistic sense as $N \to \infty$. Thus, to realize control requires

$$k > \eta_{\max} = \sqrt{2NP}\sigma_0 - r \ \ \textbf{(Condition-A)}.$$

For the mixed ecosystem, $\boldsymbol{H}$'s eigenvalues are distributed in the interval

$$\big[-\sqrt{2NP[\mathbb{D}(\mathcal{Y})+\mathbb{E}^2(|\mathcal{Y}|)]}-r, \sqrt{2NP[\mathbb{D}(\mathcal{Y})+\mathbb{E}^2(|\mathcal{Y}|)]}-r\big]$$

as $N \to \infty$. Particularly, for $\mathcal{Y} \sim \mathcal{N}(0, \sigma_0^2)$, this interval becomes $\big[-\sqrt{2NP(1+2/\pi)}\sigma_0 - r, \sqrt{2NP(1+2/\pi)}\sigma_0 - r\big]$, yielding

$$k > \sqrt{2NP(1+2/\pi)}\sigma_0 - r \ \ \textbf{(Condition-B)}$$

which ensures finite-time control in the probabilistic sense. For the PP system, we have

$$k > \sqrt{2NP(1-2/\pi)}\sigma_0 - r \ \ \textbf{(Condition-C)}$$

for realizing control in the probabilistic sense.

A. Wigner semicircle law

Lemma S3.1 (Semicircle Law [4, 5])**.** *Let $\{Z_{i,j}\}_{1\le i<j}$ and $\{Y_i\}_{1\le i}$ be two independent families of i.i.d., zero mean, and real-valued random variables with $\mathbb{E}(Z_{1,2}^2) = 1$. Further, assume that for all integers $k \ge 1$,*

$$r_k \triangleq \max\left\{\mathbb{E}|Z_{1,2}|^k, \mathbb{E}|Y_1|^k\right\} < \infty.$$

Set the elements of the symmetric $N \times N$ matrix $\boldsymbol{X}_N$ as:

$$\boldsymbol{X}_N(i,j) = \boldsymbol{X}_N(j,i) = \begin{cases} Z_{i,j}/\sqrt{N}, & i<j, \\ Y_i/\sqrt{N}, & i=j. \end{cases}$$

Let the empirical measure be $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$, where λ_i $(1 \le i \le N)$ are the real eigenvalues of $\boldsymbol{X}_N$. Let the standard semicircle distribution be the probability distribution $\sigma(x)\mathrm{d}x$ on $\mathbb{R}$ with the density

$$\sigma(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathcal{I}_{|x|<2},$$

where $\mathcal{I}$ is the indication function of a given set. Then, L_N converges weakly probabilistically to the standard semicircle distribution as $N \to \infty$.

B. Eigenvalue distributions of ecological networks

May's classic ecosystem. For this system, we have $c_{ii} = -r$ and the off-diagonal elements c_{ij} are mutually independent random variables that obey the Gaussian normal distribution $\mathcal{N}(0, \sigma_0^2)$ with probability P and are zero with probability $1-P$. Denote each element of the symmetric matrix $\boldsymbol{H} = \frac{1}{2}\left[\boldsymbol{C} + \boldsymbol{C}^\top\right]$ by $\xi_{ij} = \frac{1}{2}(c_{ij} + c_{ji})$. The expectation is $\mathbb{E}(\xi_{ij}) = \frac{1}{2}\left[\mathbb{E}(c_{ij}) + \mathbb{E}(c_{ji})\right] = 0$ and the variance is given by

$$\mathbb{D}(\xi_{ij}) = \frac{1}{4}\mathbb{D}(c_{ij}+c_{ji}) = \frac{1}{2}\mathbb{D}(c_{ij}) = \frac{1}{2}\mathbb{E}(c_{ij}^2) - \frac{1}{2}\mathbb{E}^2(c_{ij}) = \frac{1}{2}P\sigma_0^2.$$

From the semicircle law for random matrices (Lemma S3.1), the eigenvalues of $\boldsymbol{H} = \frac{1}{2}(\boldsymbol{C}^\top + \boldsymbol{C})$ are located in $\left[-r - \sqrt{2NP}\sigma_0, -r + \sqrt{2NP}\sigma_0\right]$ in a probabilistic sense as $N \to \infty$. Thus, to realize control requires $k > \eta_{\max} = \sqrt{2NP}\sigma_0 - r$ **(Condition-A)**.

Mixed ecosystem. In a mixed network with competition and mutualistic interactions, we have $c_{ii} = -r$ and the off-diagonal elements (c_{ij}, c_{ji}) have the same sign, which with probability P are drawn from the distribution $(\pm|\mathcal{Y}|, \pm|\mathcal{Y}|)$ and are zero with probability $(1-P)$. We then have

$$\begin{aligned}\mathbb{D}(\xi_{ij}) &= \frac{1}{4}\mathbb{D}\big(c_{ij}+c_{ji}\big) = \frac{1}{4}\left[\mathbb{D}(c_{ij}) + \mathbb{D}(c_{ji}) + 2\mathrm{Cov}(c_{ij}, c_{ji})\right] \\ &= \frac{1}{4}\left[2P\mathbb{D}(\mathcal{Y}) + 2\mathbb{E}(c_{ij}c_{ji}) - 2\mathbb{E}(c_{ij})\mathbb{E}(c_{ji})\right] \\ &= \frac{1}{2}\left[P\mathbb{D}(\mathcal{Y}) + \mathbb{E}(c_{ij}c_{ji})\right] = \frac{1}{2}P\left[\mathbb{D}(\mathcal{Y}) + \mathbb{E}^2(|\mathcal{Y}|)\right].\end{aligned}$$

The semicircle law implies that the eigenvalues of $\frac{1}{2}(\boldsymbol{C}^\top + \boldsymbol{C})$ are located in

$$\left[-r-\sqrt{2NP\left[\mathbb{D}(\mathcal{Y}) + \mathbb{E}^2(|\mathcal{Y}|)\right]}, -r+\sqrt{2NP\left[\mathbb{D}(\mathcal{Y}) + \mathbb{E}^2(|\mathcal{Y}|)\right]}\right]$$

in the probabilistic sense as $N \to \infty$. In particular, for $\mathcal{Y} \sim \mathcal{N}(0, \sigma_0^2)$, we have $\mathbb{D}(\mathcal{Y}) = \sigma_0^2$ and

$$\mathbb{E}(|\mathcal{Y}|) = \int_{-\infty}^{+\infty} |y| \frac{1}{\sqrt{2\pi}\sigma_0} \mathrm{e}^{-\frac{y^2}{2\sigma_0^2}} \mathrm{d}y = \sqrt{\frac{2}{\pi}}\sigma_0.$$

In this case, the eigenvalues of $\frac{1}{2}(\boldsymbol{C}^\top + \boldsymbol{C})$ are located in

$$\left[-r-\sqrt{2NP\left(1+\frac{2}{\pi}\right)}\sigma_0, -r+\sqrt{2NP\left(1+\frac{2}{\pi}\right)}\sigma_0\right]$$

in the probabilistic sense as $N \to \infty$. Figure S2 shows the accuracy of the control criterion $k > k^* = \sqrt{2NP\left(1+\frac{2}{\pi}\right)}\sigma_0 - r$ (**Condition-B**) obtained from the above estimated interval for the eigenvalue distributions. Figure S2 also shows how the growth of population size N affects the required control time and energy cost. These results agree well with the analytical estimates.

Predator-prey ecosystem. In this system, we have $c_{ii} = -r$ and the off-diagonal elements (c_{ij}, c_{ji}) have the opposite sign, which with probability P are drawn from the distribution $(\pm|\mathcal{Y}|, \mp|\mathcal{Y}|)$, and are zero with probability $(1-P)$. We have

$$\mathbb{D}(\xi_{ij}) = \frac{1}{2}\left[P\mathbb{D}(\mathcal{Y}) + \mathbb{E}(c_{ij}c_{ji})\right] = \frac{1}{2}P\left[\mathbb{D}(\mathcal{Y}) - \mathbb{E}^2(|\mathcal{Y}|)\right].$$

Applying the semicircle law, we have that the eigenvalues of $\boldsymbol{H} = \frac{1}{2}(\boldsymbol{C}^\top + \boldsymbol{C})$ are located in

$$\left[-r-\sqrt{2NP\left[\mathbb{D}(\mathcal{Y}) - \mathbb{E}^2(|\mathcal{Y}|)\right]}, -r+\sqrt{2NP\left[\mathbb{D}(\mathcal{Y}) - \mathbb{E}^2(|\mathcal{Y}|)\right]}\right]$$

in the probabilistic sense as $N \to \infty$. Especially, for $\mathcal{Y} \sim \mathcal{N}(0, \sigma_0^2)$, the eigenvalues of $\boldsymbol{H} = \frac{1}{2}(\boldsymbol{C}^\top + \boldsymbol{C})$ are located in

$$\left[-r-\sqrt{2NP\left(1-\frac{2}{\pi}\right)}\sigma_0, -r+\sqrt{2NP\left(1-\frac{2}{\pi}\right)}\sigma_0\right]$$

in the probabilistic sense as $N \to \infty$. The control criterion in the probabilistic sense becomes $k > \sqrt{2NP(1-2/\pi)}\sigma_0 - r$ (**Condition-C**).

IV. FLEXIBILITY OF CONTROL

We demonstrate the flexibility of control with different configurations of $\boldsymbol{C}$ and B_i using the ecosystems. In particular, the off-diagonal elements c_{ij} $(j \neq i)$ are constructed from an undirected scale-free network (SFN) [6] while the diagonal elements are chosen to be $c_{ii} = \xi - \sum_{j=1, j\neq i}^{N} c_{ij}$ with $\xi > 0$. We have $\lambda_{\max}(\boldsymbol{C}) = \xi > 0$, so the uncontrolled system is unstable. With our controller $\boldsymbol{u}^S$, setting $k > \xi$ is sufficient for achieving control if we set $b_{ii} = 1$ for all i. In applications, it is desired to reduce the number of controlled nodes. We thus randomly select N_D nodes for control (i.e., $b_{i_j i_j} = 1$ for $1 \leq j \leq N_D$) and define $n_D \equiv N_D/N$.

We find that the energy cost decreases as n_D is increased (a result consistent with that in linear network control [7, 8]), as controlling more nodes can significantly reduce the control time, and increasing the mean degree m of the network can reduce both the control time and energy (for a given n_D value), as shown in Fig. S3(a). We also find that controlling high-degree nodes can reduce the time and energy for $n_D \lesssim 0.2$. However, if many nodes are accessible to control, controlling low-degree nodes can yield better performance, as shown in Fig. S3(b).

V. HINDMARSH-ROSE NEURONAL MODEL

We consider a small-world network of Hindmash-Rose (HR) neurons y_i with the coupling scheme $\sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma} h_{ij}(y_i, y_j)$, where $\boldsymbol{\Gamma} = \mathrm{diag}[1, 0, 0]$ and $h_{ij}^S = u_i^S|_{x_i = y_j - y_i}$. In the network, the i-th neuron y_i $(1 \leq i \leq N)$ is described of the HR type [9]:

$$\begin{cases} \dot{y}_{i1} = y_{i2} - y_{i3} + 3y_{i1}^2 - y_{i1}^3 + I, \\ \dot{y}_{i2} = 1 - y_{i2} - 5y_{i1}^2, \\ \dot{y}_{i3} = -r y_{i3} + 4\nu(y_{i1} + 1.6), \end{cases}$$

where y_{i1} is the membrane potential, y_{i2} stands for the recovery variable associated with the fast current, y_{i3} is a slowly changing adaptation current, $I = 3.281$ is the external current input, and $\nu = 0.0012$ is the damping rate of the slow ion channel. Figure S4(a) shows that synchronization can be achieved rapidly through control. Comparing with the linear coupling scheme $h_{ij}^L = u_i^L|_{x_i = y_j - y_i}$, our controller h_{ij}^S leads to a faster transition, regardless of the network size N, as shown in Fig. S4(b).

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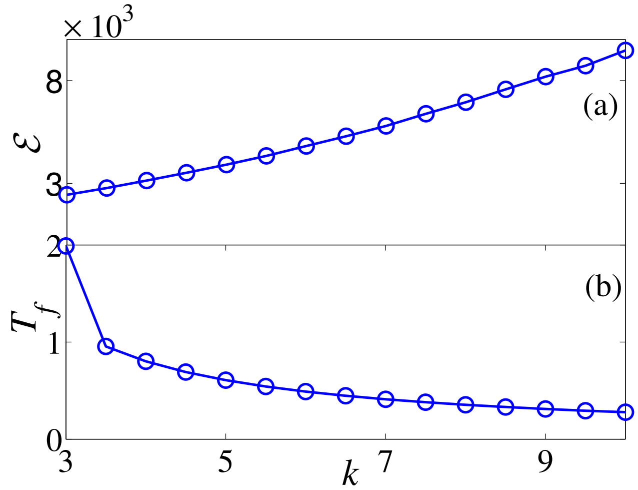
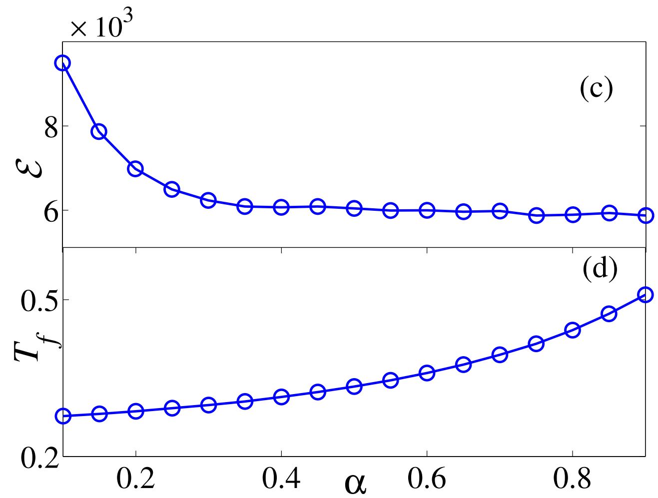
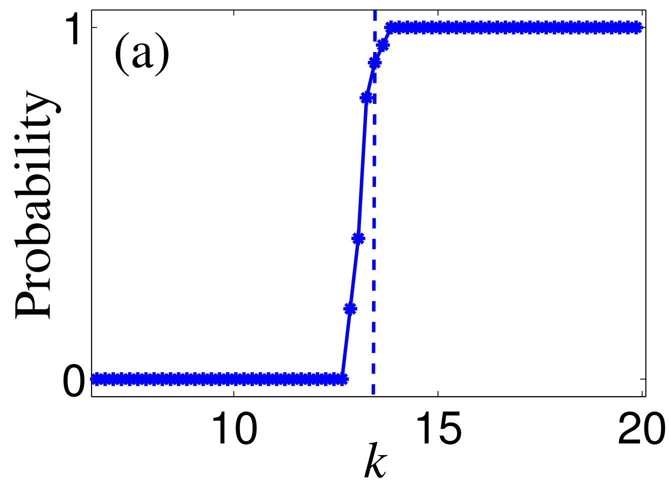


(a)

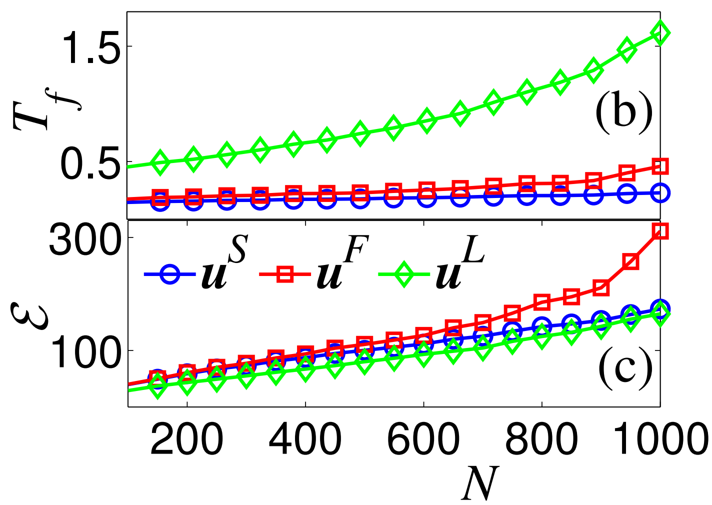


(b)

FIG. S1. **Trade-off between required control time and energy cost**. Effects of increasing k and α on control time and energy cost for the Florida food-web network: (a) energy cost versus k, (b) control time versus k, (c) energy cost versus α, and (d) control time versus α. The initial state values are randomly taken from the interval $[0, 5]$.



(a)



(b)

FIG. S2. **Eigenvalue distribution and estimates of the required control time and energy cost for mixed ecosystems**. (a) The probability of successfully controlling a mixed ecosystem when feedback control strength k passes through the critical value $k^* = \sqrt{2NP\left(1+\frac{2}{\pi}\right)}\sigma_0 - r$ (indicated by the vertical dashed line). The probability is calculated by simulating 100 random matrices with $N = 250$, $P = 0.25$, $\sigma_0 = 1$, and $r = 1$. (b,c) Required control time and energy cost, respectively, for the controlled mixed ecosystem subject to controllers $\boldsymbol{u}^S$ (circles), $\boldsymbol{u}^F$ (squares) and $\boldsymbol{u}^L$ (diamonds). The parameters are $P = 0.25$, $\sigma_0 = 1$, $k = 1.1k^*$, $\alpha = 0.8$, and $N \in [50, 1000]$. All the initial state values of the networked system are randomly chosen from the interval $[-5, 5]$.

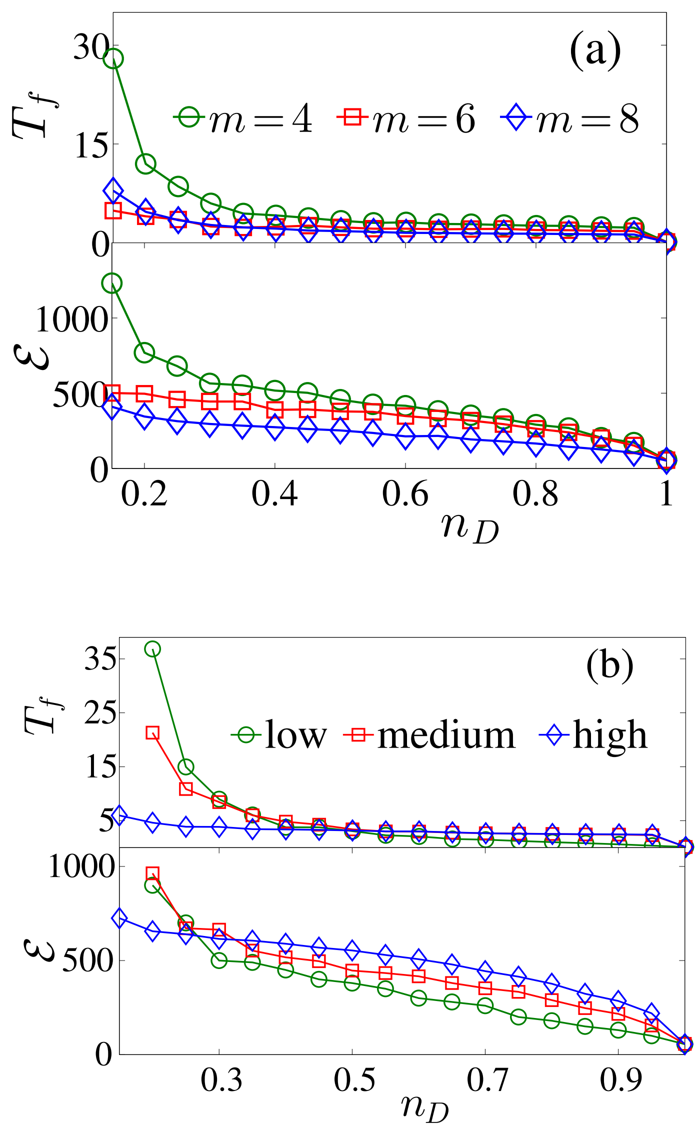


FIG. S3. **Flexibility performance with different control configurations**. For scale-free networks, control time and energy versus the density n_D of driver nodes for (a) mean degrees $m = 4,\ 6,\ 8$ and (b) $m = 6$ and driver nodes of high, medium, and low degrees. The network size is $N = 500$ and controller parameters are $\xi = 1$ and $k = 30$. Other parameters are the same as those in Fig. 4 in the main text.

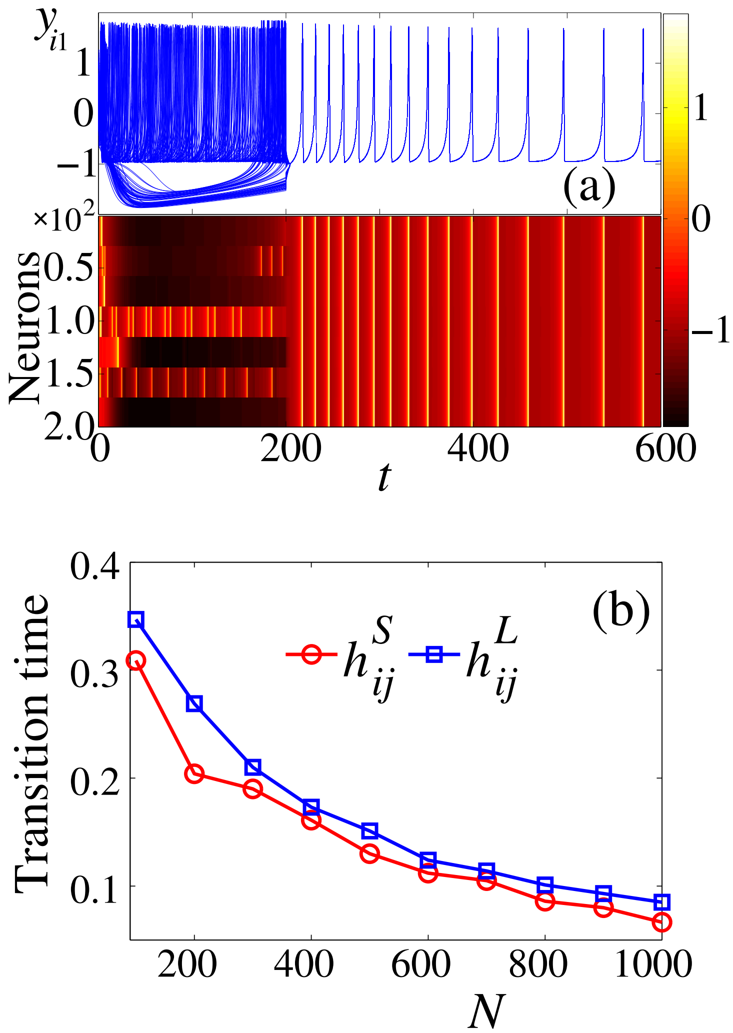


FIG. S4. **Controlled generation synchronization of spiking HR neuronal networks**. (a) Time course (upper) and color map (lower) of all potentials y_{i1} of a HR neuronal network, where $\alpha = 1/2$, $k = 0.15$, h_{ij}^S is activated at $t = 200$, and the rewiring probability 0.1 and $N = 200$ are used for generating the small-world network. (b) Synchronization transition time for different values of N.