Strange Nonchaotic Attractors in Random Dynamical Systems

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Whether strange nonchaotic attractors (SNAs) can occur typically in dynamical systems other than quasiperiodically driven systems has long been an open question. Here we show, based on a physical analysis and numerical evidence, that robust SNAs can be induced by small noise in autonomous discrete-time maps and in periodically driven continuous-time systems. These attractors, which are relevant to physical and biological applications, can thus be expected to occur more commonly in dynamical systems than previously thought.

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The subject of strange nonchaotic attractors (SNAs) has attracted continuous interest in the nonlinear and statistical physics community [1–13]. Here "strange" refers to the nontrivial, complicated geometry of the attractor, and "nonchaotic" indicates that the maximum Lyapunov exponent of the attractor is nonpositive and there is thus no sensitive dependence on initial conditions. In principle, strange nonchaotic attractors occur in all dissipative dynamical systems that exhibit the period-doubling route to chaos, where the attractors formed at the accumulation points of period-doubling cascades are fractal sets with zero Lyapunov exponent. Such attractors are, however, not physically observable because the set of parameter values for them to arise has Lebesgue measure zero in the parameter space. Situations where SNAs can arise typically were described by Grebogi *et al.* [1], who discovered that quasiperiodically driven dynamical systems admit SNAs in parameter regions of positive Lebesgue measure. Since then, there have been many studies on SNAs in quasiperiodic systems [2–7,9–13], including the experimental observations of these attractors [8]. Of particular physical interest is the study that SNAs can occur in quantum systems with a quasiperiodic potential [2]. SNAs are also relevant to biological situations [4]. Mathematical issues concerning SNAs have begun to be addressed recently [12,13]. In this regard, it is useful to note that SNAs in quasiperiodic systems can be typical (in the sense of positive Lebesgue measure), but they can be either robust or nonrobust, depending on whether they persist under small perturbations or not, respectively. A robust SNA is typical but not vice versa. For instance, SNAs arising in quasiperiodic quantum systems are typical but not robust because they occur on a positive Lebesgue measure Cantor set in the parameter space [2,3]. Robust SNAs appear, however, to be quite common [5–11].

So far, robust SNAs have been identified and studied *exclusively* in quasiperiodically driven dynamical systems. Since many physical, biological, and engineering systems do not fall in this category, it is natural to ask whether SNAs can arise in situations where the underlying system is autonomous or periodically driven. Identification of possible SNAs there would be of great interest because such dynamical systems are extremely common in many applications. There has been some (unsuccessful) attempt in this regard [14], which makes the problem even more intriguing and appealing.

In this Letter, we report robust SNAs in both autonomous and periodically driven dynamical systems. Our interest is in discrete-time, autonomous maps such as the logistic map, and continuous-time, periodically driven systems such as the kicked Duffing's oscillator. These systems are representative and well studied in nonlinear dynamics. The key feature that leads to our success in uncovering SNAs in these systems is that we consider small noise. The situation we study is therefore highly physically relevant, as small noise is inevitable in experimental or realistic applications. In particular, we consider maps and periodically driven systems in periodic windows and show that, for any such window, under noise robust SNAs must necessarily arise in a finite range of the noise amplitude. The notion that periodic windows occur commonly in nonlinear systems is well accepted. Our results thus indicate that SNAs can be expected to be observable and robust in very general settings in nonlinear dynamical systems that are not quasiperiodically driven. To be more specific, let p be a bifurcation parameter varying, which leads to the occurrences of various periodic windows, and let D be the noise amplitude. We are able to argue, based on a physical analysis, that there exist open sets of finite areas in the two-dimensional parameter space (p, D) for which the asymptotic attractor is strange but nonchaotic. Extensive numerical evidence will be provided to support our analysis.

We consider general discrete-time maps $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p)$ and periodically driven systems described by differential equations of the following form: $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, z, p)$ and $dz/dt = \omega$, where $\mathbf{x} \in \mathcal{R}^N$ and p is the bifurcation parameter. For the continuous-time system, z is a time variable and the velocity field $\mathbf{f}$ depends periodically on z. We choose p such that the system is in a periodic window of period m and the asymptotic attractor of the system is a periodic attractor of period $2^k m$, where $k = 0, 1, \ldots$. Let p_m be the parameter value for the beginning of the window, which is triggered by a saddle-node bifurcation that creates a period-m stable orbit, and let p_{m*} be the parameter value for the end of the period-doubling cascade of the original stable period-m orbit. We focus on the parameter interval $p_m < p < p_{m*}$ in which the attractor is periodic. Note that the intervals $\{\Delta p_m \equiv |p_{m*} - p_m| > 0\}_m$ are open and dense on the parameter axis [15]. In such a setting, for maps the maximum Lyapunov exponent is negative, except for a set of parameter values of Lebesgue measure zero where the period-doubling bifurcations occur. For periodically driven systems, there is a null Lyapunov exponent generated by $dz/dt = \omega$, but in a periodic window the maximum nontrivial exponent is negative. Now consider additive, *bounded* noise of amplitude D (for simplicity). Our goal is to show that for $p_m < p < p_{m*}$, there exists a range of the noise amplitude $\Delta D_m > 0$ for which the asymptotic attractor [16] is nonchaotic but strange, and robust with respect to small perturbations.

In a periodic window, a periodic attractor and a nonattracting chaotic invariant set (chaotic saddle) coexist. Trajectory from a random initial condition typically moves toward the chaotic saddle along its stable manifold, stays near the saddle for a finite time, and leaves the saddle along its unstable manifold before finally approaching the periodic attractor. There is thus transient chaos for $p_m < p < p_{m*}$. In the absence of noise, the asymptotic attractor is periodic, despite transient chaos. If noise is not strong enough to kick a trajectory on the attractor to a nearby region where the stable manifold of the chaotic saddle lies, the final attractor is still approximately periodic with a negative maximum Lyapunov exponent. Only when the noise amplitude D exceeds a critical value D_m is the probability finite for a trajectory on the original periodic attractor to be perturbed to the vicinity of the stable manifold of the chaotic saddle and move toward the chaotic saddle. Because the saddle is nonattracting, the trajectory can spend only a finite amount of time near it before approaching the original periodic attractor again, and so on. For $D \gtrsim D_m$, a trajectory switches intermittently between the original periodic attractor and the chaotic saddle. There is then a sudden change in the structure of the asymptotic attractor at D_m: for $D \gtrsim D_m$ the attractor contains both a periodic and a chaotic component, in contrast to the perturbed periodic attractor for $D < D_m$.

For discrete maps, the maximum Lyapunov exponent of the periodic attractor is $\lambda_1^P < 0$. As a trajectory begins to visit the chaotic saddle for $D \gtrsim D_m$, the maximum exponent λ_1 of the new attractor starts to increase from λ_1^P. It has been shown recently [17] that the increase in λ_1 obeys the following universal algebraic scaling law: $\lambda_1 - \lambda_1^P \sim (D - D_m)^\alpha$, where the scaling exponent $\alpha > 0$ depends on the phase-space dimension of the system and the dynamical invariants of the chaotic saddle such as its average lifetime and the Lyapunov spectrum [17]. We see that λ_1 can remain negative for a range of the noise amplitude above D_m: $D_m < D < D_{*m}$, where D_{*m} is the noise amplitude for which $\lambda_1 = 0$. We thus have $\Delta D_m \equiv D_{*m} - D_m \sim |\lambda_1^P|^{1/\alpha} > 0$. In this noise range, the attractor of the system may be geometrically complicated but its maximum Lyapunov exponent remains negative. The same consideration [18] applies to periodically driven systems for which a null Lyapunov exponent always exists but the maximum nontrivial Lyapunov exponent of the periodic attractor remains negative for $D < D_{*m}$. Thus, for $D_m < D < D_{*m}$, the asymptotic attractor of the system can have a strange geometry because it contains a chaotic component (the original chaotic saddle in the periodic window), yet the maximum Lyapunov exponent is nonpositive.

We now argue that the attractors created for $D_m < D < D_{*m}$ possess the fundamental properties of SNAs. We first consider the finite-time behavior of the maximum Lyapunov exponent. It is known [7,9–11] that an SNA, while having a nonpositive maximum Lyapunov exponent, possesses regions in the phase space in which infinitesimal vectors in fact grow in length. That is, in any finite-time interval, there is a finite probability that the maximum exponent is temporally positive. The asymptotic exponent can be regarded as the weighted sum of the temporally positive exponent when the trajectory visits the expanding regions and the temporally negative exponent when the trajectory is in regions in which vectors contract [9–11]. The asymptotic exponent becomes negative when the negative component weighs over the positive one [19]. Here the existence of the two sets with distinct behaviors for the evolution of infinitesimal vectors is apparent: for $D > D_m$ a trajectory visits both the original periodic attractor for which tangent vectors contract and the chaotic saddle for which the vectors expand. The maximum Lyapunov exponent can then be written as

$$\lambda_1 \approx f_P(D)\lambda_1^P + f_S(D)\lambda_1^S, \tag{1}$$

where $\lambda_1^S > 0$ is the maximum Lyapunov exponent of the chaotic saddle, $f_P(D)$ and $f_S(D)$ are the frequencies of visit to the original periodic attractor and the saddle, respectively. For $D_m < D < D_{*m}$, the first term weighs over the second term in Eq. (1), giving rise to a possible SNA. For $D > D_{*m}$, the second term begins to dominate, leading to a chaotic attractor. It is useful to note that, since both the periodic attractor and the chaotic saddle are dynamically invariant in the noiseless situation, the exponents λ_1^P and λ_1^S are well defined with respect to their respective invariant measures [20] and, hence, we expect relation (1) to be meaningful, at least for $D \gtrsim D_m$ (after the transition to a possible SNA).

Another key characteristic of an SNA lies in its Fourier spectrum. It is well established that an SNA necessarily possesses a singular-continuous spectrum that contains both discrete and continuous components [6]. For the attractors for $D_m < D < D_{*m}$, their spectra naturally contain these distinct components for an apparent reason: such an attractor consists of a periodic component for which the spectrum is discrete and a chaotic component for which the spectrum is broadband. Thus, we expect the attractors for $D_m < D < D_{*m}$ to have singular-continuous Fourier spectra.

Our analysis thus establishes that, in the parameter plane (p, D), there are open areas of the various sizes $(\Delta p_m, \Delta D_m)$, where m denotes the period of every possible periodic window, in which the attractors are SNAs. These are induced by noise, and they are typical in the parameter space. In addition, since the nonpositivity of the Lyapunov exponent for $D_m < D < D_{*m}$ and the strangeness of the noise-induced attractors, as characterized by fluctuations of the finite-time Lyapunov exponent into the positive side and the singular-continuous spectrum, are statistical properties of the attractors under random perturbations, they are robust. The noise-induced SNAs are thus physically observable.

We now provide numerical support. Our first example is the logistic map $x_{n+1} = rx_n(1 - x_n)$, perhaps the best studied autonomous, discrete-time chaotic system. We choose $r = 3.8008$ so that there is a period-8 window and the maximum Lyapunov exponent is $\lambda^P \approx -0.127$. When additive noise is present and its amplitude D is increased from zero, the asymptotic attractor remains periodic until $D = D_8 \approx 8.3 \times 10^{-6}$, where we observe that the Lyapunov exponent λ starts to increase from λ^P. The attractor becomes chaotic for D above $D_{*8} \approx 1.1 \times 10^{-5}$, for which λ is positive. Thus, for $D_8 < D < D_{*8}$, the attractor has a negative Lyapunov exponent. To provide evidence for the strangeness of the attractor, we show in Fig. 1(a) the finite-time Lyapunov exponent $\lambda(i)$ computed from trajectory segments of length 1000. We see that $\lambda(i)$ remains negative for most of the time but it can be positive intermittently. Figure 1(b) shows a histogram of $\lambda(i)$, where its fluctuations into the positive side are apparent. To examine the spectral characteristics of the attractor, we note that for a singular-continuous spectrum, the number $N(\delta)$ of discrete peaks with intensity larger than δ obeys the following scaling law [6]: $N(\delta) \sim \delta^{-\kappa}$, where $1 < \kappa < 2$. Figures 2(a) and 2(b) show, for $D = 10^{-5}$, Fourier spectrum of the attractor and $N(\delta)$ versus δ on a logarithmic scale, respectively. From Fig. 2(b) we obtain $\kappa \approx 1.2$. To provide more solid evidence for the singular-continuous nature of the spectrum, we compute the time-dependent Fourier transform $X(\Omega, T) = \sum_{n=1}^{T} x_n e^{i2\pi n\Omega}$. For a proper frequency Ω [typically chosen to be the golden mean $(\sqrt{5} - 1)/2$], the following scaling relation $|X(\Omega, T)|^2 \sim T^\beta$ holds and, for SNAs the scaling exponent satisfies $1 < \beta < 2$ [6]. This behavior is shown in Fig. 2(c), where we observe a relatively robust power-law behavior with $\beta \approx 1.5$. It was also suggested [6] that for SNAs, the spectral trajectory with respect to T in the complex plane (ReX, ImX) should exhibit a fractal behavior ("fractal walk"). This is indeed observed for the noise-induced attractor in the logistic map, as shown in Fig. 2(d). Figures 1(a) and 1(b) and 2(a)–2(d) can arguably be regarded as strong evidence for noise-induced SNAs in the logistic map.

We now consider the periodically driven Duffing's oscillator under noise: $d^2x/dt^2 + 0.1dx/dt + (1.0 + 0.45\cos t)x - x^3 + D\xi(t)$, where $\xi(t)$ is an effectively bounded, Gaussian stochastic process of zero mean and unit variance [21]. At the chosen set of parameter values, for $D = 0$ the system is in a period-4 window with the maximum nontrivial Lyapunov exponent $\lambda_1^P \approx -0.047$. The range of noise amplitude for which SNAs can possibly occur is $D_4(\approx 0.03) < D < D_{*4}(\approx 0.08)$. Figure 3(a) shows, for $D = 0.06$, a trajectory of 10 000 iterations on the stroboscopic surface of section defined by $t_n = 2n\pi(n = 1, 2, \ldots)$. The attractor appears to be geometrically strange, although its nontrivial maximum Lyapunov exponent is $\lambda_1 \approx -0.03$. Figures 3(b) and 3(c) show the evolution of the finite-time exponent $\lambda_1(t)$,

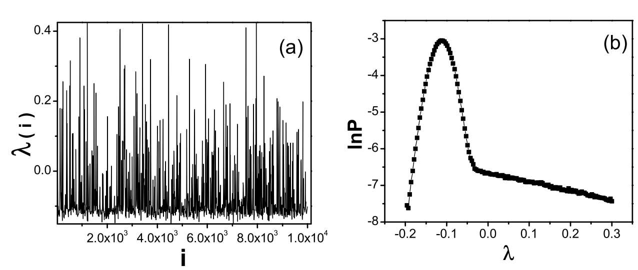

FIG. 1. For the logistic map in a period-8 window under noise of amplitude $D = 10^{-5}$: (a) evolution and (b) distribution of the finite-time Lyapunov exponent.

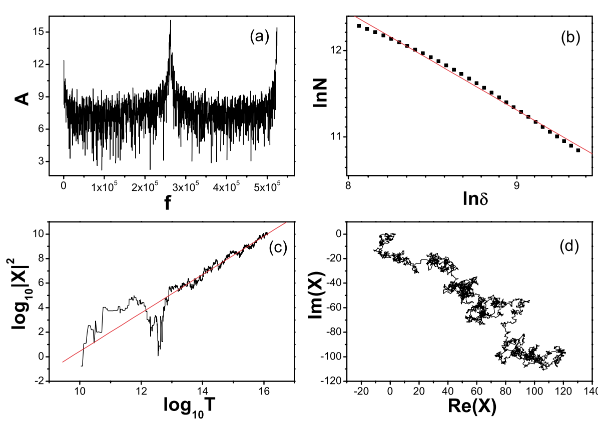

FIG. 2 (color online). For the logistic map in a period-8 window under noise of amplitude $D = 10^{-5}$: (a) Fourier spectrum, (b) spectral intensity distribution of discrete peaks on a logarithmic scale, (c) finite-time Fourier power $|X(\Omega, T)|^2$ versus T on a logarithmic scale, and (d) fractal walk of the spectral trajectory in the complex plane (ReX, ImX).

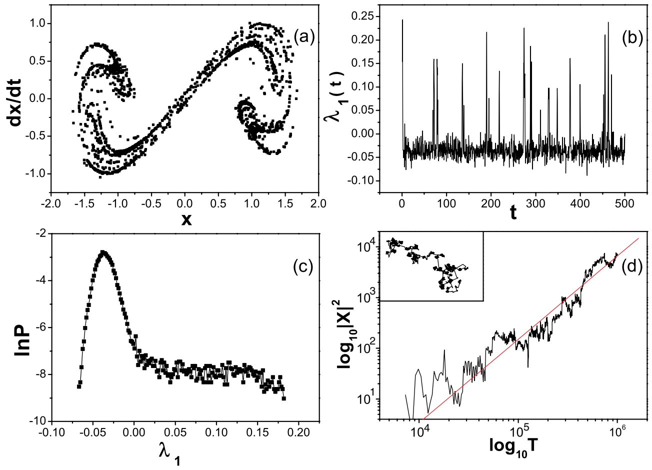

FIG. 3 (color online). For the periodically driven Duffing's oscillator under additive noise of amplitude $D = 0.06$: (a) attractor on a stroboscopic surface of section, (b) evolution of the finite-time exponent $\lambda_1(t)$, (c) histogram of $\lambda_1(t)$, and (d) power-law scaling of $|X(\Omega, T)|^2$ with T, where the scaling exponent is $1 < \beta \approx 1.6 < 2$. The inset in (d) shows a fractal walk in the complex plane (ReX, ImX).

computed using time window of length 200, and histogram of the exponent, respectively. There are apparently fluctuations of the finite-time exponent into the positive side. Figure 3(d) shows $\log_{10}|X(\Omega, T)|^2$ versus $\log_{10}T$, where we observe approximately a power-law behavior with the scaling exponent $\beta \approx 1.6$. The behavior of fractal walk of a spectral trajectory in the complex plane of $X(\Omega, T)$ is shown in the inset in Fig. 3(d). All behaviors in Figs. 3(a)–3(d) indicate strongly that the noise-induced attractor is an SNA for the Duffing's system.

In summary, we have shown that robust SNAs can occur in autonomous or periodically driven, random dynamical systems. This is contrary to the presently acknowledged belief that such attractors can be observed only in quasiperiodically driven systems. SNAs are interesting not only dynamically, but they are also physically relevant. Our study indicates that the interplay between noise and deterministic dynamics can lead to unexpected, yet important phenomenon.

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[16] Attractors in dynamical systems under random perturbations were considered and studied in the past. In fact, natural invariant measures for the attractor can be defined [F. Ladrappier, Commun. Math. Phys. **81**, 229 (1981); L.-S. Young, Ergod. Theory Dyn. Syst. **2**, 109 (1982)]. Under noise, the fractal structure of the attractor is smeared up to scales determined by the noise amplitude. It should be emphasized that, since arbitrarily large noise can move a trajectory out of the basin of attraction, attractors in stochastic systems are meaningful only for small, bounded noise. The types of attractors studied in our Letter are no exception.

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[21] In our numerical experiments we used *finite* realizations of a Gaussian random noise generator. We also found that the occurrence of strange nonchaotic attractors does not depend upon the nature of the noise. For instance, utilizing noise uniformly distributed in $[-1, 1]$ gives essentially the same results as in Figs. 1–3.