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Observability of lag synchronization of coupled chaotic oscillators

Saeed Taherion[1] and Ying-Cheng Lai[1,2]
[1]*Department of Physics and Astronomy, The University of Kansas, Lawrence, Kansas 66045*
[2]*Department of Mathematics, The University of Kansas, Lawrence, Kansas 66045*
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Lag synchronization is a recently discovered theoretical phenomenon where the dynamical variables of two coupled, nonidentical chaotic oscillators are synchronized with a time delay relative to each other. We investigate experimentally and numerically to what extent lag synchronization can be observed in physical systems where noise is inevitable. Our measurements and numerical computation suggest that lag synchronization is typically destroyed when the noise level is comparable to the amount of *average* system mismatch. At small noise levels, lag synchronization occurs in an intermittent fashion. [S1063-651X(99)50506-0]

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Synchronization of coupled chaotic oscillators has become an area of intense interest in the nonlinear dynamics community [1]. Quite recently, it has been discovered that two coupled *nonidentical* chaotic oscillators can exhibit the phenomenon of lag synchronization in which the dynamical variables of the two systems become synchronized but with a time lag with respect to each other [2]. Specifically, given two slightly different chaotic oscillators $d\mathbf{x}_1/dt=\mathbf{F}_1(\mathbf{x}_1)$ and $d\mathbf{x}_2/dt=\mathbf{F}_2(\mathbf{x}_2)$, if there is a coupling between them with a coupling strength ϵ, then one expects $\mathbf{x}_1(t)$ to synchronize with $\mathbf{x}_2(t+\tau)$ in a range of ϵ values, where $\tau\neq 0$ is the time lag which depends on both ϵ and the parameter characterizing the difference between the two oscillators. As such, lag synchronization cannot be observed if two oscillators are completely identical. Since in reality, it is not possible to have identical nonlinear oscillators, it was speculated that the phenomenon of lag synchronization would be typical in systems of coupled chaotic oscillators [2].

We have performed a series of experiments, using electronic circuits that replicate the dynamics of the chaotic Rössler oscillator [3], to detect lag synchronization. Unfortunately, we have encountered great difficulty in generating robust lag synchronization of two coupled Rössler oscillators in the laboratory. What has typically been observed in experiments is that the time-delayed variables of one oscillator tend to follow the variables of another oscillator only *intermittently* in time in some range of the coupling strength. In particular, if one measures the difference $\Delta\mathbf{x}_\tau(t)\equiv|\mathbf{x}_2(t+\tau)-\mathbf{x}_1(t)|$ with τ chosen to minimize the root-mean-square, normalized average of $\Delta\mathbf{x}_\tau(t)$, one observes that $\Delta\mathbf{x}_\tau(t)$ tends to exhibit intermittency with frequent large bursts away from zero. Increasing the coupling strength often leads to a transition to complete synchronization, i.e., $|\mathbf{x}_1(t)-\mathbf{x}_2(t)|\rightarrow 0$. This difficulty has led us to speculate that the inevitable presence of small random disturbance during the experiments may be a key factor that obstructs the observation of lag synchronization, a phenomenon which relies on a precise timing between the dynamics of the coupled chaotic oscillators. In order to test this conjecture, we have performed numerical computations to investigate the influence of small random noise on lag synchronization. Our analyses suggest that lag synchronization is destroyed when the noise level is larger than or comparable to the amount of the average mismatch between the two chaotic oscillators. At small noise levels, lag synchronization appears in an intermittent fashion over many orders of magnitude of the noise amplitude, a result that is consistent with our experimental observation.

We first describe the experimental setup. Our experiments are conducted using a pair of electronic oscillators whose dynamics mimic those of the chaotic Rössler attractor [3]. To have robust chaos for individual oscillators, we construct the circuits so that they contain components for which the voltage-current relations are piecewise linear [4,5]. The schematic diagram of both chaotic oscillators and the coupling circuit is shown in Fig. 1. The equations describing the experimental system can be written in the following dimensionless form: $dx_{1,2}/dt=-\gamma x_{1,2}-\alpha y_{1,2}-z_{1,2}+\epsilon(x_{2,1}-x_{1,2})$, $dy_{1,2}/dt=-\beta x_{1,2}+a_{1,2}y_{1,2}$, $dz_{1,2}/dt=g(x_{1,2})-z_{1,2}$, where $g(x_{1,2})=0$ if $x_{1,2}\leqslant 3$, $g(x_{1,2})=\mu(x_{1,2}-3)$ if $x_{1,2}>3$ and the parameters are as follows: $\alpha=0.5$, $\beta=1$, $\gamma=0.05$, $a_1=0.113$, and $a_2=0.129$ with about $\pm 5\%$ uncertainty due to the tolerance of the components in the circuit and $\mu=15$. Each circuit has three voltage outputs, corresponding to the dynamical variables x, y, and z in the Rössler system [3]. To stipulate nonidentity of the two chaotic oscillators, resistors R_1 and R_2 in the circuit are chosen to be $75k\Omega$ and $67k\Omega$ respectively. This difference in R_1 and R_2 corresponds to approximately a 10% difference in the parameters a_1 and a_2.

The typical oscillating frequencies of the circuits are in audio frequency range. We use a simple linear scheme for coupling between the two oscillators, i.e., two terms $\pm\epsilon(x_2-x_1)$, in the form of voltage, are applied to the derivatives of the x variables in both circuits, where ϵ is the parameter characterizing the coupling strength. In the experiments, ϵ can be changed systematically with the accuracy of 0.5% of the change. The electronic components in each circuit are carefully chosen, and the circuits are assembled on high-quality printed-circuit boards in order to minimize the effect of internal and environmental noise. Both oscillators and the coupling circuits are operated by a low-ripple and low-noise power supply (HPE3631A). The voltages from x, y, and z are recorded by using a 12-bit data acquisition board (DAS1800AO, Keithley) at the sampling frequency of 100 kHz. The noise voltage of the circuit is measured by having

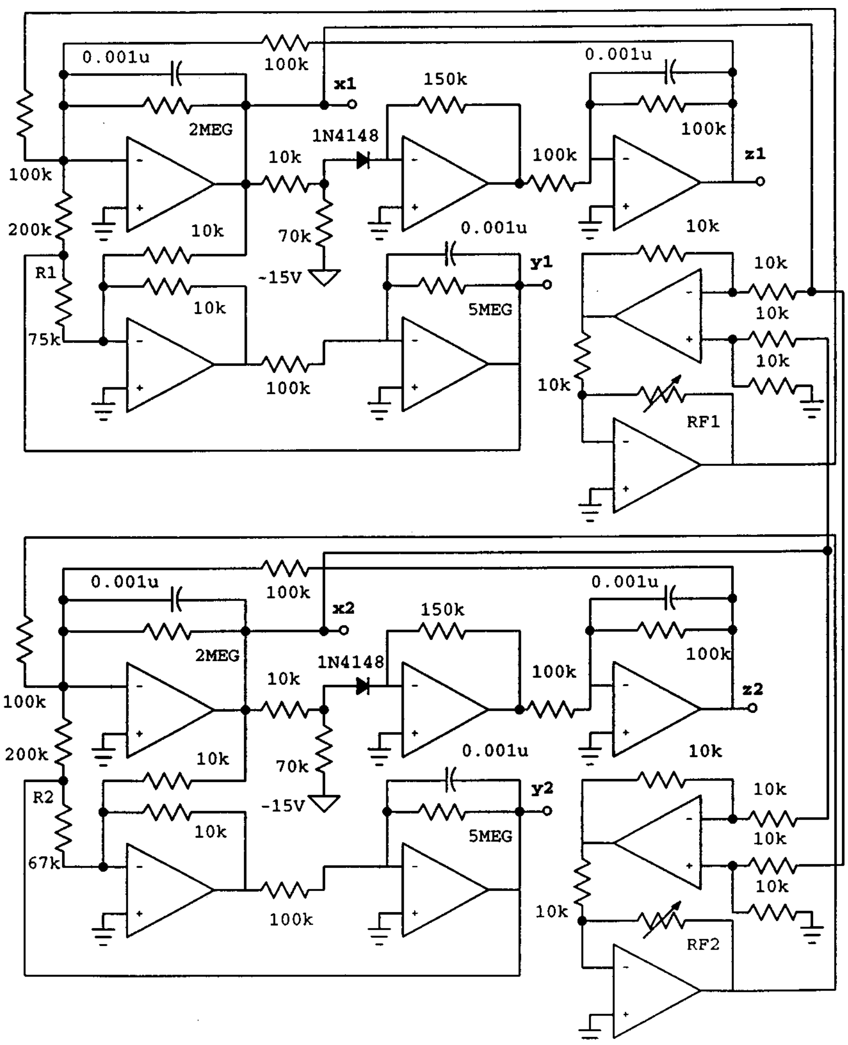

FIG. 1. Schematic diagram of two coupled chaotic piecewise linear Rössler circuits, where we set $R_1=75k\Omega$ and $R_2=67k\Omega$, so that the two circuits are nonidentical. The variable resistors RF_1 and RF_2 ($RF_1=RF_2$) are used to change the coupling. The operational amplifiers are type 741. All resistors are metal film and capacitors are polyester type with 1% and 5% tolerance respectively. The circuit is run by ± 15 volts.

the circuit operated in a steady state. The noise level is defined to be the ratio of the root-mean-square values of the noise to that of the chaotic signal.

To quantify lag synchronization, we use the following similarity function defined with respect to one dynamical variable, say x, of the chaotic oscillators [2],

$$S(\tau)=\sqrt{\frac{\langle[x_2(t+\tau)-x_1(t)]^2\rangle}{[\langle x_1^2(t)\rangle\langle x_2^2(t)\rangle]^{1/2}}}, \qquad (1)$$

where τ is the lag time. Let $S_{\min}$ be the minimum value of $S(\tau)$ and let $\tau_{\min}$ be the amount of lag when $S_{\min}$ is achieved. Lag synchronization between the two oscillators is characterized by the conditions $S_{\min}=0$ and $\tau_{\min}\neq 0$, while complete synchronization is by $S_{\min}=0$ and $\tau_{\min}=0$. Numerically, in the absence of noise, as the coupling strength is increased, one observes the transition from asynchronous chaos to lag synchronization and then to complete synchronization [2]. Let ϵ_S be the critical value of ϵ at which $S_{\min}$ reaches zero, and let ϵ_τ be the ϵ value at which $\tau_{\min}$ becomes zero. In order to be able to observe lag synchronization, one must have $\epsilon_S<\epsilon_\tau$, so that lag synchronization occurs in the parameter interval $[\epsilon_S,\epsilon_\tau]$. However, if $\epsilon_\tau>\epsilon_S$, no lag synchronization can be observed because the lag time has already become zero before synchronization occurs ($S_{\min}=0$). Figures 2(a) and 2(b) show, from a typical experiment, $S_{\min}$ and $\tau_{\min}$ versus ϵ, respectively, for $0<\epsilon_S<0.1$, where $S(\tau)$ is evaluated using the x variables of the Rössler oscillations. We observe that synchronization occurs at $\epsilon\approx 0.08$. It should be noted that due to the inevitable random noise such as the thermal noise of the circuit components, small time delay (3 μs), and the finite resolution of the data acquisition device, the plot of $x_1(t)$ versus $x_2(t)$ is a ‘‘fattened’’ line segment along the diagonal $x_1(t)=x_2(t)$ even in the complete synchronization regime. This ‘‘fattened’’ line also results in $S_{\min}\gtrsim 0$ in the synchronization regime. In our measurements, we observe that when synchronization is achieved, the typical value of $S_{\min}$ is about 0.03. The key feature in Figs. 2(a) and 2(b) is that $\tau_{\min}$ becomes zero at $\epsilon_\tau\lesssim 0.06$ (there is a periodic window for $0.02<\epsilon<0.06$), before synchronization is achieved. This means that no robust lag synchronization can be observed in this experiment. Intermittent lag synchronization, however, can be observed for $0.004\lesssim\epsilon\lesssim 0.02$. Figures 3(a) and 3(b) show, for $\epsilon=0.009$, the plots of $x_2(t)$ versus $x_1(t)$ and $x_2(t+\tau_{\min})$ versus $x_1(t)$, respectively. Apparently, $x_2(t)$ does not synchronize with $x_1(t)$[Fig. 3(a)], but $x_2(t+\tau_{\min})$ appears to synchronize with $x_1(t)$, but only in intermittently short time intervals, as can be seen in the large spread of the points off the diagonal where $x_2(t+\tau_{\min})=x_1(t)$. This also leads to the relatively large value of $S_{\min}\approx 0.16$ at $\tau_{\min}$. The

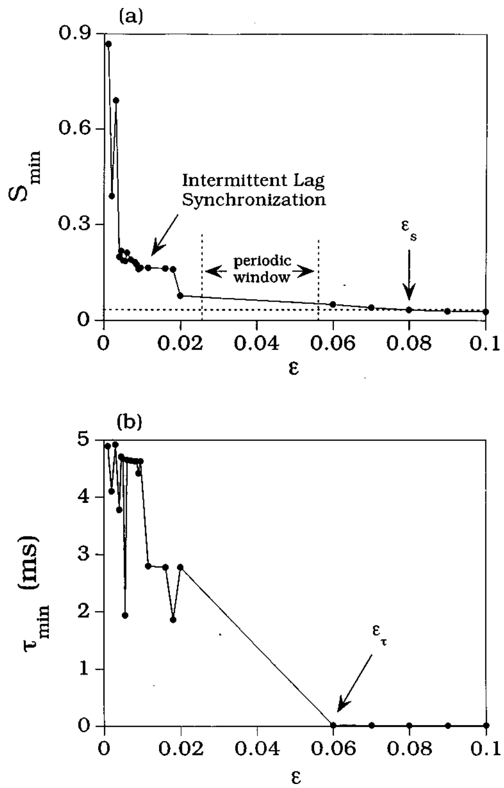

FIG. 2. For the experimental system of two coupled Rössler oscillators, (a) $S_{\min}$ and (b) $\tau_{\min}$ vs ϵ for $0<\epsilon<0.1$. The dashed line in (a) at $s_{\min}=0.03$ denotes the experimental threshold below which $x_1(t)$ and $x_2(t)$are regarded as being synchronized.

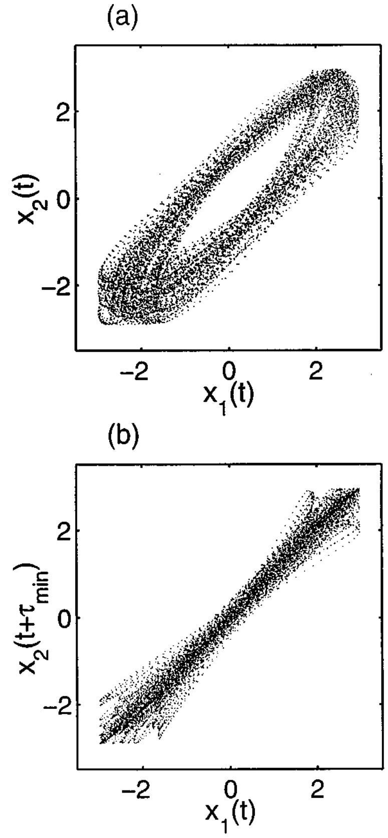

FIG. 3. For $\epsilon=0.009$ in experiments: (a) $x_2(t)$; (b) $x_2(t+\tau_{\min}$ $-x_1(t)$ vs $x_1(t)$.

intermittent behavior can be seen more clearly in the plot of $\Delta x_{\tau_{\min}}(t)=x_2(t+\tau_{\min})-x_1(t)$, as shown in Fig. 4. We find that, when the noise level is approximately 4% (smaller than the system mismatch defined in this circuit) [7], the probability distribution of time interval T between adjacent bursts exhibits a power-law decay behavior, which is typical of on-off intermittency [6]. When the noise level is about the same as the system mismatch (10%), even intermittent lag synchronization disappears. This can be seen from the amplitude-phase representation of the circuit equations. In this representation, trajectories on the chaotic attractor can be regarded as generalized rotations in the phase space. Adding uncorrelated noise e_x, e_y, and e_z to the x, y, and z variables of the circuit equations, the phase equations correspond to rotations in the x-y plane can be written

$$\frac{d\phi_{1,2}}{dt}=\alpha\sin^2\phi_{1,2}+\beta\cos^2\phi_{1,2}+(\gamma+a_{1,2})\sin\phi_{1,2}\cos\phi_{1,2}+\frac{\hat{e}_{1,2}}{A_{1,2}}+\frac{z_{1,2}}{A_{1,2}}\sin\phi_{1,2}+\epsilon H(A_1,A_2,\phi_1,\phi_2)$$
,

where $A(t)=\sqrt{x^2(t)+y^2(t)}$, $\phi(t)=\tan^{-1}[y(t)/x(t)]$, H is smooth function and $\hat{e}_{1,2}(t)=e_y(t)\cos\phi_{1,2}(t)-e_x(t)\sin\phi_{1,2}(t)$. On average, the term $\hat{e}_{1,2}(t)/A_{1,2}$ is the noise level relative to the amplitude of the chaotic signal (this is how we have defined the noise level in this Rapid Communication). The lag time τ between x_1 and x_2 can be related to the phase difference [2]: $\tau=\langle\phi_1(t)-\phi_2(t)\rangle/\langle\dot{\phi}\rangle$, where $\langle\dot{\phi}_{1,2}\rangle$'s are mean frequencies, which can be altered by a change in parameters α, β, γ, or a of each Rössler oscillator and the noise level. Mismatch between the two Rössler circuits can be achieved by choosing different parameter values for each circuit. From the circuit equation, we see that it is meaningful to compare, then, the parameter mismatch with the relative noise level.

We have conducted a large number of experiments, using an additional circuit configuration of the Rössler oscillator with no piecewise linear elements and using different cou-

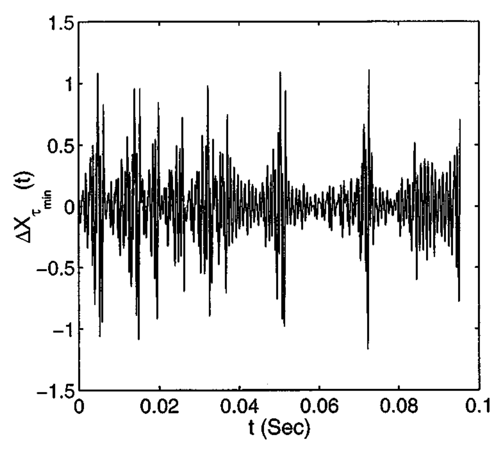

FIG. 4. For $\epsilon=0.009$ in experiments, plot of time series $\Delta x_{\tau_{\min}}(t)=x_2(t+\tau_{\min})-x_1(t)$. The intermittent behavior of lag synchronization is apparent.

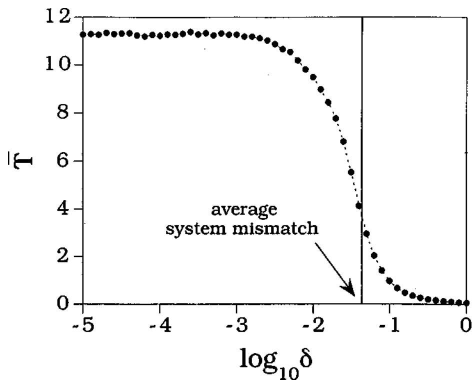

FIG. 5. For $\epsilon=0.5$ in the numerical model, the average time interval between burst $\bar{T}$ vs the noise amplitude. The vertical line indicates the amount of the average system mismatch.

pling schemes including unidirectional coupling, in order to realize robust lag synchronization. The behaviors in Figs. 2–4 appear to be common.

We now present numerical confirmation for the experimental observation. We use the following system of two coupled Rössler oscillators which was used in Ref. [2] to first report lag synchronization: $dx_{1,2}/dt=-\omega_{1,2}y_{1,2}-z_{1,2}+\epsilon(x_{2,1}-x_{1,2})$, $dy_{1,2}/dt=\omega_{1,2}x_{1,2}+ay_{1,2}$, $dz_{1,2}/dt=b+z_{1,2}(x_{1,2}-c)$, where $a=0.165$, $b=0.2$, $c=10.0$, and $\omega_{1,2}$ are parameters of oscillators 1 and 2, respectively. In Ref. [2], the authors used $\omega_{1,2}=0.97\pm0.02$ to stipulate nonidentity of the two chaotic oscillators. In order to mimic the influence of random disturbances and to maintain nonidentity (in the average sense) between the two oscillators, we choose $\omega_1=0.99+\delta\sigma_{1,2}(t)$ and $\omega_2=0.95+\delta\sigma_{1,2}(t)$, where δ is the noise amplitude, and $\sigma_1(t)$ and $\sigma_2(t)$ are random numbers uniformly distributed in $[-1,1]$. Under the influence of noise, the *average* system mismatch is thus $\overline{\Delta\omega}=0.04$ and we vary δ from 10^{-5} to 10^0, a wide range that covers the magnitude of the system mismatch. We find that, at small noise levels, lag synchronization is only temporal and appears in an intermittent fashion, while when the noise level is comparable to $\overline{\Delta\omega}$, the bursts occur so frequently that lag synchronization disappears practically. To quantify this behavior, we choose 50 noise levels uniformly distributed on a logarithmic scale in $[10^{-5},10^0]$ and for each noise level, we compute the average time interval $\bar{T}$ between adjacent burst by setting a threshold $\Delta x=\pm0.08$. The distributions of the time intervals T are apparently exponential so that $\bar{T}$ is well defined. Figure 5 shows $\bar{T}$ versus $\log_{10}\delta$. For $\delta\ll\overline{\Delta\omega}$, $\bar{T}$ remains at a constant, indicating an almost unchanged behavior of intermittency at small noise levels. As δ increases, we see that $\bar{T}$ drops quickly to zero when δ exceeds $\overline{\Delta\omega}$. Thus, at larger noise levels, bursts occurs more and more frequently, causing a practical disappearance of lag synchronization. Adding noise to other parameters or to dynamical variables of the system yields similar results (data not shown) [8].

Intuitively, lag synchronization is a phenomenon that depends on a precise timing between the two chaotic oscillators, but such a timing is usually destroyed by inevitable random factors present in the environment. As a consequence, lag synchronization can only occur intermittently, even when noise is small compared with the amount of mismatch between the systems. At large noise, lag synchronization is no longer possible and one observes a direct transition to complete synchronization at sufficiently large coupling strength. These results suggest that one should be cautious when attempting to observe lag synchronization in laboratory experiments or in practical systems.

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[7] In our experiments, the quantification of the noise [J. Engberg and T. Larsen, *Noise Theory of Linear and Nonlinear Circuits* (Wiley, New York, 1995)] is as follows. The noise is in-band; that is, its frequency spectrum covers that of the chaotic signal. Its root-mean-square voltage is about 4% of that of the chaotic signal.

[8] Here in numerical experiments, the noise is uniformly distributed in $[-\epsilon,\epsilon]$, and it is added at each integration step when the differential equations are integrated.