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Route to high-dimensional chaos

Mary Ann Harrison* and Ying-Cheng Lai†

Department of Physics and Astronomy, The University of Kansas, Lawrence, Kansas 66045

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We present a route to high-dimensional chaos, that is, chaos with more than one positive Lyapunov exponent. In this route, as a system parameter changes, a subsystem becomes chaotic through, say, a cascade of period-doubling bifurcations, after which the complementary subsystem becomes chaotic, leading to an additional positive Lyapunov exponent for the whole system. A characteristic feature of this route, as suggested by numerical evidence, is that the second largest Lyapunov exponent passes through zero continuously. Three examples are presented: a discrete-time map, a continuous-time flow, and a population model for species dispersal in evolutionary ecology. [S1063-651X(99)50704-6]

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Transition to chaos, i.e., how a system becomes chaotic as a system parameter changes, has been a fundamental and central problem in the study of nonlinear dynamics. In low-dimensional chaotic systems, i.e., systems with only one positive Lyapunov exponent [1], this transition often occurs via the following four known routes: (i) the period-doubling cascade route [2], (ii) the intermittency transition route [3], (iii) the crisis route [4], and (iv) the route to chaos in quasiperiodically driven systems [5].

There has been growing interest in high-dimensional chaotic systems [6]. Such a system is characterized by more than one positive Lyapunov exponent for typical trajectories in the phase space, although the phase-space dimension can be arbitrarily high. So far, the route to high-dimensional chaos, i.e., how high-dimensional chaos arises as a system parameter changes, remains a less-studied area. The purpose of this paper is to address this issue. Our main contribution is the identification of a general route to high-dimensional chaos: one from regular to low-dimensional chaos and then to high-dimensional chaos. This route can be observed for systems exhibiting, say, a period-doubling cascade to low-dimensional chaos [7]. We give conditions and investigate the characteristic features for this route to high-dimensional chaos. Our study is motivated partly by the recent extensive studies of techniques to control [8] and to synchronize [9] high-dimensional chaotic systems, which are highly nontrivial and challenging problems. A good understanding of how nonlinear systems develop high-dimensional chaos may help us gain insight into devising methods to manipulate such systems. This may have practical implications.

We consider a general nonlinear system described either by an N-dimensional discrete-time map: $\mathbf{x}_{n+1}=\mathbf{F}[\mathbf{x}_n,\mathbf{p}]$, or by an N-dimensional continuous-time flow: $d\mathbf{x}/dt=\mathbf{F}(\mathbf{x},\mathbf{p})$, where $\mathbf{x}\in\mathbf{R}^N$ and $\mathbf{p}$ is a set of bifurcation parameters. For concreteness, we focus on high-dimensional chaos with two positive Lyapunov exponents [10]. Since the system can exhibit *two* positive exponents, we decompose the system into *two* subsystems A and B that interact with each other: $\mathbf{x}=(\mathbf{x}^1,\mathbf{x}^2)$, where $\mathbf{x}^1\in\mathbf{R}^{N_1}$, $\mathbf{x}^2\in\mathbf{R}^{N_2}$, and $N_1+N_2=N$:

$$A{:}\mathbf{x}^1_{n+1}=\mathbf{f}(\mathbf{x}^1_n,\mathbf{x}^2_n,\mathbf{p}_A),\quad B{:}\mathbf{x}^2_{n+1}=\mathbf{g}(\mathbf{x}^1_n,\mathbf{x}^2_n,\mathbf{p}_B),\tag{1}$$

where $\mathbf{F}=(\mathbf{f},\mathbf{g})$, and $(\mathbf{p}_A,\mathbf{p}_B)=\mathbf{p}$. With this setting, the route to high-dimensional chaos can be described as follows. Consider a line in the parameter space with two end points $\mathbf{p}_1$ and $\mathbf{p}_2$, where the system is not chaotic at $\mathbf{p}_1$ and chaotic with two positive Lyapunov exponents at $\mathbf{p}_2$. Let p be the distance from $\mathbf{p}_1$ along this line, and let p_{c1} and p_{c2} be the transition points to low- and high-dimensional chaos, respectively. For $p<p_{c1}$, both subsystems A and B are not chaotic, e.g., with stable periodic attractors. Regarding the influence of the subsystem A to subsystem B as a driving (or vice versa), we assume that B is in a state where it can become chaotic only when the driving is chaotic. As p increases towards p_{c1}, the subsystem A, say, undergoes a cascade of period-doubling bifurcations [7] and becomes chaotic with one positive Lyapunov exponent for $p>p_{c1}$. Subsystem B is now driven chaotically. As p increases further passing through some critical value p_c, B becomes chaotic, after which the full system can become high-dimensionally chaotic with two positive Lyapunov exponents at $p=p_{c2}$, where $p_{c2}\gtrsim p_c$.

A characteristic feature of this scenario to high-dimensional chaos, as evidenced by numerical computations, is that the second nontrivial Lyapunov exponent passes through zero *continuously*. Qualitatively, this can be understood by realizing that near the transition, the subsystem B is driven chaotically and it can thus be regarded as a randomly driven dynamical system. It is known that the largest Lyapunov exponent of a random dynamical system changes through zero from the negative side in a continuous fashion [11]. In the following we present numerical examples and a simple analyzable model to illustrate this feature.

We first consider the following three-dimensional map, which we have constructed to illustrate the transition and its characteristic features:

$$\begin{aligned} x_{n+1}&=\left[x_n+\frac{1-e^{-a}}{a}y_n\right]\mathrm{mod}(2\pi),\\ y_{n+1}&=e^{-a}y_n+k\sin(x_{n+1}+2\pi z_n),\\ z_{n+1}&=rz_n(1-z_n)+\epsilon y_n,\end{aligned}\tag{2}$$

where a, k, r, and ϵ are parameters. The (x,y) dynamics is the Zaslavsky map when the term $2\pi z_n$ in the y equation is absent, which can exhibit chaotic attractors [12]. The term

*Electronic address: harrison@poincare.math.ukans.edu

†Also at Department of Mathematics, University of Kansas, Lawrence, KS 66045. Electronic address: lai@poincare.math.ukans.edu

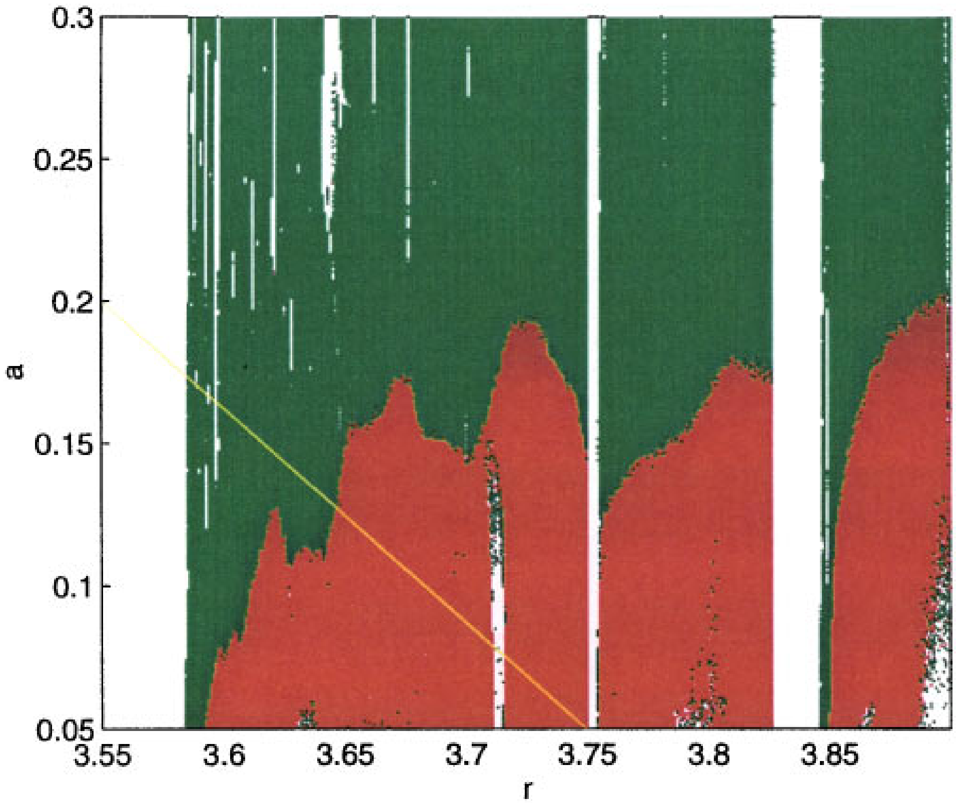

FIG. 1. (Color) Regions in the (r,a) plane in which the asymptotic attractor of Eq. (2) has zero (blank), one (green), and two (red) positive Lyapunov exponents.

$2\pi z_n$ represents the coupling from the z dynamics, which is the logistic map with a term ϵy_n, describing the coupling from the (x,y) dynamics. The (x,y) equation and the z equation are thus naturally two interacting subsystems that constitute the full three-dimensional map. Although we have constructed Eq. (2) as a simple model to study high-dimensional chaos, a physical consideration is that Eq. (2) represents an idealized situation in the context of passive particle advection on the surface of an incompressible fluid [11]. In numerical experiments, we fix $k=0.5$ and $\epsilon=0.01$, and study the transition in the two-dimensional parameter space (r,a).

Figure 1 shows the regions in the (r,a) plane in which the asymptotic attractor of Eq. (2) has zero (blank), one (green), and two (red) positive Lyapunov exponents. The figure is obtained by distributing a grid of 400×400 parameter pairs

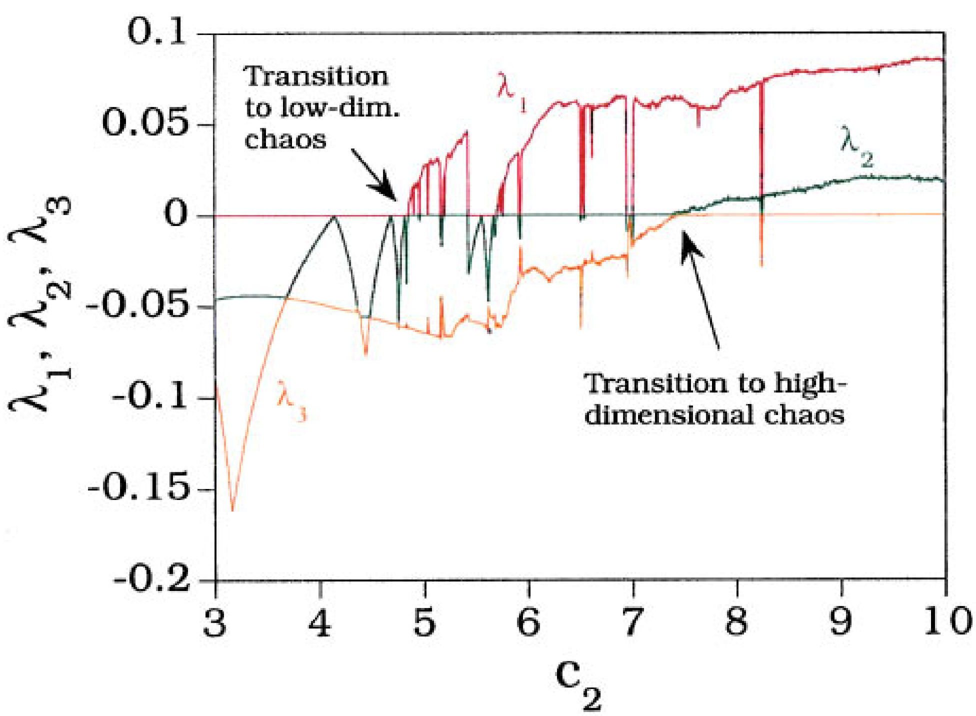

FIG. 3. (Color) First three Lyapunov exponents of the coupled Rössler system.

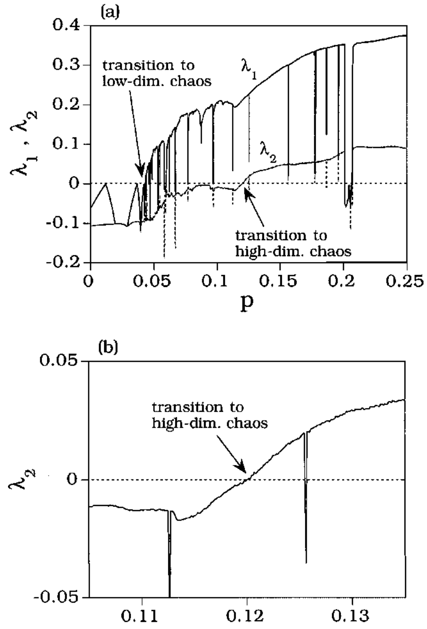

FIG. 2. For the model Eq. (2): (a) the two largest Lyapunov exponents vs p (see text for definition of p), and (b) a blowup of part of (a) near the transition to high-dimensional chaos.

in the region defined by $3.55 \leqslant r \leqslant 3.9$ and $0.05 \leqslant a \leqslant 0.3$, and computing the Lyapunov spectrum [13] of Eq. (2) for each parameter pair by using 10^6 iterations from a random initial condition chosen from the phase-space region ($0 \leqslant x \leqslant 2\pi$, $-2.5 \leqslant y \leqslant 2.5$, $0 \leqslant z \leqslant 1$) (after disregarding 10^5 initial iterations). We see that there are an infinite number of paths in the parameter region where the transition from zero to one and to two positive Lyapunov exponents can occur. For concreteness, we choose the path from $\mathbf{p}_1 \equiv (r_1, a_1) = (3.55, 0.2)$ to $\mathbf{p}_2 \equiv (r_2, a_2) = (3.75, 0.05)$ (the yellow line in Fig. 1 passing through all three regions). Let p be the distance from $\mathbf{p}_1$ along the line. We see that the system can be nonchaotic, and low- and high-dimensionally chaotic when p is in the white, green, and red regions, respectively. Figure 2(a) shows the first two Lyapunov exponents versus p. As p is increased from $\mathbf{p}_1$, the parameter r in the logistic equation increases so that the z subsystem undergoes a series of period-doubling bifurcations and becomes chaotic at $r_{c1} \approx 3.580$, which corresponds to $p_{c1} \approx 0.038$. Thus, for $p < p_{c1}$, the full system has no positive Lyapunov exponent. For $p > p_{c1}$, the full system has one positive Lyapunov exponent λ_1. As p is increased through p_{c1}, the z dynamics becomes chaotic so that the (x,y) subsystem is now driven by a chaotic variable. The (x,y) subsystem becomes chaotic at $p_c \approx 0.119$ and the full system becomes high-dimensionally chaotic with two positive Lyapunov exponents at $p_{c2} \approx 0.120$. The route for the (x,y) subsystem to become chaotic is similar to the transition to chaos in random dynamical systems [11], which is typically a continuous transition. As such, the second Lyapunov exponent of the full system becomes positive continuously from the negative side as p increases through p_{c2}, as shown in Fig. 2(b), a blowup of a part of Fig. 2(a) near the transition point p_{c2}.

The continuous behavior of the second largest Lyapunov exponent λ_2 near the transition can be understood more quantitatively as follows. Under the chaotic driving from the z subsystem, a tangent vector along a typical trajectory in the (x,y) subspace experiences both time intervals of expansion and time intervals of contraction. For $p \lesssim p_c$, the largest Lyapunov exponent of the (x,y) subsystem, λ_{xy}, is slightly negative so that contraction dominates over expansion. For $p \gtrsim p_c$, λ_{xy} is slightly positive so that expansion weighs over contraction. At p_c, expansion and contraction are balanced. The exponent λ_{xy} thus passes through zero continuously at p_c due to the existence of two competing phase-space regions (expansion versus contraction). Since $p_{c2} \gtrsim p_c$, we expect the second largest Lyapunov exponent of the full system to pass through zero in a continuous fashion. To be illustrative, we consider the following simple analyzable two-dimensional map in the unit square:

$$x_{n+1} = 2x_n \bmod(1), \quad y_{n+1} = \begin{cases} 2y_n \bmod(1) & \text{if } x_n < a \\ y_n/2 & \text{if } x_n \geqslant a, \end{cases} \tag{3}$$

where $0 < a < 1$ is the bifurcation parameter. The x dynamics is the doubling transformation with a positive Lyapunov exponent $\lambda_1 = \ln 2$. In addition, the invariant density of x is uniform in the unit interval. The y dynamics is a simple expansion-contraction map. The probability for expansion is a and the probability for contraction is $(1-a)$. The Lyapunov exponent of the y subsystem which, for this simple model, is the second Lyapunov exponent of the full system, is given by

$$\lambda_2 = a \ln 2 + (1-a)\ln(\tfrac{1}{2}) = (2a-1)\ln 2. \tag{4}$$

We see that λ_2 passes through zero continuously as a is increased through $\frac{1}{2}$.

How general is the above route to high-dimensional chaos? To address this question, we now give two more numerical examples.

(1) *A high-dimensional continuous-time flow.* We consider the following system of two coupled Rössler oscillators [14]: $dx_{1,2}/dt = -(y_{1,2} + z_{1,2}) + \epsilon(x_{2,1} - x_{1,2})$, $dy_{1,2}/dt = x_{1,2} + a_{1,2}y_{1,2}$, and $dz_{1,2}/dt = b_{1,2} + x_{1,2}z_{1,2} - c_{1,2}z_{1,2}$, where we fix the parameters $a_{1,2} = b_{1,2} = 0.2$, $c_1 = 3.5$, and $\epsilon = 0.05$, and we choose c_2 to be the bifurcation parameter. We find the following decomposition of this system into two subsystems: $A - \{x_2, y_2, z_2\}$ and $B - \{x_1, y_1, z_1\}$. Figure 3 shows the first three Lyapunov exponents, among six, of the coupled Rössler system. Features similar to those in Figs. 2(a) and 2(b) are observed. In particular, the second nontrivial Lyapunov exponent becomes positive in a continuous fashion [15].

(2) *A model of species dispersal in evolutionary ecology.* We consider a realistic model of populations of two species (clones) in two patches [16]. The model was investigated recently by Holt and McPeek to address the important ecological question: how does dispersal (spatial movement of species in the environment) affect the population? The dynamical variable is $N_{ij}(t)$—the density of population of clone i in patch j at generation t, where $i,j = 1,2$. Clones 1 and 2 differ only in a fixed dispersal rate, e_i, which is defined as the fraction of individuals dispersing from their natal patch at each generation. Assume that the realized fitness in patch j, $W_j[N_{Tj}(t)]$, is identical for individuals of both clones and depends functionally on the summed abundances, $N_{Tj}(t) = N_{1j}(t) + N_{2j}(t)$, in patch j. Without dispersal, the dynamics of clone i in patch j are governed by the recursion:

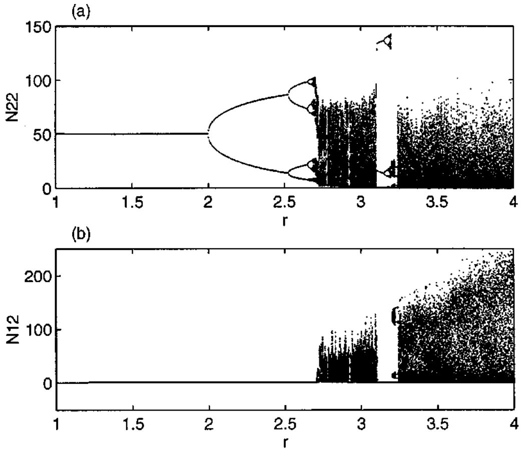

FIG. 4. Bifurcation diagrams of (a) N_{22} and (b) N_{12} vs the parameter r in the ecological model Eq. (5).

$N_{ij}(t+1)=W_J[N_{Tj}(t)]N_{ij}(t)$ [16]. With dispersal, a fraction m of dispersers survives to enter their non-natal patch, inwhich case the dynamics of clone 1 can be described by

$$
\begin{aligned}
N_{11}(t+1)&=(1-e_1)W_1[N_{T1}(t)]N_{11}(t)\\
&\quad+me_1W_2[N_{T2}(t)]N_{12}(t),\\
N_{12}(t+1)&=(1-e_1)W_2[N_{T2}(t)]N_{12}(t)\\
&\quad+me_1W_1[N_{T1}(t)]N_{11}(t),
\end{aligned}
\tag{5}
$$

where similar equations describe clone 2. The fitness is given by the following relation [17]: $W_j=\exp[r_j(1-\Sigma_i N_{ij}/K_j)]$, where K_j is the carrying capacity of patch j. Regarding $\{N_{11}(t),N_{12}(t)\}$ and $\{N_{21}(t),N_{22}(t)\}$ as two interacting subsystems, where subsystem A is the low-dispersal clone and subsystem B is the high-dispersal clone, we find that the full system, which is a four-dimensional map, becomes high-dimensionally chaotic through the route described above for models Eq. (2) and the coupled Rössler system. Figures 4(a) and 4(b) show the bifurcation diagrams of N_{22} and N_{12} versus the parameter $r(r_1=r_2)$ at $K_1=100$, $K_2=50$, $e_1=0.5$ (large dispersal rate), $e_2=0.01$ (small dispersal rate), and $m=1$. Clearly, $N_{22}(t)$ undergoes a period-doubling cascade and becomes chaotic, after which $N_{12}(t)$ becomes chaotic, and the whole system possesses two positive Lyapunov exponents. The ecological significance of Figs. 4(a) and 4(b) is that chaotic dynamics in fact favor the evolution of dispersal. When the dynamics of clone 2 are not chaotic, the population of clone 1 becomes zero asymptotically, indicating extinction of the species with high-dispersal rate. However, when the dynamics of clone 2 becomes chaotic, the dynamics of clone 1 becomes chaotic, too, with *nonzero* population densities in both patches.

In summary, we have presented a route to high-dimensional chaos in nonlinear systems by using models arising in different contexts. Numerical evidence suggests that, in this scenario to high-dimensional chaos, the second largest Lyapunov exponent passes through zero continuously. Physically, a subset of dynamical variables becomes chaotic via some (perhaps) known route to low-dimensional chaos such as the period-doubling bifurcation route, after which the complementary subset becomes chaotic. As a consequence, in a bifurcation diagram, the latter subset of dynamical variables appears to become chaotic in a relatively abrupt fashion [see, for example, Fig. 4(b)]. The mechanism by which the system becomes high-dimensionally chaotic via this route suggests an effective way to control it: by stabilizing the driving chaotic subsystem around some periodic orbits, the whole system becomes periodic.

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[1] While there has been no formal definitions of low-dimensional versus high-dimensional chaos, here we take the notion that low-dimensional chaos is characterized by one positive Lyapunov exponent, and high-dimensional chaos by more than one.

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