Characterization of blowout bifurcation by unstable periodic orbits

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Blowout bifurcation in chaotic dynamical systems occurs when a chaotic attractor, lying in some invariant subspace, becomes transversely unstable. We establish quantitative characterization of the blowout bifurcation by unstable periodic orbits embedded in the chaotic attractor. We argue that the bifurcation is mediated by changes in the transverse stability of *an infinite number of unstable periodic orbits*. There are two distinct groups of periodic orbits: one transversely stable and another transversely unstable. The bifurcation occurs when some properly weighted transverse eigenvalues of these two groups are balanced.
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Recently, a novel type of bifurcation has been discovered in chaotic dynamical systems [1,2]. This is the so-called ''blowout bifurcation'' that occurs in dynamical systems with a symmetric invariant subspace. Let **S** be the invariant subspace in which there is a chaotic attractor. Since **S** is invariant, initial conditions in **S** result in trajectories that remain in **S** forever. Whether the chaotic attractor in **S** is also an attractor in the full phase space depends on the sign of the largest Lyapunov exponent $\Lambda_\perp$ computed for trajectories in **S** with respect to perturbations in the subspace **T** which is *transverse* to **S**. When $\Lambda_\perp$ is negative, **S** attracts trajectories transversely in the vicinity of **S** and, hence, the chaotic attractor in **S** is an attractor in the full phase space. If $\Lambda_\perp$ is positive, trajectories in the neighborhood of **S** are repelled away from it and, consequently, the attractor in **S** is transversely unstable and it is hence not an attractor in the full phase space. Blowout bifurcation occurs when $\Lambda_\perp$ changes from negative to positive values. There are distinct physical phenomena associated with the blowout bifurcation. For example, near the bifurcation point where $\Lambda_\perp$ is negative, if there are other attractors in the phase space, then typically, the basin of the chaotic attractor in **S** is riddled [3]. When $\Lambda_\perp$ is slightly positive, if there are no other attractors in the phase space, the dynamics in the transverse subspace **T** exhibits an extreme type of temporally intermittent bursting behavior, the on-off intermittency [4,5]. Recent study has also revealed that blowout bifurcation can lead to symmetry breaking in chaotic systems [6].

In the study of chaos theory, it is important to be able to understand a bifurcation in terms of unstable periodic orbits of the system because the knowledge of periodic orbits usually yields a great deal of information about the dynamics [7–9]. Periodic orbits are known to be responsible for many different types of bifurcations in chaotic systems. For example, the period-doubling bifurcation [10] and the saddle-node bifurcation are bifurcations of periodic orbits. Catastrophic events in chaotic systems such as crises [11] and basin boundary metamorphoses [12] are triggered by collision of periodic orbits, usually of low period, embedded in different dynamical invariant sets. The birth of Wada basin boundaries, meaning common boundaries of more than two basins of attraction, is caused by a saddle-node bifurcation on the basin boundary [13]. More recent study indicates that the riddling bifurcation, bifurcation that gives birth to a riddled basin, is triggered by the loss of transverse stability of some periodic orbit of low period embedded in the chaotic attractor in **S** [14]. In view of the role of periodic orbits played in these major bifurcations, it is desirable to study the blowout bifurcation by periodic orbits. In this regard, Ashwin, Buescu, and Stewart have noticed that as a system parameter changes towards the blowout bifurcation point, more and more atypical invariant measures become transversely unstable [2]. At the bifurcation, the natural measure of the chaotic attractor in **S** becomes unstable.

In this paper, we establish a *quantitative characterization* of the blowout bifurcation by unstable periodic orbits embedded in the chaotic attractor in the invariant subspace **S**. In particular, we argue that near the bifurcation, there exist two groups of periodic orbits Σ_s and Σ_u, each having an infinite number of members, one transversely stable and another transversely unstable, respectively. The sign of the largest transverse Lyapunov exponent $\Lambda_\perp$ is determined by the relative weights of Σ_s and Σ_u: $\Lambda_\perp$ is negative (positive) when Σ_s (Σ_u) weighs over Σ_u (Σ_s). (A precise definition of the ''weights'' will be described in the sequel.) At the bifurcation, the weights of Σ_s and Σ_u are balanced. In contrast to most known bifurcations in chaotic systems that usually involve only one or a few periodic orbits [10–14], blowout bifurcation is induced by a change in the transverse stability of *an infinite number of unstable periodic orbits*. The number $\hat{N}_p$ of the unstable periodic orbits of period p that change transverse stability in an arbitrarily small neighborhood of the bifurcation points grown as $\hat{N}_p \sim e^{h_T p}$, where h_T is the topological entropy of the chaotic attractor in **S**.

We consider the following general class of N-dimensional dynamic systems,

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n), \tag{1}$$

$$\mathbf{z}_{n+1} = F(\mathbf{x}_n, \alpha)\mathbf{G}(\mathbf{z}_n),$$

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where $\mathbf{x}\in\mathbb{R}^{N_\parallel}$ $(N_\parallel\geqslant 1)$, $\mathbf{z}\in\mathbb{R}^{N_\perp}$ $(N_\perp\geqslant 1)$, $N_\parallel+N_\perp=N$, and α is the bifurcation parameter. Assume that the function $\mathbf{G}(\mathbf{z})$ satisfies $\mathbf{G}(\mathbf{0})=\mathbf{0}$ so that $\mathbf{z}=\mathbf{0}$ is the invariant subspace **S**. The dynamics in **S** is described by the map $\mathbf{f}(\mathbf{x})$ which has a chaotic attractor. There is an infinite number of unstable periodic orbits embedded in this attractor. Equation (1) represents a typical system for which blowout bifurcation can occur [1,2]. The largest transverse Lyapunov exponent $\Lambda_\perp$ is given by $\Lambda_\perp=\lim_{L\to\infty}(1/L)\times\Sigma_{n=1}^{L}\ln|F(\mathbf{x}_n,\alpha)\mathbf{DG}(\mathbf{z}_n)|_{\mathbf{z}_n=0}\cdot\mathbf{u}|$, where $\mathbf{u}$ is a randomly chosen vector in $\mathbb{R}^{N_\perp}$. Let α_c be the blowout bifurcation point so that as the parameter α passes through α_c, $\Lambda_\perp$ crosses zero from the negative side. The goal of this paper is to relate, quantitatively, the transverse stability of the infinite number of unstable periodic orbits embedded in **S** to $\Lambda_\perp$.

Our approach is to study a model system that is simple but captures typical features of the blowout bifurcation, for which unstable periodic orbits of high periods and their dynamical properties can be computed numerically. In particular, we consider the following three-dimensional version of Eq. (1):

$$x_{n+1}=a-x_n^2+by_n, \tag{2}$$

$$y_{n+1}=x_n,$$

$$z_{n+1}=(\alpha x_n+\beta y_n)g(z_n),$$

where the invariant subspace **S** is defined by $z=0$ and, hence, it is two dimensional $[\mathbf{x}\equiv(x,y)]$. The dynamics in **S** is given by the Hénon map $\mathbf{f}(\mathbf{x})$, and we choose $a=1.4$ and $b=0.3$, a parameter setting for which it is believed that the map has a chaotic attractor [15]. The function $g(z)$ satisfies $g(0)=0$ and $g'(0)=$const (which for simplicity is chosen to be 1). There are many choices for $g(z)$, e.g., $g(z)=z(1-z)$ (the logistic function) [5], $g(z)=(1/2\pi)\sin(2\pi z)$ [6], etc. The transverse Lyapunov exponent of Eq. (2) is $\Lambda_\perp=\lim_{n\to\infty}(1/n)\Sigma_{j=1}^{n}\ln|\alpha x_j+\beta y_j|$. Restricting our study to the case where $\beta=0$ (without loss of generality), we find numerically that a blowout bifurcation occurs at $\alpha_c\approx1.333$, where $\Lambda_\perp<0$ for $\alpha<\alpha_c$ and $\Lambda_\perp\geqslant0$ for $\alpha\geqslant\alpha_c$. To see the transverse stability of distinct unstable periodic orbits, we note that at a given parameter value α, the x-interval $[x_{\min},x_{\max}]$ in which the attractor lies can be divided into two subintervals: $[x_{\min},x_c(\alpha)]$ (the transversely attracting interval) and $(x_c(\alpha),x_{\max}]$ (the transversely repelling interval), where $x_c(\alpha)=1/\alpha$. In these two subintervals, the instantaneous derivative in the z equation evaluated at $z=0$ satisfies $\partial z_{n+1}/\partial z_n=\alpha x_n<1$ (attracting) and $\partial z_{n+1}/\partial z_n=\alpha x_n>1$ (repelling). Periodic orbits of high periods typically have their orbit points in both the attracting and repelling regions. A periodic orbit can then be either transversely stable or transversely unstable. Let $\mathbf{x}_1(j),\mathbf{x}_2(j),\ldots,\mathbf{x}_p(j)$ be the jth period-p orbit, where $j=1,2,\ldots,N_p$ (N_p being the total number of the period-p orbits), $\mathbf{x}_{n+1}(j)=\mathbf{f}[\mathbf{x}_n(j)]$ for $n=1,2,\ldots,p-1$, and $\mathbf{f}[\mathbf{x}_p(j)]=\mathbf{x}_1(j)$. We define the following transverse Lyapunov exponent for this orbit:

$$\lambda_p(j)=\frac{1}{p}\sum_{n=1}^{p}\ln\left|\frac{\partial z_{n+1}}{\partial z_n}\right|_{z_n=0,\mathbf{x}_n=\mathbf{x}(j)}\Bigg|$$
$$=\ln\alpha+\frac{1}{p}\sum_{n=1}^{p}\ln x_n(j). \tag{3}$$

If $\lambda_p(j)<0$ (>0), this period-p orbit is transversely stable (unstable). Thus, all the period-p orbits can be divided into two groups: one transversely stable and another transversely unstable. We then define the following period-p transversely stable and unstable weights,

$$\Lambda_p^s(\alpha)=\sum_{j=1}^{N_p^s}\mu_p(j)\lambda_p(j)|_{\lambda_p(j)<0}, \tag{4}$$

$$\Lambda_p^u(\alpha)=\sum_{j=1}^{N_p^u}\mu_p(j)\lambda_p(j)|_{\lambda_p(j)>0},$$

where N_p^s and N_p^u are the numbers of the transversely stable and unstable period-p orbits, respectively, $N_p^s+N_p^u=N_p$, and $\mu_p(j)$ is the natural measure of typical trajectories on the chaotic attractor which stay close to the jth period-p orbit. As $p\to\infty$, intuitively the probability measure of all the period-p orbits becomes a good approximation of the natural measure of the attractor [9]. Let $\Lambda^{s,u}(\alpha)=\lim_{p\to\infty}\Lambda_p^{s,u}(\alpha)$. Our claims are as follows: (i) $\Lambda^u(\alpha)<|\Lambda^s(\alpha)|$ for $\alpha<\alpha_c$, and $\Lambda^u(\alpha)>|\Lambda^s(\alpha)|$ for $\alpha>\alpha_c$; (ii) the blowout bifurcation occurs when $\Lambda^u(\alpha)=|\Lambda^s(\alpha)|$.

We now present numerical verifications. To start, we compute, within the limitation of our computing source, all the unstable periodic orbits of the Hénon chaotic attractor up to period-31 by using the method of Biham and Wenzel [16]. To compute $\mu_p(j)$, we use the theory developed by Grebogi, Ott, and Yorke [9], which relates the natural measure of an area A containing part of the attractor to the expanding eigenvalues of all the unstable periodic orbits enclosed in this area. Specifically, for a two-dimensional hyperbolic map $\mathbf{f}(\mathbf{x})$, the natural measure of A is given by [9]

$$\mu(A)=\lim_{p\to\infty}\sum_{\mathbf{x}_{jp}\in A}\frac{1}{L_1(\mathbf{x}_{jp},p)}, \tag{5}$$

where $L_1(\mathbf{x}_{jp},p)$ is the largest expanding eigenvalue of the jth period-p orbit, and the summation is taken over all fixed points of $\mathbf{f}^{\,p}(\mathbf{x})$ in A. It was conjectured in Ref. [9] that Eq. (5) holds approximately for nonhyperbolic maps, and relevant works lended credence to this conjecture [8]. We thus use the following approximation for the quantity $\mu_p(j)$ in Eq. (4):

$$\mu_p(j)\approx\frac{1/L_1(\mathbf{x}_{jp},p)}{\sum_{j=1}^{N_p}[1/L_1(\mathbf{x}_{jp},p)]}. \tag{6}$$

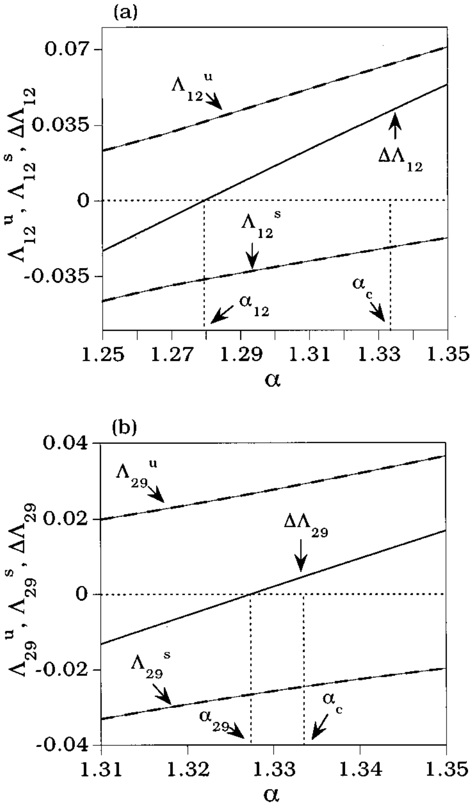

FIG. 1. (a) For all the period-12 orbits, $\Lambda_{12}^{s}(\alpha)$ (the transversely stable weight), $\Lambda_{12}^{u}(\alpha)$ (the transversely unstable weight) and $\Delta\Lambda_{12}(\alpha)\equiv\Lambda_{12}^{u}(\alpha)-|\Lambda_{12}^{s}(\alpha)|$ versus α near the blowout bifurcation point. We see that $\Delta\Lambda_{12}(\alpha)$ crosses zero at $\alpha_{12}\approx1.279$ ($|\alpha_{12}-\alpha_c|\approx0.054$). (b) For all the period-29 orbits, $\Delta\Lambda_{29}(\alpha)$ versus α. Now, $|\alpha_{29}-\alpha_c|\approx6\times10^{-3}$.

Next, we choose a small parameter interval about α_c and evenly distribute a large number of parameter values α in this interval. For each α value, we compute $\Lambda_p^s(\alpha)$ and $\Lambda_p^u(\alpha)$ for those computed distinct periodic orbits. (For $p=31$, there are 37 936 distinct orbits.) Figure 1(a) shows the period-12 weights $\Lambda_{12}^s(\alpha)$ and $\Lambda_{12}^u(\alpha)$ (dotted lines) versus α for $\alpha\in[1.25,1.35]$. The solid line in Fig. 1(a) is $\Delta\Lambda_{12}(\alpha)\equiv\Lambda_{12}^u(\alpha)-|\Lambda_{12}^s(\alpha)|$ versus α. We obtain $\alpha_{12}\approx1.279$, the critical parameter value at which $\Delta\Lambda_{12}(\alpha)=0$. The difference between α_{12} and bifurcation point α_c is $\Delta\alpha_{12}\equiv|\alpha_{12}-\alpha_c|\sim0.054$, which is somewhat large. But as we examine higher periodic orbits, the difference tends to decrease rapidly. Figure 1(b) shows $\Delta\Lambda_{29}(\alpha)$ versus α for $\alpha\in[1.31,1.35]$. We obtain $\alpha_{29}\approx1.327$ so that $\Delta\alpha_{29}\approx6\times10^{-3}$. Figure 2 shows the error $\Delta\alpha_p\equiv|\alpha_p-\alpha_c|$ versus the period p for $p\in[10,31]$ on a semilogarithmic scale. The data can be roughly fitted by a straight line with a slope of about -0.11, indicating that $\Delta\alpha_p\sim e^{-0.11p}$ [17]. Thus, we expect that as the period p increases, the collective behavior of all the period-p orbits, quantitatively described by the transversely stable and unstable weights in Eq. (4), more and more precisely characterizes the blowout bifurcation.

As the period increases, the number of the periodic orbits that changes transverse stability in the vicinity of α_c increases exponentially. Figure 3 shows $\ln\hat{N}_p$ versus p for p

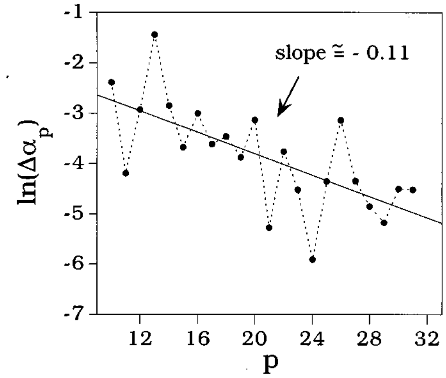

FIG. 2. $\ln(\Delta\alpha_p)$ versus p.

$\in[15,31]$, where $\hat{N}_p$ is the number of period-p orbits that changes from being transversely stable to being transversely unstable as α is increased from 1.31 to 1.35. We have $\hat{N}_p\sim e^{0.45p}$ (the solid line). As a comparison, the dotted line in Fig. 3 shows the total number of distinct period-p orbit N_p versus p (also on a semilogarithmic scale), the slope of which is an estimate of the topological entropy. When smaller parameter intervals about α_c are examined, we observe that the plot of $\ln\hat{N}_p$ versus p is shifted downwards parallelly, indicating a robust scaling $\hat{N}_p\sim e^{h_T p}$. The key observation is that as $p\rightarrow\infty$, an infinitge number of distinct periodic orbits change from being transversely stable to being transversely unstable in a parameter interval about α_c, no matter how small the interval is. Thus, *blowout bifurcation is mediated by change in the transverse stability of an infinite number of unstable periodic orbits embedded in the chaotic attractor in the invariant subspace.*

We stress that in order to characterize the blowout bifurcation by unstable periodic orbits, it is necessary to compute the locations of all periodic orbits up to reasonably high periods, which is in general a difficult task. However, we believe that our results are general because the Hénon map, which we use to model the dynamics in the invariant subspace, has been a paradigm in the study of chaotic systems.

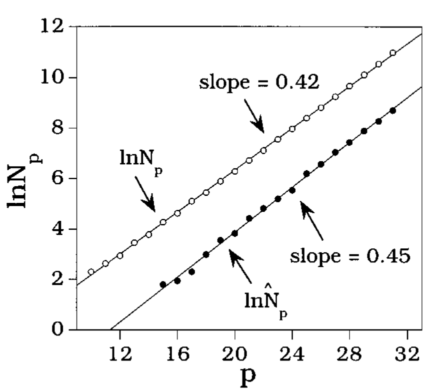

FIG. 3. $\ln\hat{N}_p$ versus p (filled circles), where $\hat{N}_p$ is the number of period-p orbits that changes from being transversely stable to transversely unstable when α is increases from 1.31 to 1.35. The upper curve is $\ln N_p$ versus p (open circles), where N_p is the number of distinct period-p orbits. We have $N_p\sim e^{h_T p}$ and $\hat{N}_p\sim e^{h_T p}$.

Furthermore, our model system Eq. (2) captures the essential features of the blowout bifurcation [1,2]. We have also tested cases where the dynamics in the invariant subspace is one dimensional and have obtained similar results [18]. Thus, we feel safe in concluding that blowout bifurcation is a unique type of bifurcation that is mediated by *an infinite number* of unstable periodic orbits.

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[17] The scaling $\Delta\alpha \sim e^{-Cp}$ $(C>0)$ can be qualitatively understood as follows. The quantity $\Delta\Lambda_p(\alpha)$ is an approximation of the transverse Lyapunov exponent $\Lambda l_\perp$. At the blowout bifurcation point, $\Lambda_\perp = 0$ and, hence, $\Delta\alpha_p \sim \Delta\Lambda_p(\alpha_c)$. The number of period-$p$ orbits $N(p)$ is roughly $e^{h_T p}$. Imagine we compute the transverse Lyapunov exponent for each period-p orbit at α_c. Using the average value of these exponents as an estimate of $\Lambda_\perp$ involves an error which scales like $[N(p)]^{-C_0} \sim e^{-C_p}$, where $C \equiv C_0 h_T$ and C_0 is a constant $(0<C_0<1)$ which depends on the probability measure of the periodic orbits. The rather small value of C and large fluctuations in Fig. 2 are probably due to nonhyperbolicity of the Hénon attractor.

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