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Distinct small-distance scaling behavior of on-off intermittency in chaotic dynamical systems

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On-off intermittency in chaotic dynamical systems refers to the situation where some dynamical variables exhibit two distinct states in their course of time evolution. One is the ''off'' state, where the variables remain approximately a constant, and the other is the ''on'' state, where the variables temporarily burst out of the off state. Previous work demonstrates that there appears to be a universal scaling behavior for on-off intermittency. In particular, the length of off time intervals, or the length of the laminar phase, obeys the algebraic scaling law. We present evidence that there are in fact distinct classes of on-off intermittency. Although the statistics of their laminar phase obeys the algebraic scaling, quantities such as the average transient time for trajectories to fall in a small neighborhood of the asymptotic off state exhibit qualitatively different scaling behaviors. The dynamical origin for producing these distinct classes of on-off intermittency is elucidated. [S1063-651X(96)08907-6]

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I. INTRODUCTION

Recently, there has been a growing interest in the phenomenon of on-off intermittency in nonlinear dynamical systems [1–6]. On-off intermittency refers to the situation where some dynamical variables of the system exhibit two distinct states as the system evolves in time. One is the ''off'' state, where the dynamical variables remain approximately constant values in various time intervals. There can also be occasional bursts of the dynamical variables away from their constant values in the off state. These bursts are referred to as the ''on'' state, which occurs intermittently as time progresses. Mechanisms for generating on-off intermittency and characterizations of on-off time series have been investigated [1–3]. It has also been shown that on-off intermittency is in fact closely related to the phenomenon of riddled basins [7,4,5].

A general class of dynamical systems that can generate on-off intermittency is systems that are driven either randomly or chaotically [1–3]. A simple but representative class of systems is the discrete map investigated by Heagy, Platt, and Hammel [3]

$$\mathbf{y}_{n+1}=G(\mathbf{x}_n,p)\mathbf{g}(\mathbf{y}_n), \tag{1}$$

where $\mathbf{y}_n$ is an N-dimensional state vector, $\mathbf{g}(\mathbf{y}_n)$ is a nonlinear function, $G(\mathbf{x}_n,p)$ is a scalar function that models the external driving to the $\mathbf{y}$ dynamics, and p is a parameter of the driving function. Without loss of generality, we assume that the asymptotic value of the off state is defined by $\mathbf{y}=\mathbf{0}$ and therefore the nonlinear function $\mathbf{g}(\mathbf{y})$ has the property that $\mathbf{g}(\mathbf{0})=\mathbf{0}$. The dynamics of $\mathbf{x}$ can be either a stochastic process or a deterministic chaotic process; the latter can be written as

$$\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n), \tag{2}$$

where $\mathbf{f}(\mathbf{x})$ is a chaotic map. Systems so described can produce on-off intermittency in $\mathbf{y}$ for a wide class of driving functions $G(\mathbf{x}_n)$ [3]. Heagy, Platt and Hammel specifically considered the case where both $\mathbf{y}$ and $\mathbf{x}$ are one dimensional, $g(y)=y(1-y)$ and $G(x)=ax$, with a being a parameter characterizing the driving strength. Different driving dynamics $f(x)$ in Eq. (2), including uniform random variables, the tent map, the $2x$ mod(1) map, and the logistic map, were tested, all producing on-off intermittency in y when the driving strength a is larger than some critical value a_c [the value of a_c generally depends on $f(x)$] [3].

Statistical properties of on-off intermittent time series have also been investigated [3,6]. A remarkable result is that certain characterizations of on-off intermittency appear to follow universal scaling laws in the sense that they hold regardless of the type of driving dynamics $\mathbf{f}(\mathbf{x})$. Among these characterizations, a natural one is the probability distribution of laminar phases, i.e., the distribution $P(T)$, where T is the time interval for which the trajectory stays in the off state. It was shown by Heagy, Platt, and Hammel [3] that when $\mathbf{x}$ is a random variable with smooth density, $P(T)$ obeys an algebraic scaling $P(T)\sim T^{-\gamma}$, where the scaling exponent γ attains a universal value of $\frac{3}{2}$ when $a\gtrsim a_c$ (a_c is the critical value for the birth of on-off intermittency). Numerical computation indicates that the same result appears to hold even if $\mathbf{x}$ is a deterministically chaotic variable [3,8]. It was also shown that asymptotically, $P(T)$ decays exponentially [3]. Subsequent work by Venkataramani *et al.* [6] shows that on-off intermittent time series are in fact fractal time series in that the set of intersecting points of the time series $y(t)$ with $y=\text{const}\gtrsim 0$ is a fractal set in certain time scales. For $a\gtrsim a_c$, such a fractal set possesses a box-counting dimension of $\frac{1}{2}$, an equivalent of the fact that γ is equal to $\frac{3}{2}$ in the scaling of $P(T)$.

In this paper, we present evidence that there can in fact be distinct classes of on-off intermittency in chaotic dynamical systems. In particular, different dynamics governing the driving may lead to *qualitatively different* types of on-off inter-

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mittent time series. While the conventional laminar-phase statistics for these different types of on-off intermittency still obey the universal algebraic scaling law [provided that one carefully sets the threshold for counting $y(t)$ as being in the off state, as we will discuss later], there are other statistical properties of the on-off intermittent time series that exhibit qualitatively different scaling behaviors. When the invariant density $\rho(\mathbf{x})$ of the driving variable $\mathbf{x}$ appears to contain an infinite number of singularities (numerically) so that the probability distribution $P(m)$ of the time intervals m during which a typical trajectory experiences attraction towards the invariant subspace decays exponentially, it would take a prohibitively long time for the system to get arbitrarily close to the off state. Thus, if one plots the time series of finite length, one observes a ''gap'' in the distance of the trajectory from the off state. We call this *class-I* on-off intermittency. If, on the other hand, $\rho(\mathbf{x})$ is smooth so that $P(m)$ decays algebraically, one expects to be able to observe the system being as close to the off state as one practically wishes. We call this *class-II* on-off intermittency. To quantitatively distinguish these two different classes of on-off intermittency, we examine the scaling behavior of the average transient time $\tau(\epsilon)$, defined to be the time for a typical trajectory to enter the ϵ neighborhood of the off state. Our main results can be summarized as follows. For class-I on-off intermittency, $\tau(\epsilon)$ scales with ϵ algebraically,

$$\tau(\epsilon)\sim\epsilon^{-\alpha}, \tag{3}$$

where $\alpha>0$ is the scaling exponent. For class-II on-off intermittency, $\tau(\epsilon)$ scales algebraically with $|\ln \epsilon|$,

$$\tau(\epsilon)\sim|\ln \epsilon|^{\beta}, \tag{4}$$

where $\beta>0$ is the scaling exponent and the scaling relation is valid for small ϵ values ranging over many orders of magnitude. We provide heuristic arguments and numerical evidence for Eqs. (3) and (4).

This paper is organized as follows. In Sec. II we present numerical evidence that qualitatively different types of on-off intermittency can be generated by different types of driving. In Secs. III and IV we give theoretical arguments and numerical results for the scaling of $\tau(\epsilon)$ for class-I and class-II on-off intermittency, respectively. In Sec. V we present a discussion.

II. EFFECT OF DRIVING ON CHARACTERISTICS OF ON-OFF INTERMITTENT TIME SERIES

A powerful tool to study the dynamical mechanism for on-off intermittency to occur in the general class of system Eqs. (1) and (2) is to make use of the idea of symmetry and invariant subspace. To see this, we regard the driving variable $\mathbf{x}$ in Eq. (2) and the dynamical variable $\mathbf{y}$ as two interconnected components of a single variable $\mathbf{z}\equiv(\mathbf{x},\mathbf{y})$. Since the asymptotic value of the off state is $\mathbf{y}=\mathbf{0}$ and $\mathbf{g}(\mathbf{0})=\mathbf{0}$, we see that $\mathbf{y}=\mathbf{0}$ defines an invariant subspace in the full phase space $\mathbf{z}$. We say a subspace is invariant if initial conditions in the subspace result in trajectories that remain in the subspace forever. The variable $\mathbf{x}$ is produced by the dynamics in this invariant subspace that can be either deterministically chaotic or stochastic. The variable $\mathbf{x}$ thus provides driving to the $\mathbf{y}$ subspace that is transverse to the invariant subspace. On-off intermittency in $\mathbf{y}$ occurs when the dynamics near the invariant subspace is *weakly unstable* with respect to transverse perturbations $\delta\mathbf{y}$ to trajectory points in the invariant subspace [4,5]. To quantify this instability one can define the transverse Lyapunov spectrum [4] as

$$\lambda_{\perp}^{i}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln\left|G(\mathbf{x}_n)\mathbf{Dg}(\mathbf{y}_n)|_{\mathbf{y}_n=\mathbf{0}}\cdot\mathbf{u}_i\right|, \tag{5}$$

where $\mathbf{Dg}(\mathbf{y}_n)|_{\mathbf{y}_n=\mathbf{0}}$ is the Jacobian matrix of the map $\mathbf{g}$ evaluated at $\mathbf{y}_n=\mathbf{0}$ and $\mathbf{u}_i$ is one of the eigenvectors in the eigenspace of $\Pi_{n=1}^{\infty}\mathbf{Dg}(\mathbf{y}_n)|_{\mathbf{y}_n=\mathbf{0}}$. For a randomly chosen unit vector $\mathbf{u}$, Eq. (5) yields the largest transverse Lyapunov exponent, which we denote by $\lambda_{\perp}$. The dynamics in the vicinity of the invariant subspace is weakly unstable when $\lambda_{\perp}$ is only slightly positive. In this case, on average, trajectories are repelled away from the invariant subspace $\mathbf{y}_n=\mathbf{0}$ so that $\mathbf{y}_n\neq\mathbf{0}$ can occur. This corresponds to the on behavior. But since $\lambda_{\perp}$ is only slightly positive, in any finite time trajectories can be attracted towards and then stay in the vicinity of the invariant subspace. This leads to the off behavior. These behaviors can be more precisely quantified by fluctuations in the values of $\lambda_{\perp}$ computed for an ensemble of trajectories restricted to the invariant subspace in finite times [4,5].

With this dynamical picture of on-off intermittency in mind, we now consider the following version of Eqs. (1) and (2):

$$x_{n+1}=f(x_n),\quad y_{n+1}=\frac{1}{2\pi}(px_n)\sin 2\pi y_n, \tag{6}$$

where both x and y are one dimensional, so that they define a two-dimensional map, and $f(x)$ is a chaotic map. The y equation is invariant under the symmetric operation $y\to -y$ and hence $y=0$ defines the one-dimensional invariant subspace. In order to see the on-off intermittent behavior, we restrict our investigation to cases where the dynamics in the invariant subspace described by $f(x)$ generates a chaotic attractor with invariant density $\rho(x)$. In the following we present numerical experiments for two cases where (i) $\rho(x)$ appears to contain an infinite number of singularities such as that produced by the logistic map for most of the parameter values in the chaotic regime and (ii) $\rho(x)$ is smooth.

A. Class-I on-off intermittency

We choose $f(x)$ to be the logistic map $f(x)=rx(1-x)$ at $r=3.8$. In this case, numerical computation indicates that $\rho(x)$ appears to contain an infinite number of singularities, as shown in Fig. 1, where $\rho(x)$ is computed using a trajectory of 10^7 points. These singularities come from the successive iterations of the critical point of the map $x_c=0.5$ [9]. The transverse Lyapunov exponent is

$$\lambda_{\perp}=\int_0^1\ln|px|\rho(x)dx. \tag{7}$$

Thus we have $p_c=\exp[-\int_0^1\ln|x|\rho(x)dx]$, where $\lambda_{\perp}\geqslant 0$ for $p\geqslant p_c$ and $\lambda_{\perp}<0$ for $p<p_c$. On-off intermittency occurs

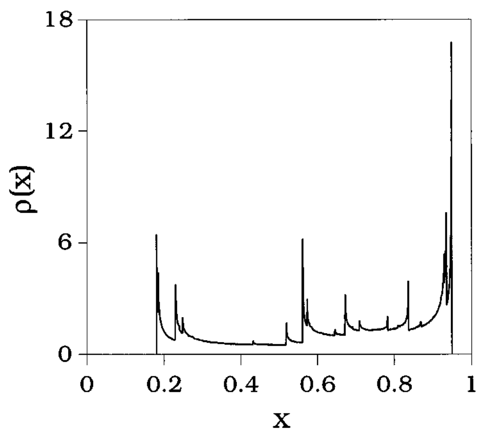

FIG. 1. Invariant density $\rho(x)$ of the logistic map at $r=3.8$ obtained from a trajectory of 10^7 points. Numerically, $\rho(x)$ appears to contain an infinite number of singularities.

when $p \gtrsim p_c$. For $r=3.8$, we numerically find that $p_c \approx 1.725$. Figure 2(a) shows a time series y_n of 5000 iterations (after 10^6 preiterations) resulting from an arbitrary initial condition $0<x_0<1$ and $0<y_0<0.5$ for $p=1.75$ ($|p-p_c| \approx 2.5\times 10^{-2}$). Clearly, there are time intervals when y_n stays near $y=0$ (the off state), but there are also intermittent bursts of y_n (the on state) away from the off state. This is due to the fact that $\lambda_\perp$ is only slightly positive when $p \gtrsim p_c$ ($\lambda_\perp \approx 0.014$). Imagine we choose an ensemble of initial conditions in x, compute $\lambda_\perp$ for each initial condition at a finite time, and then construct a histogram of these exponents. Since the asymptotic value of $\lambda_\perp$ is only slightly positive, there is a spread of the histogram into the negative side, indicating that a trajectory can spend long stretches of time near $y=0$ in finite times. But since $\lambda_\perp$ is positive, occasionally the trajectory can be repelled away from $y=0$. Thus on-off intermittency occurs. To see how close a typical trajectory can be to $y=0$ while in the off state, we plot the same time series on a semilogarithmic scale $\log_{10} y_n$ versus n, as shown in Fig. 2(b), where we clearly see a gap.

B. Class-II on-off intermittency

We choose $f(x)$ to be the $2x\,\mathrm{mod}(1)$ map: $x_{n+1}=2x_n\mathrm{mod}(1)$, where $x\geqslant 0$. This map produces a uniform invariant density $\rho(x)=1$ for $0\leqslant x\leqslant 1$. The transverse Lyapunov exponent is therefore given by $\lambda_\perp=\ln p-1$, so on-off intermittency occurs when $p \gtrsim p_c=e=2.718\,28\ldots$. Figure 3(a) shows such an on-off intermittent time series y_n of 5000 iterations (after 10^6 preiterations) from an arbitrary initial condition ($0<x_0<1$ and $0<y_0<0.5$) for $p=2.743\,28$. Here, again, we have $(p-p_c)\approx 2.5\times 10^{-2}$ and $\lambda_\perp \approx 0.012$. Figure 3(b) shows the same time series on the semilogarithmic scale $\log_{10}(y_n)$ versus n.

We now compare class-I with class-II on-off intermittency. Although both cases produce on-off intermittent time series as shown in Figs. 2 and 3, there is a subtle difference between them. For class-I [Fig. 2(a)] on-off intermittency, there appears to be a very small gap between the minimum value of y_n and the invariant subspace $y=0$. The gap is particularly clear when y_n is plotted on a logarithmic scale, as shown in Fig. 2(b). We see that for the 5000 iterations

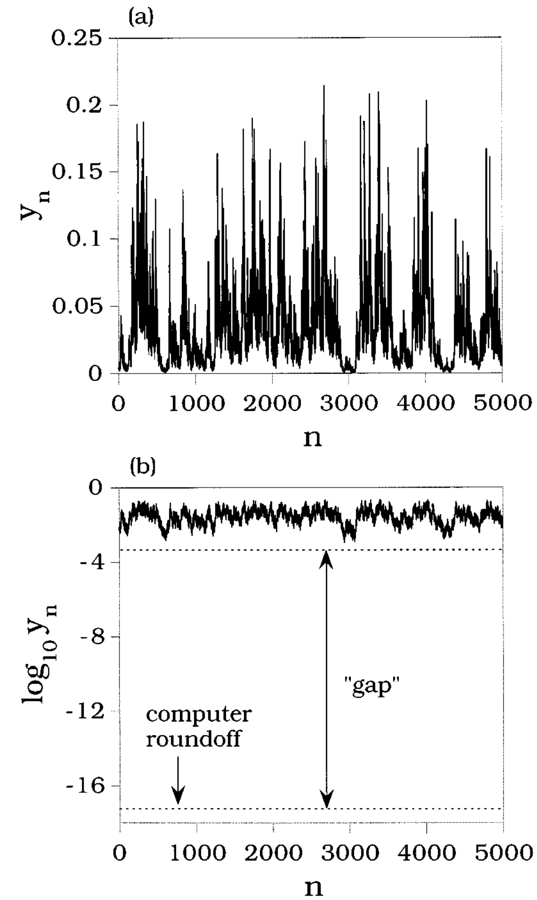

FIG. 2. (a) On-off intermittent time series y_n generated by Eq. (6), where $p=1.75$ [$(p-p_c)\approx 2.5\times 10^{-2}$]. The x dynamics is the logistic map at $r=3.8$. (b) Same time series plotted on the semilogarithmic scale $\log_{10} y_n$ versus n. For the 5000 iterations shown, there is a gap between y_n and the computer roundoff, indicating that it is difficult for the trajectory to get arbitrarily close to the invariant subspace.

shown, the minimum value of y_n is about 10^{-4}. If y_n could get arbitrarily close to the asymptotic off state $y=0$, the numerically computed value of y_n would attain values that are arbitrarily close to the computer roundoff about $\lesssim 10^{-16}$, which indeed occurs frequently for class-II on-off intermittency, as shown in Fig. 3(b). Thus, for class-I on-off intermittency, it is very difficult for a trajectory to get arbitrarily close to $y=0$. In order to see y_n to fall within less than 10^{-4} of the off state, it is necessary to iterate Eq. (6) for at least more than 5000 iterations, while for class-II on-off intermittency, a trajectory can easily get extremely close to $y=0$, which is apparent even in the linear plot [Fig. 3(a)]. For the 5000 iterations shown, the minimum value of y_n is about 10^{-16} (the computer roundoff). Thus y_n may even be closer to $y=0$ than that shown in Figs. 3(a) and 3(b). Note that both time series are generated at the parameter values about 2.5×10^{-2} above the critical parameter value p_c with similar values of $\lambda_\perp$. Thus the qualitatively different time series in Figs. 2 and 3 may be due to the different characteristics in the invariant density $\rho(x)$ of the chaotic process in the invariant subspace $y=0$. In fact, we have examined a large number of different $\rho(x)$ to test this hypothesis. For class I, the examples examined include the logistic map at various

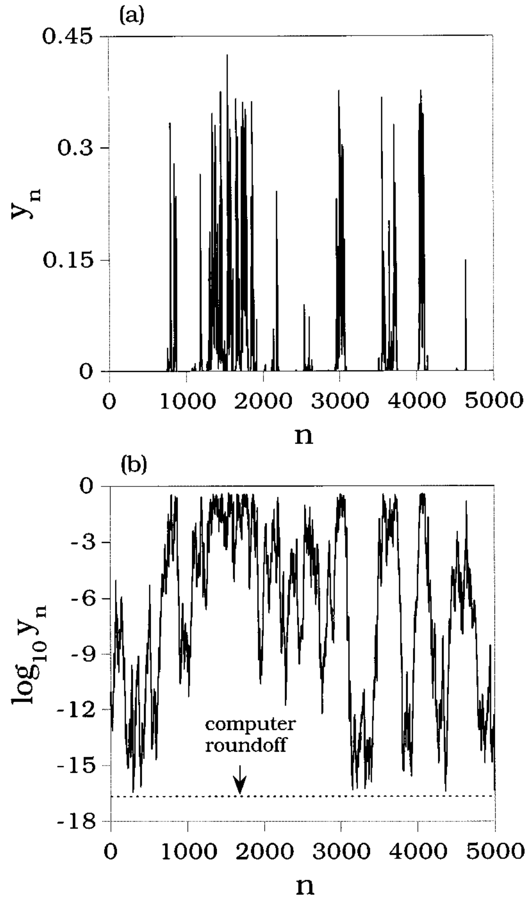

FIG. 3. (a) On-off intermittent time series y_n generated in the y equation in Eq. (6) when the x dynamics is the $2x$ mod(1) map that produces a uniform probability distribution in $x \in [0,1]$. The parameter is $p=2.743\,28$ [$(p-p_c)\approx 2.5\times 10^{-2}$]. (b) Same time series plotted on the semilogarithmic scale $\log_{10} y_n$ versus n. In this case, the trajectory can get arbitrarily close to the invariant subspace since the minimum value of y_n can be as small as the computer roundoff.

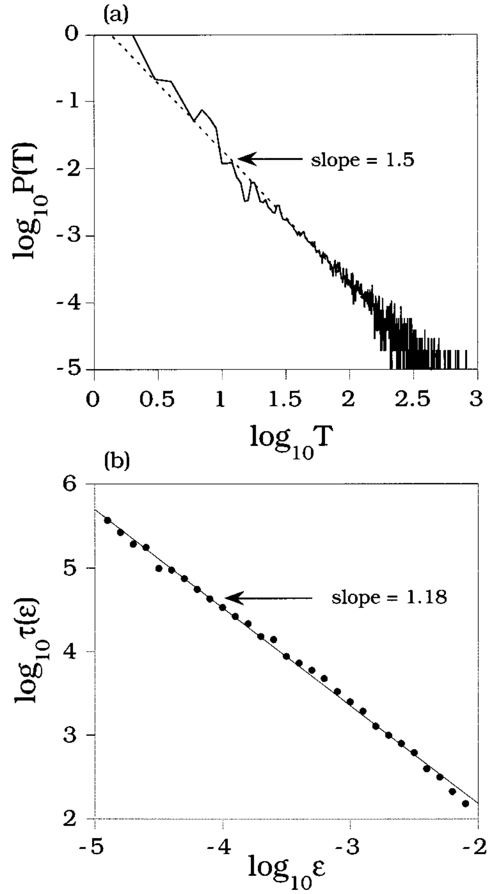

FIG. 4. (a) Laminar-phase distribution $P(T)$ for the on-off intermittent time series in Fig. 2(a) computed at the threshold $\epsilon=10^{-3}$. Roughly, we have $P(T)\sim T^{-3/2}$. (b) Average transient time $\tau(\epsilon)$ versus ϵ on a logarithmic scale. Clearly, $\tau(\epsilon)\sim\epsilon^{-\alpha}$, where $\alpha>0$ is the algebraic scaling exponent.

values of the parameter r that seem to generate $\rho(x)$ with an infinite number of singularities when examined numerically. For class II, the examples examined include the tent map and stochastic processes that produce smooth probability densities. For all the cases examined, we obtain similar results, as exemplified by Figs. 2 and 3. Therefore, we propose the conjecture that characteristically different invariant densities of the driving variable produce distinct on-off intermittent time series. In what follows we shall quantify the scaling behavior of these distinct on-off intermittent processes and give heuristic arguments to support the scaling.

III. SCALING BEHAVIOR OF CLASS-I ON-OFF INTERMITTENCY

Two convenient ways to characterize on-off intermittency reported in the literature are (i) to examine the statistics of laminar phases [3] and (ii) to compute the fractal dimension of the on-off intermittent time series [6]. In method (i), a small distance ϵ from the invariant subspace is set so that one can distinguish the off state from the on state. If a trajectory falls within ϵ of the invariant subspace, it is regarded as being in the off state; otherwise it is considered in the on state. In method (ii), a threshold at a distance ϵ from the invariant subspace is set so that the fractal dimension of the set of intersecting points of the trajectory at the threshold distance can be computed. In both approaches, it is essential that the threshold ϵ be set properly so that there are either sufficiently many laminar phases or sufficiently many intersecting points at the threshold to guarantee a meaningful numerical computation of the laminar-phase statistics or the fractal dimension.

For systems that exhibit class-I on-off intermittency, it is extremely difficult for a trajectory to get arbitrarily close to the invariant subspace. Thus, to examine the laminar-phase statistics or to compute the fractal dimension in a computationally feasible way, it is necessary to set ϵ at somewhat larger values. Indeed, for example, the laminar phase statistics so obtained obeys the algebraic scaling law. Figure 4(a) shows, on a logarithmic scale, a histogram of 10^8 laminar phases for the on-off intermittent time series in Fig. 2(a), where $\epsilon=10^{-3}$. Apparently, the histogram exhibits the algebraic scaling behavior $P(T)\sim T^{-3/2}$. If one sets the threshold ϵ to be much smaller than, say, 10^{-3}, it is computationally difficult to determine whether the laminar-phase scaling would still be algebraic in short time scales because it would take a prohibitively long time to observe a typical trajectory to fall below the threshold, let alone to accumulate enough statistics to extract the correct scaling behavior. In this sense, the laminar-phase statistics is inadequate to capture the scaling behavior of the class-I on-off intermittent time series *at small distances*, at least from the standpoint of performing the computation or experimental observation in realistic time.

We thus seek to use alternative scaling quantities to characterize the small distance behavior of class-I on-off intermittency. We propose to study the scaling of the average transient time $\tau(\epsilon)$ for a typical trajectory to first fall in the ϵ neighborhood of the off state. Figure 4(b) shows, for the same parameter setting as in Fig. 2, $\tau(\epsilon)$ versus ϵ on a logarithmic scale. To obtain this plot, 5000 trajectories resulting from random initial conditions uniformly chosen from $x_0 \in (0,1)$ and $y_0 \in (0,0.5)$ are used to compute the average value of $\tau(\epsilon)$ for each value of ϵ. We see that the plot can be fitted by a straight line, indicating a robust algebraic scaling behavior [Eq. (3)]. For $\epsilon < 10^{-5}$, the transient time for some trajectories become prohibitively long for $\tau(\epsilon)$ to be computed in reasonable time.

We now give a heuristic argument for the algebraic scaling of $\tau(\epsilon)$. Take a trajectory that starts with an initial condition $y_0 \sim 1$. In order for the trajectory to fall within ϵ of the invariant subspace $y=0$, on average the trajectory must experience attraction towards $y=0$ in time $\tau(\epsilon)$. It is thus insightful to study the statistics of the time intervals in which trajectories experience contraction on average. For simplicity we consider the dynamics in the vicinity of $y=0$. For y_n small we have $y_{n+1} \approx p x_n y_n$ from Eq. (6), so $y_m \approx (p^m \prod_{i=0}^{m-1} x_i) y_0$. Thus we are led to consider the sequence in x: $\{x_0, x_1, \ldots, x_{m-1}\}$, which satisfies

$$p^m \prod_{i=0}^{m-1} x_i \equiv (p\overline{x_m})^m \sim \epsilon, \tag{8}$$

where $(\overline{x_m})^m \equiv x_0 x_1 \cdots x_{m-1}$ and $p\overline{x_m} < 1$. The integer m is in fact the time interval that a trajectory is attracted towards the invariant subspace on average. As a crude approximation we assume that $\overline{x_m}$ is independent of m and write $\overline{x_m} = \overline{x}$. We then ask, What is the probability distribution $P(m)$ for the length m of the sequence? To answer this question, we observe that points in the sequence $\{x_0, x_1, \ldots, x_{m-1}\}$ can be divided into two groups: one with $px_i \geq 1$ (or $x_i \geq x_c \equiv 1/p$) and one with $px_i < 1$ (or $x_i < x_c$). For the logistic map at $r=3.8$, we observe that for most times, a typical trajectory visits the region $x < x_c$ and $x \geq x_c$ in an alternative fashion. Numerical computation using a trajectory of 10^8 iterations reveals that the probability for a trajectory to visit $x < x_c$ for two consecutive iterations is negligible and no event for the trajectory of this length to visit $x < x_c$ consecutively for more than two times has been observed. Thus we have the crude estimation

$$P(m) \sim q^{m/2},$$

where q is the probability for two consecutive trajectory points x_1 and x_2, one on left of x_c and the other on right of x_c (or vice versa), to satisfy $(px_1)(px_2) < 1$. We thus see that $P(m)$ decays exponentially:

$$P(m) \sim \exp(-|\ln q| m/2). \tag{9}$$

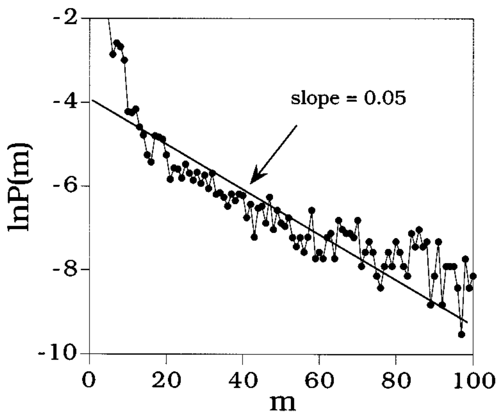

FIG. 5. Distribution $P(m)$ [Eq. (9)] for the logistic map at $r=3.8$ plotted on a semilogarithmic scale. Clearly, $P(m)$ decays exponentially, which gives rise to the algebraic scaling of $\tau(\epsilon)$ seen in Fig. 4(b).

Figure 5 shows such an exponential decay computed numerically for the logistic map at $r=3.8$, where $\epsilon = 10^{-3}$ and 10^7 values of m are accumulated to compute the histogram $P(m)$. The exponential decay indicates that it is highly unlikely for m, the average time interval in which a trajectory experiences net attraction towards the invariant subspace, to be large. Combining Eqs. (8) and (9) yields

$$P(\epsilon) \sim P(m) \sim \epsilon^{|\ln q|/(2|\ln p\overline{x}|)}. \tag{10}$$

Since $\tau(\epsilon) \sim 1/P(\epsilon)$, Eq. (10) immediately yields Eq. (3), the algebraic scaling law for $\tau(\epsilon)$, with the scaling exponent given by $\alpha \approx |\ln q|/(2|\ln p\overline{x}|)$. From Eq. (9) and Fig. 5, we see that $|\ln q|/2 \approx 0.05$. To estimate the quantity $|\ln(p\overline{x})|$, we make use of Fig. 5, where the average value of m for $(p\overline{x})^m \sim \epsilon = 10^{-3}$ to be satisfied is $\overline{m} \sim 20 + 1/0.05 = 40$. Thus $|\ln p\overline{x}| \sim \ln 10^{-3}/40 \approx 0.17$. We obtain $\alpha \approx 0.29$.

We stress that the argument leading to Eq. (3) is only heuristic. There are several very crude approximations used in arriving at Eq. (3). Thus, naturally our argument does not yield a good estimate of the algebraic scaling exponent α [about 1.18 from Fig. 4(b)]. Nonetheless, the argument serves to establish the algebraic scaling relation between $\tau(\epsilon)$ and ϵ, which is supported by extensive numerical experiments.

IV. SCALING OF $\tau(\epsilon)$ FOR CLASS-II ON-OFF INTERMITTENCY

For class-II on-off intermittency, trajectories can get arbitrarily close to the invariant subspace. Therefore, the conventional laminar-phase statistics [3] or fractal dimension [6] characterization suffices to quantify this type of on-off intermittency. Computationally, an arbitrarily small threshold ϵ can be used to determine the laminar-phase statistics. After extensive numerical experiments, we find that this class of on-off intermittency is most likely to be generated by chaotic or random variables that have smooth invariant density in the invariant subspace. In this case, a trajectory in the invariant subspace can stay in the contracting region ($px < 1$) for a large number of iterations, in contrast to the previous case in Sec. III where trajectories must visit both the contracting and

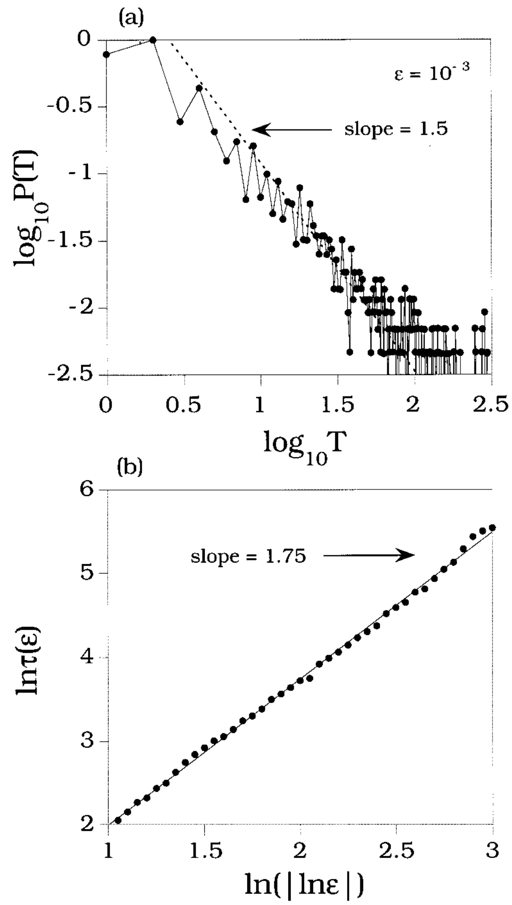

FIG. 6. (a) Laminar-phase distribution $P(T)$ for the on-off intermittent time series in Fig. 3(a) computed at the threshold $\epsilon=10^{-3}$. Again, we have $P(T)\sim T^{-3/2}$. (b) In this case, the average transient time $\tau(\epsilon)$ obeys the scaling law Eq. (4), as shown in the plot of $\ln\tau(\epsilon)$ versus $\ln|\ln\epsilon|$. The scaling exponent β is close to $\frac{3}{2}$.

expanding regions alternatively. For instance, consider the case in Fig. 3, where $\rho(x)$ is uniform in $x\in[0,1]$. The contracting region is given by $p.x<1$ or $x<x_c=1/p$. The probability for x to stay in $x<x_c$ *consecutively* for n iterations is p^{-n}. As a consequence, the probability $P(m)$ defined in Sec. III no longer decays exponentially, but rather it can attain appreciable values even when m is large.

To derive the scaling of the average transient time for class-II on-off intermittency, we make use of the previously established algebraic scaling of the laminar-phase $P(T)\sim T^{-\gamma}$ [3]. Since T is the time that a typical trajectory stays in the off state, while m in Eq. (9) is the time during which a trajectory experiences contraction towards the invariant subspace, we have $m\sim T$. Thus we expect $P(m)$ to follow a similar algebraic scaling law. We write $P(m)\sim m^{-\beta}$. Using Eq. (8) to express m in terms of $|\ln\epsilon|$ and using $\tau(\epsilon)\sim 1/P(\epsilon)$, we immediately obtain Eq. (4). One implication is that since $P(m)\sim P(T)$, the scaling exponent β in Eq. (4) should be close to the scaling exponent $\frac{3}{2}$ in $P(T)$ when $a\gtrsim a_c$. Figure 6(a) shows $P(T)$ versus T on a logarithmic scale for the on-off intermittency in Fig. 3, where we set $\epsilon=10^{-3}$ and use 10^6 laminar phases to calculate the histogram [11]. Figure 6(b) shows $\tau(\epsilon)$ versus $|\ln\epsilon|$ on a logarithmic scale, where for each ϵ, 5000 trajectories are used to compute $\tau(\epsilon)$. Note that the range of abscissa in the plot ($\ln|\ln\epsilon|\in[1.0,3.0]$) corresponds to approximately the range $\epsilon\in[0.06,1.8\times10^{-9}]$. The robust fitting of the data to a straight line in this ϵ range indicates that scaling relation Eq. (4) is valid for at least seven orders of magnitude in ϵ. When $\epsilon\approx0.06$, $\tau(\epsilon)\approx8$. When ϵ is decreased to about 10^{-9}, we have $\tau(\epsilon)=220$. Thus the increase in $\tau(\epsilon)$ is only incremental compared to the decrease in ϵ, indicating that it is not significantly more difficult for a typical trajectory to get within 0.06 than to get within 10^{-9} of the invariant subspace $y=0$. This behavior is qualitatively different from that shown in Fig. 4(b), where $\tau(\epsilon)$ increases faster than the rate that ϵ decreases. Furthermore, we see that the slope of the fitted line in Fig. 6(b) is about 1.75, which is close to the exponent $\frac{3}{2}$ in the scaling of $P(T)$.

V. DISCUSSION

The main point of the paper is that there can be distinct small-distance scaling behaviors associated with on-off intermittency in chaotic dynamical systems. Numerical results and qualitative arguments support the conjecture that these distinct scaling behaviors are caused by the different types of dynamical processes in the invariant subspace that provide the ''driving'' to generate on-off intermittency in dynamical variables in the transverse subspace. In particular, if the driving is such that the time intervals during which a typical trajectory is attracted towards the asymptotic off state obey an exponentially decaying law, it is very difficult for trajectories to get arbitrarily close to the invariant subspace. In this case, the conventional characteristics of on-off intermittency reported in the literature, such as the laminar-phase distribution, are inadequate to capture the statistical behavior of the trajectories near the invariant subspace. We quantify this small-distance behavior by studying the scaling of the average transient time $\tau(\epsilon)$. We argue, with numerical support, that $\tau(\epsilon)$ scales with ϵ algebraically for this type of on-off intermittency (class I). If, on the other hand, the invariant density of the driving variables is smooth, it appears that a typical trajectory can get arbitrarily close to the invariant subspace. This class of on-off intermittency is the one that has been investigated extensively in the literature. We show that the laminar-phase statistics in this case does yield the correct small-distance scaling behavior. The average transient time $\tau(\epsilon)$ scales with $|\ln\epsilon|$ algebraically (class II) over many orders of magnitude in ϵ, which is a direct consequence of the previously established algebraic scaling for the length of the laminar phases [3].

The numerical tests for the small-distance scaling behaviors [Eqs. (3) and (4)] are only performed on limited distance scales. Thus the question remains of whether Eqs. (3) and (4) would hold in the asymptotic limit $\epsilon\to0$. We conjecture that the asymptotic scaling law for both class-I and class-II intermittency should be algebraic. To see why $\tau(\epsilon)$ decays algebraically for class-II on-off intermittency as $\epsilon\to0$, we note that the laminar-phase distribution $P(T)$ appears to decay exponentially as $T\to\infty$ [3]. Thus, asymptotically, $P(m)$ also decays exponentially, leading to the algebraic scaling behavior in $\tau(\epsilon)$ as $\epsilon\to0$. Consequently, we expect the scaling law Eq. (4) for class-II on-off intermittency to be valid only in a finite range of distance scales. Nonetheless, from a practical point of view, it is impossible to observe the asymptotic behavior of $P(T)$ at $T\rightarrow\infty$ or that of $\tau(\epsilon)$ at $\epsilon\rightarrow 0$ in numerical or physical experiments. Thus our main results Eqs. (3) and (4) are important in practical situations.

The distinct scaling behaviors observed for class-I and class-II on-off intermittency have direct implications in practical applications such as controlling chaos. Suppose that for a system that exhibits on-off intermittency, the desirable operational state corresponds to the off state. One thus wishes to stabilize a trajectory in the vicinity of the invariant subspace to achieve better system performance by using arbitrarily small perturbations to an accessible system parameter or state (the controlling chaos idea proposed by Ott, Grebogi, and Yorke [10]). Assuming there is a maximum allowed magnitude for the parameter control $\delta \ll 1$. In order to achieve control, one sets a controlling neighborhood of the off state with size ϵ proportional to δ. Feedback control law can then be designed for a trajectory in the ϵ neighborhood of the invariant subspace [12]. In realizing the control, one waits until a trajectory resulting from a random initial condition to fall in the ϵ neighborhood to activate the parameter perturbations. The average waiting time is precisely the average transient time $\tau(\epsilon)$ whose scaling behavior is investigated in this paper. We see that for class-I on-off intermittency, the waiting time increases drastically as ϵ is decreased. This puts a practical limit to how small the magnitude of the feedback control can be, as there is a tradeoff between the smallness of the parameter perturbation one wishes to apply and the time one has to wait, while for class-II on-off intermittency, the average waiting time scales with ϵ as some power of $|\ln(\epsilon)|$. This indicates that the required waiting time increases only incremently even if ϵ is decreased by many orders of magnitude. Therefore, it is possible to apply extremely small parameter perturbations to achieve the desirable system performance *in a relatively short time* when one controls class-II on-off intermittency.

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