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Intermingled basins and two-state on-off intermittency

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We consider dynamical systems which possess two low-dimensional symmetric invariant subspaces. In each subspace, there is a chaotic attractor, and there are no other attractors in the phase space. As a parameter of the system changes, the largest Lyapunov exponents transverse to the invariant subspaces can change from negative to positive: the former corresponds to the situation where the basins of the attractors are *intermingled*, while the latter corresponds to the case where the system exhibits a *two-state on-off intermittency*. The phenomenon is investigated using a physical example where particles move in a two-dimensional potential, subjected to friction and periodic forcing.

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In this paper we consider nonlinear dynamical systems that possess *two* symmetric low-dimensional invariant subspaces. Denote the two subspaces by $\mathscr{S}_1$ and $\mathscr{S}_2$. Assume there is a chaotic attractor in each subspace for the parameter regime of interest. Since each subspace is invariant, initial conditions in each subspace result in trajectories which remain in the subspace forever. Furthermore, since we assume the only attractor in each subspace is a chaotic one, the largest Lyapunov exponent is positive for almost all initial conditions in each subspace. Whether the chaotic attractors in $\mathscr{S}_1$ and $\mathscr{S}_2$ are also attractors for the whole system depends on the sign of the largest Lyapunov exponent computed for trajectories in $\mathscr{S}_1$ and $\mathscr{S}_2$ with respect to perturbations in the subspace $\mathscr{T}$ which is *transverse* to both $\mathscr{S}_1$ and $\mathscr{S}_2$. In particular, when the largest transverse Lyapunov exponent is negative, both $\mathscr{S}_1$ and $\mathscr{S}_2$ attract trajectories transversely in the phase space, and the chaotic attractors in $\mathscr{S}_1$ and $\mathscr{S}_2$ are the global attractors for the whole phase space. When the largest transverse Lyapunov exponent is positive, trajectories in the vicinity of $\mathscr{S}_1$ and $\mathscr{S}_2$ are repelled away from the subspaces and, consequently, the attractors in $\mathscr{S}_1$ and $\mathscr{S}_2$ are transversely unstable and they are hence not the attractors for the whole phase space. They are, nevertheless, attractors for trajectories restricted to $\mathscr{S}_1$ and $\mathscr{S}_2$ in this case. We are interested in the parameter regime where, as some system parameter changes, the largest transverse Lyapunov exponent changes from negative to positive. The purpose of this Rapid Communication is to study the dynamics of the system for parameter values near the transition point.

Our work is motivated by a series of recent studies on riddled basins [1–5] and related bifurcations [6]. The phenomenology of riddle basins was introduced in Ref. [1] where the authors established that for a certain class of discrete maps with an invariant subspace, (1) if there is a chaotic attractor in the subspace, (2) if there is another attractor in the phase space, and (3) if the Lyapunov exponent transverse to the subspace is negative, then the basin of the chaotic attractor in the invariant subspace is riddled. That is, for every initial condition that asymptotically approaches the chaotic attractor in the subspace, there are initial conditions arbitrarily nearby that asymptotically approach the other attractor. Rigorous results on riddled basins for discrete maps were presented in Refs. [1,2]. The dynamics of riddled basins was subsequently investigated in [3,4] using a more realistic physical model. The authors of Ref. [6] also found that the phenomenon of riddled basins is in fact related to the phenomenon of "on-off intermittency" [7,8], an extreme form of temporarily intermittent bursting in the dynamical variables of the system, which occurs when there is no other attractor in the phase space and the transverse Lyapunov exponent is positive. In their studies of the riddling, only the basin of the chaotic attractor in the invariant subspace is riddled; the basins of the other attractors (typically nonchaotic) still contain open sets with finite measure. Furthermore, previous studies of on-off intermittency [7,8,6] considered only one-state intermittency in which the chaotic orbit typically spends long stretches of time in only one state (e.g., near the invariant subspace) with occasional bursting out of this state.

In Ref. [1], a more extreme type of basin structure was introduced, namely, a basin structure called *intermingled*. In this case, two or more basins of attraction are riddled, which usually occurs when there is more than one invariant subspace in the system. In this Rapid Communication, we demonstrate that for systems that possess two symmetric invariant subspaces, the presence of intermingled basins in some parameter regime may imply the existence of a type of intermittent behavior, which we refer to as *two-state on-off intermittency*, in nearby parameter regimes. This type of bifurcation occurs when there are no other attractors in the system except the ones in the invariant subspaces. When the largest transverse Lyapunov exponent with respect to both subspaces is negative, the basin of each attractor is riddled and, consequently, the two basins are intermingled. When the largest transverse Lyapunov exponent passes through zero from being negative, the attractors in both subspaces lose transverse stability simultaneously. Immediately after the bifurcation, since the exponent is still close to zero (only

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slightly positive), chaotic trajectories would spend long stretches of time near both subspaces with intermittent periods of bursts and switches between them. We illustrate these phenomena using a physical model in which particles are under the influence of a two-dimensional potential, and are subjected to both periodic forcing and friction [9]. As a by-product, our studies demonstrate that intermingled basins can also occur in more realistic physical systems. So far, to our knowledge, the study of intermingled basins has been restricted to abstract models of discrete maps [1,4].

We consider a mechanical system where particles move under the influence of the following potential in the plane:

$$V(\mathbf{x})=(1-x^2)^2+(y^2-a^2)^2(x-d)+b(y^2-a^2)^4, \quad (1)$$

where $\mathbf{x}\equiv(x,y)$, a, d, and b (>0) are parameters. We assume particles are also subjected to friction and periodic forcing of the form $f_0\sin(\omega t)$ in the x direction. This model modifies the one studied in [3,4] in that there are now two symmetric lines defined by $y=\pm a$ on which $V(\mathbf{x})$ is independent of the coordinate y and reduces to the Duffing's two-well potential in x [10]. By introducing the two symmetric lines, our model, Eq. (1), allows for intermingled basins and two-state on-off intermittency, as we shall see below. The equation of motion is

$$\frac{d^2\mathbf{x}}{dt^2}=-\gamma\frac{d\mathbf{x}}{dt}-\nabla V(\mathbf{x})+f_0\sin(\omega t)\mathbf{x}_0, \quad (2)$$

where γ is the friction coefficient, and $\mathbf{x}_0$ is the unit vector in x. The system can be written as five first-order autonomous differential equations in terms of the dynamical variables x, $v_x\equiv dx/dt$, y, $v_y\equiv dy/dt$, and $z=\omega t$,

$$\begin{aligned}\frac{dx}{dt}&=v_x,\\ \frac{dv_x}{dt}&=-\gamma v_x+4x(1-x^2)-(y^2-a^2)^2+f_0\sin z,\\ \frac{dy}{dt}&=v_y, \quad (3)\\ \frac{dv_y}{dt}&=-\gamma v_y-4y(y^2-a^2)(x-d)-8by(y^2-a^2)^3,\\ \frac{dz}{dt}&=\omega.\end{aligned}$$

Note that on the two lines $y=\pm a$, if $v_y=0$, the equations of motion reduce to

$$\begin{aligned}\frac{dx}{dt}&=v_x,\\ \frac{dv_x}{dt}&=-\gamma v_x+4x(1-x^2)+f_0\sin z, \quad (4)\\ \frac{dz}{dt}&=\omega,\end{aligned}$$

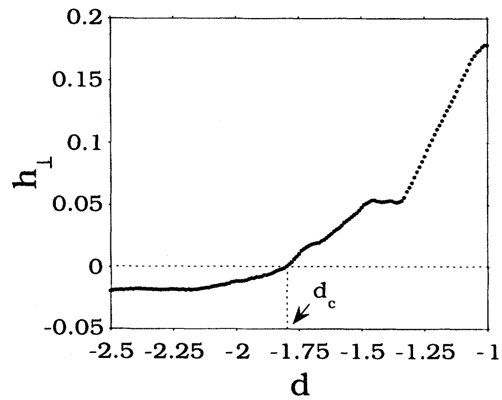

FIG. 1. The transverse Lyapunov exponent $h_\perp$ versus the parameter d for $-2.5\leq d\leq-1.0$. The exponent changes from negative to positive at $d_c\approx-1.805$. Other parameters are: f_0=2.3, γ=0.05, ω=3.5, a=0.8, and b=0.008.

which is the set of equations describing a forced-damped Duffing oscillator [10] in which chaos can occur. Since Eq. (4) is independent of y and v_y, we see that a trajectory with initial condition in the subspaces $y=\pm a$ and $v_y=0$ will remain in the subspaces forever. The conditions $y=\pm a$ and $v_y=0$ thus define two three-dimensional invariant subspaces, where the two chaotic attractors are located, in the five-dimensional phase space. The largest Lyapunov exponent $h_\perp$ transverse to the two invariant subspaces is computed by taking a variation of the y and v_y equations in (3) and setting $y=\pm a$ and $v_y=0$. We obtain

$$\begin{aligned}\frac{d\delta y}{dt}&=\delta v_y,\\ &\quad (5)\\ \frac{d\delta v_y}{dt}&=-\gamma\delta v_y-8a^2(x-d)\delta y,\end{aligned}$$

where x is a trajectory produced by Eq. (4) in the invariant subspaces, which acts like a driving signal in Eq. (5). The exponent $h_\perp$ is computed via $h_\perp=\lim_{t\to\infty}(1/t)\ln[\delta(t)/\delta(0)]$, where $\delta(t)\equiv\sqrt{[\delta y(t)^2+[\delta v_y(t)]^2}$.

As mentioned before, we are interested in the case where the attractors in the two invariant subspaces are chaotic. Therefore, we set $f_0=2.3$, $\gamma=0.05$, and $\omega=3.5$, a parameter setting for which Eq. (4) has a chaotic attractor [10]. We also require that the full system, Eq. (3), has no other attractors. This can be realized by setting the parameter b not too small. As such, the term $b(y^2-a^2)^4$ in the potential increases rapidly as y is away from $\pm a$, and particle trajectories are confined in the vicinity of the strip region in x from $y=-a$ to a if a is not too large. With forcing, particles can then overcome the potential barrier at $y=0$ and visit both regions in the vicinity of $y=\pm a$. We choose $b=0.008$ and $a=0.8$ so that, at $y=0$, the barrier term in the potential is $b(y^2-a^2)^4\approx1.34\times10^{-3}$, which is small compared with the forcing amplitude. Note that the potential barrier at $y=0$ is also determined by the parameter d. In what follows we will vary d and examine the dynamics in a region Where $h_\perp$ changes sign.

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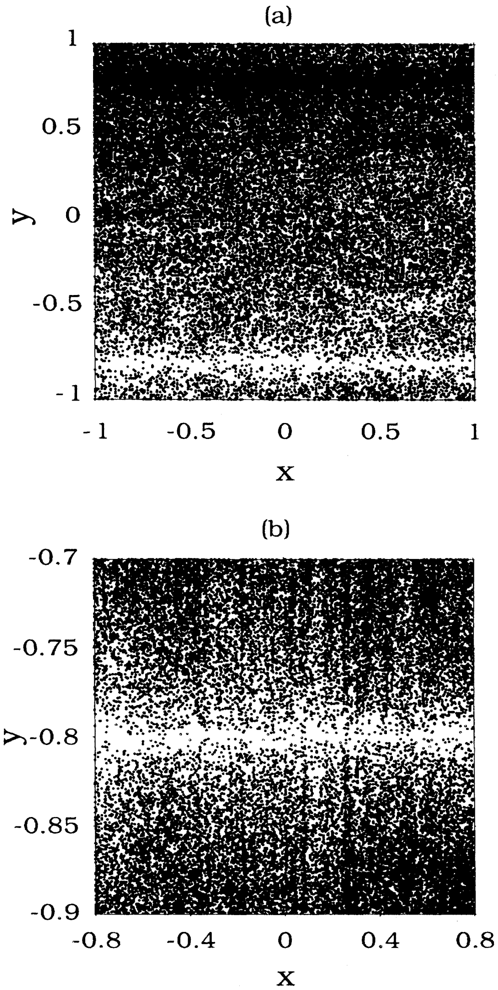

FIG. 2. (a) At $d=-0.19<d_c$ ($h_{\perp}\approx-0.0078<0$), basins of the $y=+a$ attractor (black dots) and the $y=-a$ attractor (white regions) for initial conditions chosen from the region $-1\leq[x,y]\leq1$ ($v_x=v_y=0$). (b) A blowup of (a) for $-0.8\leq x\leq0.8$ and $-0.9\leq y\leq-0.7$. The two basins are intermingled.

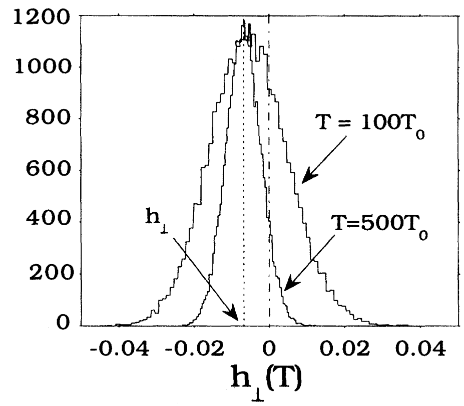

FIG. 3. For the same parameter setting as in Fig. 2 (intermingled basin), histograms of the largest transverse Lyapunov exponent $h_{\perp}(T)$ computed from 28 000 initial conditions randomly chosen in one of the subspaces at $T=100T_0$ and $500T_0$, where $T_0=2\pi/\omega$. Although the asymptotic value of $h_{\perp}$ is slightly negative, at finite times there are many trajectories which experience repulsion from the invariant subspace, reflected by the spread of these histograms into the positive region.

Figure 1 shows the largest transverse Lyapunov exponent $h_{\perp}$ versus the parameter d for $d\in[-0.25,-0.1]$ in which $h_{\perp}$ changes from being negative to positive at $d_c\approx-1.805$. For $d<d_c$, the chaotic attractors at $y=\pm a$ and $v_y=0$ are also attractors for the whole phase space [Eq. (3)]. Figure 2(a) shows the basins of the attractors at $y=+a$ (black dots) and at $y=-a$ (white regions) for initial conditions taken from the two-dimensional region $-1\leq[x,y]\leq1$ with $v_x=v_y=0$. Apparently, the two basins of attraction are intermingled. In fact, arbitrarily near any point in the basin of one attractor, there are points which belong to the basin of the other attractor. This can be seen in Fig. 2(b), where a blowup of Fig. 2(a) for $-0.9\leq y\leq-0.7$ (near the $y=-a$ attractor) is shown exhibiting black dots (in the basin of the $y=a$ attractor) very close to $y=-a$. The regions near the $y=-a$ attractor can be magnified further: black dots which belong to the basin of the $y=+a$ attractor are always seen, regardless how small the region around $y=-a$ is considered.

To see why the basins of the $y=\pm a$ attractors are intermingled, one can argue that each basin is in fact riddled. This can be understood by noting that there are fluctuations in the largest transverse Lyapunov exponent $h_{\perp}$ [3,4,6]. For the parameter setting of Fig. 2, although $h_{\perp}$ is slightly negative, trajectories typically experience finite time periods where $h_{\perp}$ is positive. Imagine that one chooses a large number of initial conditions in any one of the invariant subspaces and compute values of $h_{\perp}(T)$ for trajectories resulting from each initial condition over a time T. The histogram of all these $h_{\perp}(T)$ at a given instant of time T is usually a distribution with finite width around the asymptotic value of $h_{\perp}$ which is only slightly negative. This situation is shown in Fig. 3, where histograms of $h_{\perp}(T)$ computed from 28 000 randomly chosen initial conditions in one of the invariant subspaces are shown at time $T=100T_0$ and $T=500T_0$, where $T_0\equiv2\pi/\omega$ is the forcing period. The spread of the histogram into the positive region indicates that there are trajectories which are transversely repelled from the chaotic attractor in finite times. Asymptotically, the histogram converges to a δ function centered around $h_{\perp}$ for initial conditions chosen in the invariant subspaces. Yet there is a set of measure zero points on the attractor whose transverse Lyapunov exponents are positive. Near those points, trajectories are repelled away from the subspaces and, consequently, there are points which belong to the basin of the other attractor arbitrarily close to this attractor. By symmetry, both basins are riddled. Since apparently there is no other attractor in the phase space, the two basins must be intermingled.

As the parameter d increases through d_c, $h_{\perp}$ becomes slightly positive and the attractors in the two invariant subspaces are no longer transversely stable. Trajectories are now repelled transversely from the attractors. However, since $h_{\perp}$ is only slightly positive, the spread of $h_{\perp}(t)$ at finite time into the negative side (pictures not shown) implies that trajectories can spend long stretches of time near each of the two attractors. This leads to an intermittent behavior: a

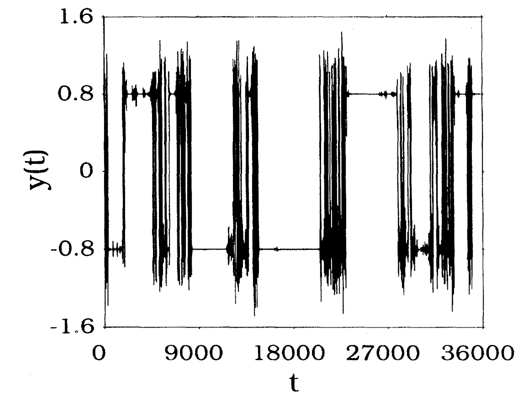

FIG. 4. At $d=-1.8>d_c$ ($h_\perp \approx 0.000\,58>0$), two-state on-off intermittency in $y(t)$ resulting from an arbitrary initial condition.

typical trajectory spends a long time near one attractor, is repelled away from this attractor, then is possibly attracted to the other attractor or the same attractor temporarily spending a long stretch of time near the attractor, repelled away, etc. Figure 4 shows, at $d=-1.8$, such an intermittent behavior in $y(t)$ for an arbitrary initial condition. This intermittent behavior differs from the conventional on-off intermittency where there is only one temporarily attracting state [7,8,6]. We call this *two-state on-off intermittency*.

In summary, we have presented a physical model which exhibits intermingled basins and two-state on-off intermittency as *a single parameter* is varied. Near the transition point, there are in fact universal scaling behaviors. For slightly negative values of $h_\perp$ (intermingled basins), these behaviors include (1) scaling of the fraction of the basin of one attractor near another with the distance away from the other attractor, (2) scaling of the probability of falling into the basin of another attractor upon small perturbation in the initial condition with the magnitude of the perturbation, and (3) the influence of small amplitude random noise. Those issues have been conjectured to be universal for the case of riddled basins based on solving diffusionlike models of two-dimensional maps [3,4]. The extension of these results to the case of intermingled basins will be reported in [11]. Conventional one-state on-off intermittent time series have been shown to exhibit interesting fractal scaling behaviors [8,12]. Results for two-state on-off intermittency will also be reported in [11].

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