Early effect in time-dependent, high-dimensional nonlinear dynamical systems with multiple resonances

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(Received 16 December 2014; published 9 February 2015)

We investigate high-dimensional nonlinear dynamical systems exhibiting multiple resonances under adiabatic parameter variations. Our motivations come from experimental considerations where time-dependent sweeping of parameters is a practical approach to probing and characterizing the bifurcations of the system. The question is whether bifurcations so detected are faithful representations of the bifurcations intrinsic to the original stationary system. Utilizing a harmonically forced, closed fluid flow system that possesses multiple resonances and solving the Navier-Stokes equation under proper boundary conditions, we uncover the phenomenon of the *early effect*. Specifically, as a control parameter, e.g., the driving frequency, is adiabatically increased from an initial value, resonances emerge at frequency values that are lower than those in the corresponding stationary system. The phenomenon is established by numerical characterization of physical quantities through the resonances, which include the kinetic energy and the vorticity field, and a heuristic analysis based on the concept of instantaneous frequency. A simple formula is obtained which relates the resonance points in the time-dependent and time-independent systems. Our findings suggest that, in general, any true bifurcation of a nonlinear dynamical system can be unequivocally uncovered through adiabatic parameter sweeping, in spite of a shift in the bifurcation point, which is of value to experimental studies of nonlinear dynamical systems.

DOI: 10.1103/PhysRevE.91.022906 PACS number(s): 05.45.−a

I. INTRODUCTION

Fundamental to nonlinear dynamical systems is a rich variety of bifurcation phenomena. As a control parameter of the system is changed, the asymptotic state of the system can exhibit transitions from one type of behavior to another, characteristically a different type. The bifurcations constitute the most pronounced feature that distinguishes nonlinear from linear dynamical systems. In fact, the history of research on nonlinear dynamics can be said to center about the study of various bifurcation or transition phenomena [1,2].

Given a dynamical system, numerical studies of bifurcations are relatively straightforward: one identifies the most relevant set of parameters responsible for the physical phenomena of interest, changes the parameters systematically, and then investigates, for each fixed parameter set, the attractors or invariant sets of the system from random initial conditions. This approach, however, often is not applicable to experimental study of bifurcations because of the practical difficulty of varying the system parameters in small increments in a controlled manner. An alternative and feasible way is to vary the most relevant parameter of the system *slowly* or *adiabatically* in time. Especially if the intrinsic time scale of the system is much faster than that of the parameter variation, the system can be regarded as constantly being in some adiabatic state. The hope is that, in any such state, the system reaches approximately some asymptotic invariant set so that appropriate measurements can be taken, based on which the system dynamics can be studied. The question is, Are bifurcations from adiabatic variations of system parameters faithful reflections of the true bifurcations of the system? This is referred to as the *slow passage* problem, which has been investigated in different physical contexts such as laser instabilities [3,4], fluctuations in mechanical systems exhibiting resonances [5], and thermal convection in fluids [6,7]. Mathematically, a dynamical system under adiabatic, time-dependent parameter variation can be modeled by using a set of nonautonomous differential equations. For convenience, we use the terms *stationary system* and *slow passage system* to describe the original dynamical system with time-independent, stationary parameters and the corresponding system with adiabatic parameter variations, respectively. A remarkable phenomenon is the so-called *early effect*, where a bifurcation occurs at a parameter value smaller than the corresponding bifurcation point in the original stationary system. For example, slow passage through resonance typically results in early onset of oscillations [8,9]. We note that, in the vast literature on nonlinear dynamical systems, there are only a few publications addressing the slow passage problem. Yet, in spite of the existing work, the mechanism leading to early bifurcation of the system due to adiabatic parameter change for nonlinear dynamical systems, especially high-dimensional systems, is not well understood.

In this paper, we seek to obtain a better understanding of the phenomenon of the early effect associated with the slow passage problem in high-dimensional nonlinear dynamical systems by focusing on the phenomenon of resonance, which is ubiquitous in many physical systems [10]. Consider a dynamical system under external forcing and suppose that the forcing frequency, denoted ω_f, is an experimentally controllable bifurcation parameter. As ω_f changes through

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ω_n, a natural frequency of the unforced system, the system response is typically dramatically enhanced, e.g., in its oscillation amplitude. Because of its relative simplicity, the phenomenon of resonance serves as a prototypical class of systems to study, understand, and exploit the slow passage problem. In this regard, recently the system of a linear, periodically forced, damped pendulum [8] was investigated, revealing the early effect. Interestingly, the shift in the onset of resonance was found to depend on the initial parameter of the system, indicating a kind of *memory* effect. Here, to significantly broaden the scope of the investigation, we consider high-dimensional nonlinear systems with multiple resonances. In particular, we study fluid flow in an enclosed circular cylinder, subject to external periodic forcing. The top lid and sidewall of the cylinder are at rest, while the bottom lid is under harmonic modulation. To investigate the resonant phenomenon from the slow passage point of view, we assume that the forcing frequency varies slowly with time. The flow is thus effectively driven periodically at a slowly time-ramped frequency for a fixed modulation amplitude of the bottom lid. Due to the nonlinear nature of the system, there are multiple resonant frequencies. We find that the early effect occurs for each and every resonant frequency, where the resonance frequency emerges between its initial value and the static resonance frequency, regardless of the direction of the variation of the forcing frequency (i.e., increasing or decreasing). Based on these results, we uncover a simple parameter scaling law characterizing the early effect, which is generally applicable to all resonances. We anticipate that these findings will have considerable value for experimental bifurcation study of nonlinear dynamical systems in general.

In Sec. II, we describe the mathematical model of the static rotating fluid system and demonstrate the phenomenon of multiple resonances. In Sec. III, we numerically investigate the slow passage problem by adiabatically varying the frequency of the external harmonic modulation. In Sec. IV, we derive the onset condition for multiple resonances using the concept of instantaneous frequency. In Sec. V, we present conclusions.

II. A FLOW SYSTEM WITH MULTIPLE RESONANCES

We consider the system of a swirling fluid flow confined in a cylinder of radius R and height H. The bottom lid is driven externally to rotate at the angular rate $\Omega[1 + A\sin(\Omega_f t)]$, where t is time in seconds, Ω (in rad/s) is the mean rotation frequency, Ω_f (in rad/s) is the forcing frequency, and A is the forcing amplitude. Using R as the length scale and the dynamic time $1/\Omega$ as the time scale, the system can be nondimensionalized with four dimensionless parameters: (i) the Reynolds number $\mathrm{Re} = \Omega R^2/\nu$ (ν is the kinematic viscosity), (ii) the forcing amplitude A, (iii) the aspect ratio H/R, and (iv) the normalized forcing frequency $\omega_f = \Omega_f/\Omega$. The resulting Navier-Stokes equation is

$$(\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}, \tag{1}$$

where $\mathbf{u} = (u_r, u_\theta, u_z, t)$ is the velocity field in the cylindrical coordinates (r,θ,z), p is the kinematic pressure, and the flow is incompressible: $\nabla\cdot\mathbf{u} = 0$. The no-slip boundary conditions are

$$\mathbf{u}(1,\theta,z,t) = (0,0,0), \tag{2}$$

$$\mathbf{u}(r,\theta,H/R,t) = (0,0,0), \tag{3}$$

$$\mathbf{u}(r,\theta,0,t) = (0, r[1 + A\sin(\omega_f t)], 0). \tag{4}$$

To solve Eqs. (1)–(4) numerically, we employ a standard second-order time splitting method, in combination with a pseudospectral method for spatial discretization based on the Galerkin-Fourier expansion in θ and Chebyshev collocation in r and z. Specifically, we use $n_r = 64$ and $n_z = 96$ Chebyshev modes in the radial and axial directions, respectively, and $n_\theta = 24$ Fourier modes in the azimuthal direction. For accurate use of spectral techniques, a smooth regularization of the boundary condition singularity that occurs in the corners is provided. The time step in the numerical integration is set to be $\delta t = 0.02$. The numerical code was previously developed and validated to study turbulent solutions in the large-Re regime [11].

For $A = 0$ (no external forcing) and a moderate aspect ratio (e.g., $H/R = 2.5$), the stationary flow system exhibits several supercritical Hopf bifurcations as Re is increased, which occur for $\mathrm{Re}_i \approx 2710$, 3044, 3122 ($i = 1,2,3$), with the corresponding natural (Hopf) frequencies $\omega_i^H \approx 0.1692$, 0.1135, 0.2182. When a harmonic force is applied to the bottom lid of the system ($A \neq 0$), the resulting flow can be in a quasiperiodic state possessing both the forcing frequency and the natural frequency of the unforced limit cycle state. If the ratio between the former and the latter is rational, the phenomenon of Arnold tongues will occur, which is typical in quasiperiodic dynamical systems [2]. In this case, the system exhibits multiple resonances.

Numerically, we find a supercritical Hopf bifurcation with the frequency $\omega_0 \approx 0.17$ for $\mathrm{Re} = 2710$ for $H/R = 2.5$. Consider the static system for Re slightly smaller than the bifurcation point, e.g., $\mathrm{Re} = 2600$. Without forcing, the system is in a steady vortex breakdown state. To examine the effect of small harmonic forcing on the flow structure, we set $A = 0.01$, choose a set of systematically increasing values of the forcing frequency, and calculate the vorticity field. The 16 panels in Fig. 1 show the difference between the azimuthal vorticity $\eta(t)$ and η_0, the corresponding vorticity associated with the steady state for $A = 0$, for different values of ω_f. We observe alternation in the vorticity structure near the disk and sidewall boundary layers, particularly near the corner where the disk meets the sidewall. This is due to the formation of junction vortices between the stationary sidewall and the modulated rotating disk. Another feature is that the junction vortices propagate up along the sidewall, collide with each other at the axis near the top, and then combine to enhance the vortex breakdown recirculation, amplifying its pulsations. This dynamical feature appears more dramatic at the 1:1 resonance with $\omega_f \approx 12\omega_b$, where $\omega_b = 0.014\,544\,027\,777$. The resonant behavior also occurs for other frequencies, e.g., $\omega_f \approx 8\omega_b$ and $15\omega_b$, as shown in Fig. 1.

To characterize the multiple resonance behavior in a quantitative manner, we calculate the total kinetic energy $E(t)$

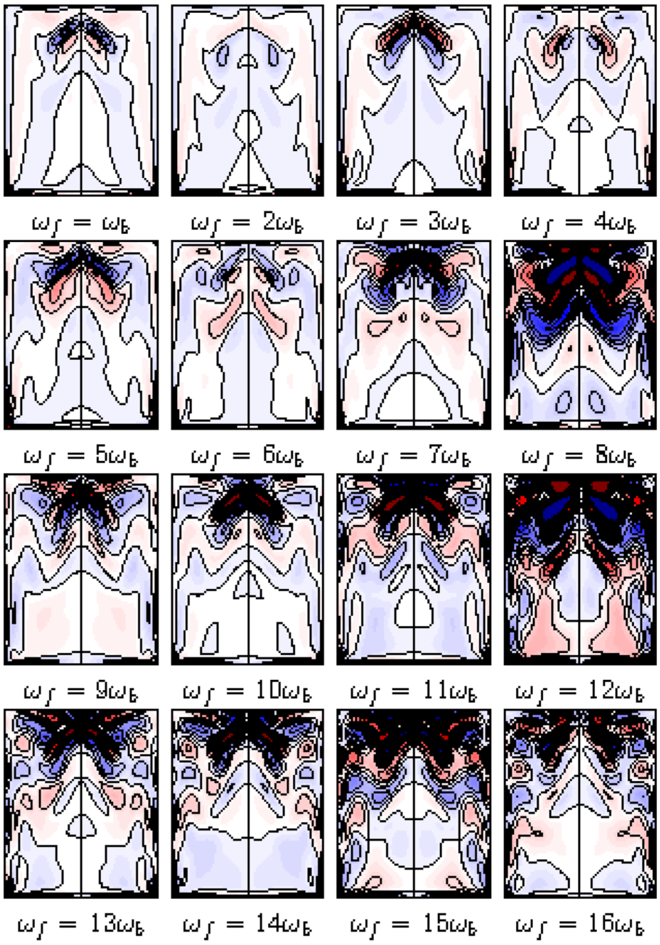

FIG. 1. (Color online) Structure of the azimuthal vorticity field in the stationary flow system. For $\mathrm{Re} = 2600$, $H/R = 2.5$, and $A = 0.01$, snapshots of the azimuthal vorticity modulation for various values of ω_f: $\omega_f = n\omega_b\,(n = 1,\ldots,16)$, where $\omega_b = 0.014\,544\,027\,777$. For $\omega_f = 8\omega_b, 12\omega_b$, or $15\omega_b$, the vorticity fields are relatively stronger.

of the flow:

$$E(t) = \frac{1}{2}\int_0^{H/R}\int_0^1 \left(u_r^2 + u_\theta^2 + u_z^2\right) r\,dr\,dz. \quad (5)$$

We can quantify the oscillation amplitude of the flow by calculating the peak-to-peak amplitude ΔE of the kinetic energy as a function of the forcing frequency ω_f. For convenience, we normalize ΔE by the kinetic energy E_0 associated with the unforced flow ($A = 0$) multiplied by the scaling factor $\sqrt{\omega_f}$. The result is shown in Fig. 2, where the quantity $\omega_f^{0.5}\Delta E/E_0$ is plotted versus the forcing frequency ω_f and the three vertical dotted lines indicate the Hopf frequencies of the first three modes bifurcated from the basic steady state for $\mathrm{Re} = 2600$. When the forcing frequency matches any of the Hopf frequencies ($\omega_f = \omega_i$), $\omega_f^{0.5}\Delta E/E_0$ exhibits a local maximum, signifying a resonance. The flow system described by Eqs. (1)–(4) under harmonic forcing can thus serve as a prototypical model for investigating the slow passage problem in high-dimensional nonlinear systems with multiple resonances.

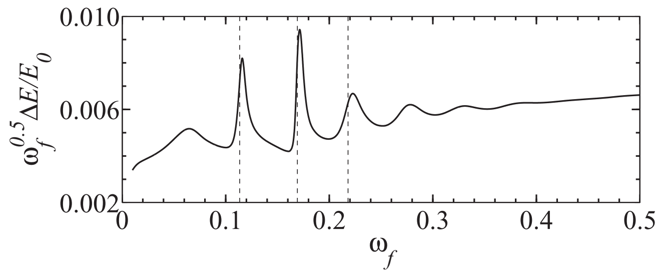

FIG. 2. Multiple resonances in the stationary driven flow system. Normalized peak-to-peak amplitude of the kinetic energy versus the forcing frequency ω_f for $A = 0.01$ in the stationary driven system, where each dashed line indicates a Hopf frequency ω_i ($i = 1,2,3$).

III. PHENOMENA ASSOCIATED WITH SLOW PASSAGE

To be concrete, we fix the Reynolds number ($Re = 2600$), the aspect ratio ($H/R = 2.5$), and the amplitude of the forcing frequency ($A = 0.01$). We vary the forcing frequency ω_f slowly according to

$$\omega_f(t) = \omega_0 + (\epsilon t)^P, \quad (6)$$

where ω_0 is an initial frequency, $P > 0$ is a power ramping exponent, and $\epsilon \ll 1$. For $P = 1$, the forcing frequency $\omega_f(t)$ is varied at a constant rate, but there is "acceleration" for $P > 1$ and "deceleration" for $P < 1$. The slow parameter variation $\omega_f(t)$ is to be incorporated into the boundary condition, Eq. (4). The initial velocity field is chosen to be that associated with a limit cycle under constant driving frequency ω_0. Our goal is to determine the dynamical response of the system as $\omega_f(t)$ passes through a series of internal (Hopf) frequencies ω_i.

A. Early effect

To demonstrate the early effect, we carry out bifurcation analysis in two ways: (i) for a stationary system for a set of systematically increasing parameter values and (2) for a time-dependent system with parameter sweeping as in Eq. (6). In both cases, we calculate the normalized, peak-to-peak kinetic energy $\omega_f^{0.5}\Delta E/E_0$ and plot it versus the driving frequency ω_f, as shown in Figs. 3(a)–3(c) for three values of the frequency ramping power P. In all cases, we observe that, in the slow passage system, the resonances (as indicated by the occurrence of local maxima in the energy) occur "earlier" than those in the corresponding stationary system with respect to the bifurcation parameter ω_f. This is thus clear evidence that the early effect can occur in nonlinear systems with multiple resonances.

Let ω_J be the resonant frequency (or the jump frequency [8]) in the slow passage system. For $P = 1$ (linear frequency ramping), we see from Fig. 3(a) that each resonant frequency ω_{J_i} occurs at the middle between the initial frequency ω_0 and the resonant frequency ω_i in the stationary system,

$$\omega_{J_i} = \frac{\omega_0 + \omega_i}{2}, \quad i = 1,2,3, \quad (7)$$

or $\omega_{J_k} - \omega_{J_i} = (\omega_k - \omega_i)/2$. For decelerated and accelerated frequency ramping, as shown in Figs. 3(b) and 3(c),

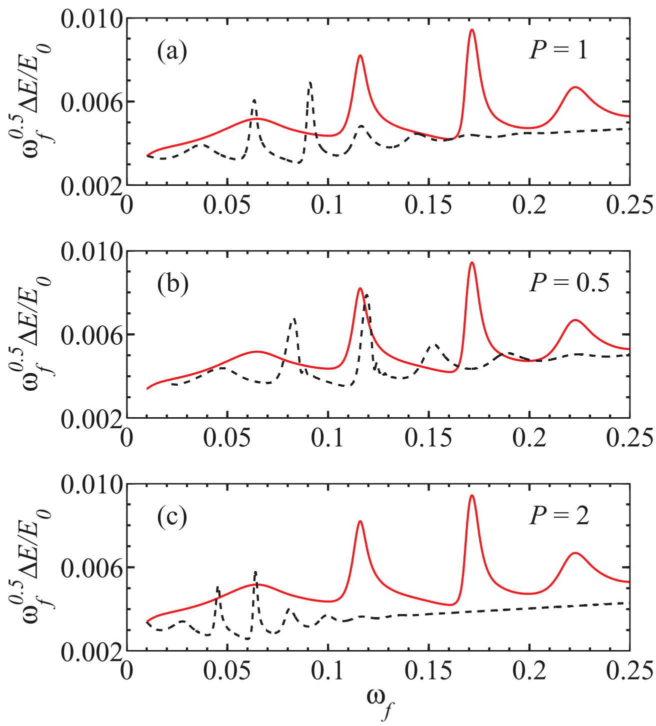

FIG. 3. (Color online) Demonstration of the early effect. Normalized kinetic energy $\omega_f^{0.5}\Delta E/E_0$ as a function of the forcing frequency for two cases: (1) from static (conventional) bifurcation analysis [solid (red) curve), as in Fig. 2, and (2) when the forcing frequency is time dependent as given in Eq. (6) (dashed black curve). Other parameters are $\epsilon = 10^{-6}$, $\omega_0 = 0.01$, and $A = 0.01$. (a–c) Three values of the power exponent P in the parameter ramping. In all three cases, the resonances occur "earlier" than those in the static, time-independent system as the driving frequency is increased from ω_0. For linear parameter variation (a), the resonances occur at the midfrequencies between the initial frequency ω_0 and the corresponding Hopf frequencies ω_i. For slowly decelerated ramping [(b); $P = 0.5$], the early effect is not as pronounced as in the case of a constant ramping rate. For slowly accelerating ramping [(c); $P = 2$], the resonances occur even earlier than in the constant ramping case.

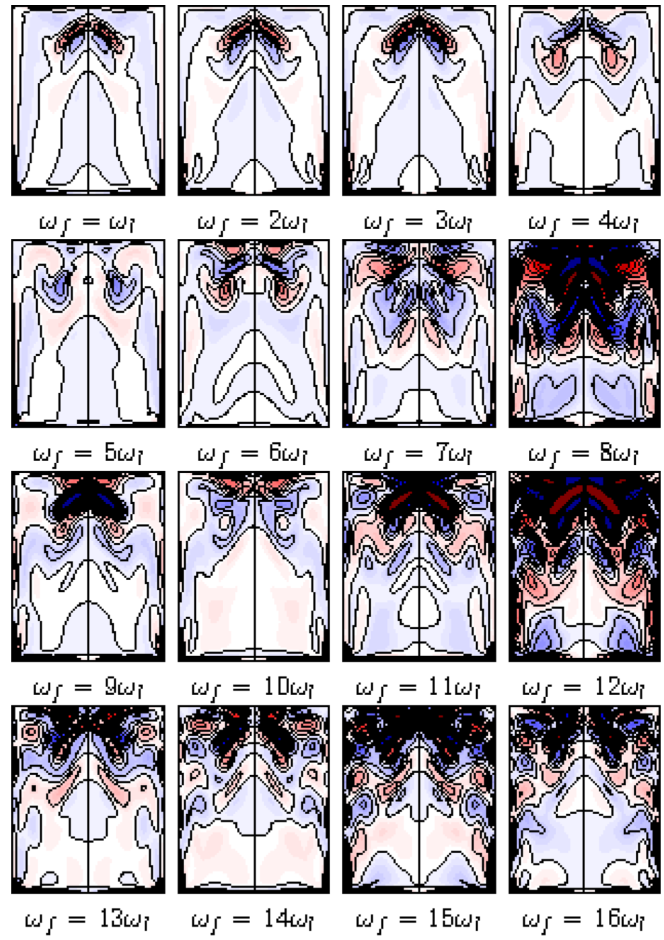

FIG. 4. (Color online) Vorticity field under linear parameter ramping. For $P = 1$ [$\omega_f(t) = \omega_0 + \epsilon t$], snapshots of the azimuthal vorticity field for various values of ω_f, where $\omega_l = \omega_b/2$. Relatively strong vorticity fields are observed for $\omega_f(t) = 8\omega_b$, $12\omega_b$, or $15\omega_b$, as in the corresponding stationary system (cf. Fig. 1).

respectively, we have

$$\omega_0 < \omega_{J_i}^{P_1} < \omega_{J_i}^{1} < \omega_{J_i}^{P_2} < \omega_i,$$

where $0 < P_2 < 1 < P_1$. In the slow passage system, the resonant frequency thus depends on the parameter ramping power: $\omega_J = \omega_J(P)$. In Sec. IV, we obtain

$$\omega_{J_i}(P) = \frac{\omega_i + P\omega_0}{(P+1)} \quad \text{as} \quad \epsilon \to 0. \tag{8}$$

Note that, for $P \to 0$ so that the slow passage system approaches its stationary counterpart, we have $\omega_{J_i}(P) \to \omega_i$.

B. Vorticity field

A key question is whether the dynamical properties of the flow under time-dependent parameter ramping reflect those of the original stationary system. An affirmative answer would provide justification for experimental investigation of bifurcations by sweeping a parameter slowly in a time-dependent fashion. While our computation indicates that, under slow parameter ramping, multiple resonances still occur and their number is preserved, it is not apparent that the dynamical properties of the original stationary system are unchanged when measurements are taken on a time scale faster than that of the adiabatic parameter change. To address this issue, we examine the structure of some key physical quantity at different time instants, each corresponding to a particular value of the driving frequency that is approximately constant in the time scale of measurement. To be concrete, we calculate the snapshots of the vorticity field.

Figures 4, 5, and 6 show, for $P = 1$ (linear ramping), $P = 1/2$ (decelerated ramping), and $P = 2$ (accelerated ramping), respectively, 16 snapshots of the azimuthal vorticity distribution, each corresponding to a specific value of the driving frequency. Due to the early effect, the increment $\delta\omega$ in the driving frequency from one snapshot to the next needs to be adjusted compared with the case of a stationary system (cf. Fig. 1), and the amount of adjustment depends on P. In particular, for Figs. 4, 5, and 6, we have $\delta\omega \equiv \omega_l = \omega_b/2$, $\omega_d = 2\omega_b/3$, and $\omega_a = \omega_b/3$, respectively. Comparing Figs. 4–6 with Fig. 1, we find that the dynamical structure

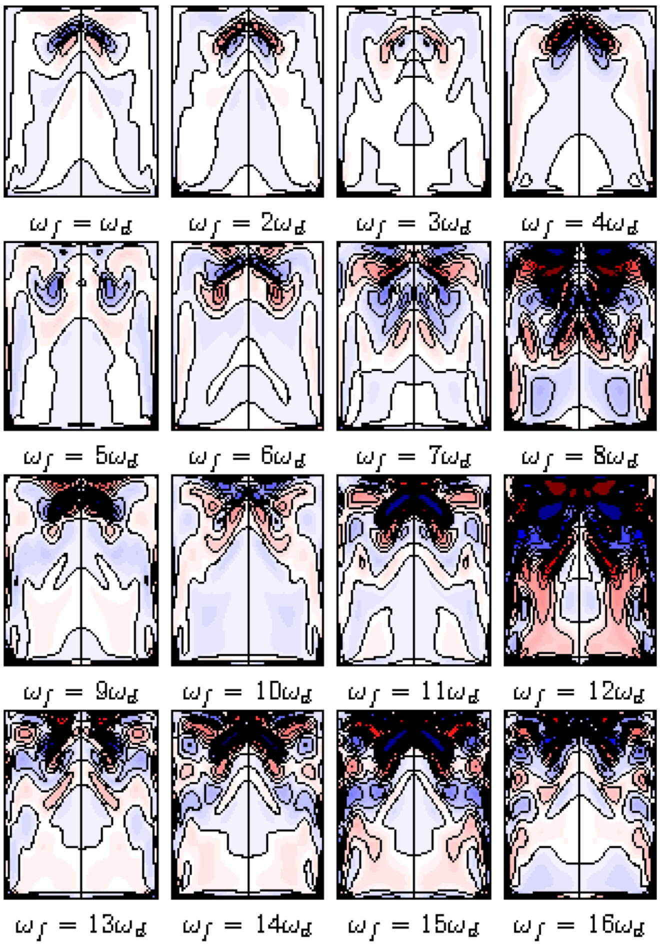

FIG. 5. (Color online) Vorticity field under decelerated parameter ramping. For $P = 1/2$, snapshots of the azimuthal vorticity field for various values of ω_f at different instants in time, where $\omega_d = 2\omega_b/3$. Relatively strong vorticity fields are observed for $\omega_f(t) = 8\omega_b$, $12\omega_b$, or $15\omega_b$, as in the corresponding stationary system (cf. Fig. 1).

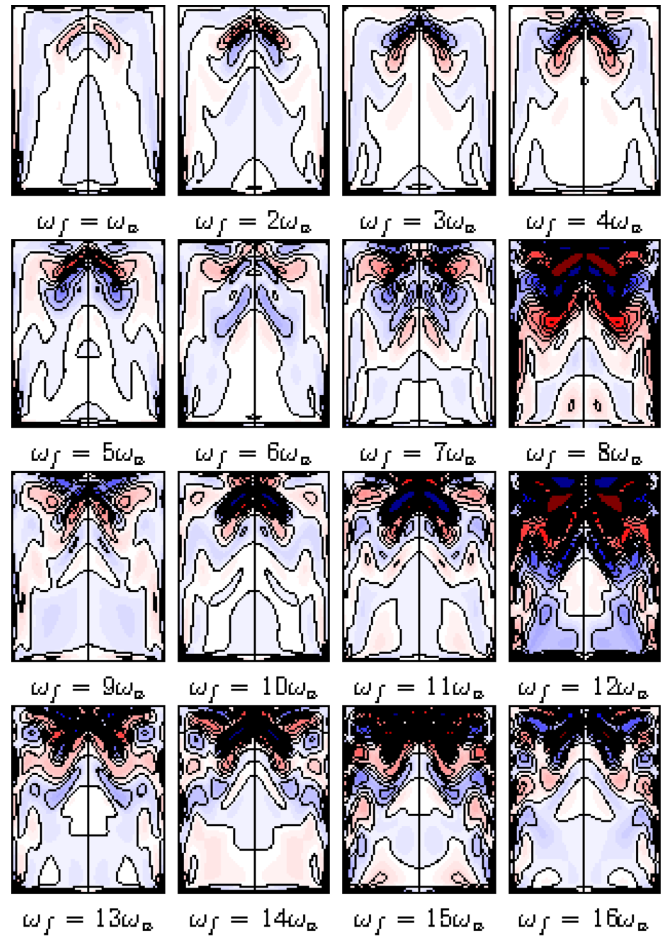

FIG. 6. (Color online) Vorticity field under accelerated parameter ramping. For $P = 2$, snapshots of the azimuthal vorticity field for various values of ω_f at different instants in time, where $\omega_a = \omega_b/3$. Relatively strong vorticity fields are observed for $\omega_f(t) = 8\omega_b$, $12\omega_b$, or $15\omega_b$, as in the corresponding stationary system (cf. Fig. 1).

of the azimuthal vorticity field remains characteristically unchanged under adiabatic parameter change. Examination of other physical quantities, such as different vorticity components and the velocity field, reveals essentially the same phenomenon, thereby justifying the use of the slow passage method to probe the bifurcation structure of the underlying system.

IV. ONSET OF RESONANCES UNDER ADIABATIC PARAMETER MODULATION

We aim to obtain the onset conditions under which multiple resonances emerge in the presence of adiabatic parameter sweeping. For linear dynamical systems under such parameter variations, there is a single resonance and its onset can be analytically understood through explicit determination of the trajectory in the phase space [8]. This approach, however, is not applicable to nonlinear systems with multiple resonances, especially in high dimensions. We resort to the concept of *instantaneous frequency*, commonly used in signal processing [12], to obtain a heuristic understanding of the emergence of multiple resonances.

Given the time-dependent frequency modulation $\omega_f(t) = \omega_0 + (\epsilon t)^P$, the *instantaneous phase* ϕ of the forcing is [12]

$$\phi(t) = \omega_f(t)t - \frac{\pi}{2} = \omega_0 t + (\epsilon t)^P t - \frac{\pi}{2}, \tag{9}$$

the instantaneous frequency is given by

$$\mathrm{IF}(t) = \frac{d\phi(t)}{dt} = \omega_0 + (P+1)(\epsilon t)^P, \tag{10}$$

where ω_0 is chosen to be smaller than the minimal resonant frequency. Let ω_{n_i} be the ith natural frequency in the stationary system. Then a resonance occurs if $\omega_{n_i} = \omega_i$. If $\mathrm{IF}(t_i) = \omega_{n_i}$, a resonance occurs at time t_i, which can be determined through Eq. (10) as

$$t_i = \sqrt[P]{\frac{\omega_{n_i} - \omega_0}{(P+1)\epsilon^P}}. \tag{11}$$

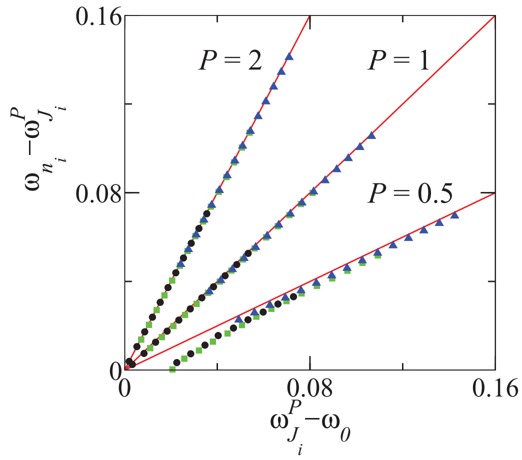

FIG. 7. (Color online) Verification of Eq. (12). Numerically obtained relation between $(\omega_{n_i} - \omega_{J_i}^P)$ and $(\omega_{J_i}^P - \omega_0)$ for $P = 1/2$, 1, and 2 and fixed $\epsilon = 10^{-6}$.

Substituting this expression of t_i into Eq. (6), we obtain the following formula for the frequency of the ith resonance:

$$\omega_{J_i}^P = \omega_f(t_i) = \frac{\omega_{n_i} + P\omega_0}{(P+1)} \quad \text{as} \quad \epsilon \to 0, \tag{12}$$

where $\omega_{J_i}^P = \omega_{n_i}$ for $P = 0$. For $P > 0$, ω_J^P depends on the initial frequency ω_0. We have $\omega_0 < \omega_J^P < \omega_{n_i}$, indicating an early effect. Equation (12) can be rewritten as

$$\left(\omega_{n_i} - \omega_{J_i}^P\right) = P\left(\omega_{J_i}^P - \omega_0\right). \tag{13}$$

Numerical verification of this relation is presented in Fig. 7.

We provide further support for the use of the instantaneous frequency. Note that Eq. (10) can be rewritten as $\text{IF}(t) = (P+1)\omega_f(t) - P\omega_0$. Thus, we can replot Figs. 3(a)–3(c), the normalized kinetic energy versus the driving frequency, by using the instantaneous frequency instead. In particular, the kinetic energy can be normalized through the instantaneous frequency as $(\text{IF})^{0.5}\Delta E/E_0$ and plotted versus this frequency, as shown in Fig. 8. The remarkable phenomenon is that all curves for different values of P collapse into a single one: the energy versus the driving frequency in the original stationary system as in Fig. 2. This means that, by using the instantaneous frequency, a full picture of the bifurcation behavior of the stationary system can be captured by one time-dependent sweeping of a relevant parameter, which is experimentally desirable.

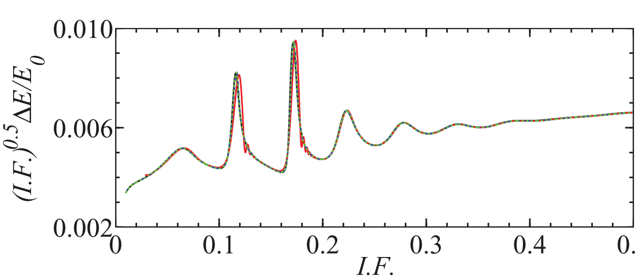

FIG. 8. (Color online) Kinetic energy normalized by instantaneous frequency. Replotting of Figs. 3(a)–3(c) by replacing the driving frequency with the instantaneous frequency. Red, blue, and green curves correspond to $P = 1/2$, 1, and 2, respectively. The black curve is the same energy-versus-frequency curve as in Fig. 2 for the original stationary flow system. All curves match each other, validating the use of the instantaneous frequency in the slow passage system to uncover the resonances in the original stationary system.

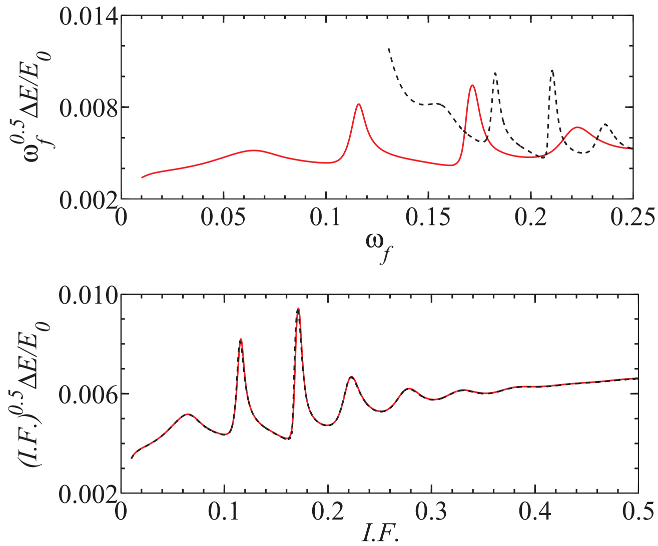

FIG. 9. (Color online) Early effect with backward parameter sweeping. For $\epsilon = 10^{-6}$, the properly normalized kinetic energy versus the driving frequency (upper panel) and the instantaneous frequency (lower panel) for backward parameter sweeping, where the driving frequency is decreased adiabatically from some initial value above all possible resonant frequencies of the stationary system, i.e., $\omega_0 = 0.25 > \omega_{n_i}$ $(i = 1,2,3)$. We observe essentially the same phenomena as for the forward parameter sweeping case.

So far we have discussed the setting where the driving frequency is adiabatically increased from some initial value well below all possible resonant frequencies of the stationary system. The time-dependent parameter sweeping can also be carried out in the opposite direction; i.e., we can adiabatically decrease the frequency from some initial value that is above all intrinsic resonant frequencies,

$$\omega_f(t) = \omega_0 - (\epsilon t)^P \quad \text{with} \quad \omega_n < \omega_0. \tag{14}$$

As shown in Fig. 9, this "back-sweeping" of the driving frequency results in essentially the same early effect, so it is equally effective for obtaining a detailed picture of the bifurcations of the system.

V. CONCLUSION

In experimental studies of nonlinear dynamical systems, it is desirable to use a single parameter sweep to detect the possible bifurcations. This can be done by letting the bifurcation parameter vary with time adiabatically, effectively producing a time-dependent, nonstationary dynamical system. Can the true bifurcations be correctly detected? and Would the dynamical properties of the system, especially those at the bifurcation points, be affected? To address these experimentally relevant questions, we study a closed, externally driven fluid flow system that exhibits multiple resonances as the driving frequency is varied. We find that the early effect, originally discovered in driven linear systems with a single resonance [8], also occurs in nonlinear systems. In particular, as the driving frequency is increased from an initial value below which there are no resonances, a particular resonance occurs for a frequency value that is lower than that for the same resonance to occur in the original time-independent, stationary system. Further, by examining the spatial distributions of physical quantities, e.g., the vorticity field, at different instants in time (corresponding to different values of the driving frequency), we find that the dynamical properties of the original system are preserved under adiabatic parameter variations. For the representative setting of power ramping of the bifurcation parameter, our analysis and numerical computations reveal a simple relation between the bifurcation points in the slow passage and stationary systems, as summarized schematically in Fig. 10. Moreover, we have shown that "forward sweeping" and "back sweeping" of the driving frequency result in essentially the *same* early effect, which provides an effective opportunity to obtain a detailed picture of the bifurcations of the system. Our findings not only establish a foundation for experimental detection of bifurcations in nonlinear dynamical systems through the method of time-dependent parameter sweeping, but also provide insights into the fundamental correspondence between stationary and nonstationary systems.

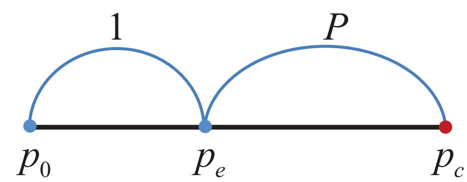

FIG. 10. (Color online) Schematic of the early effect. For a nonlinear dynamical system with adiabatic parameter variations in the form $p(t) = p_0 + (\epsilon t)^P$, where p_0 is the initial parameter value and $\epsilon \ll 1$, onset of a bifurcation occurs at $p_e < p_c$, where p_c is the corresponding bifurcation point in the original stationary system. Quantitatively, we have $(p_e - p_0)/(p_c - p_e) = 1/P$ for $\epsilon \to 0$. A similar relation holds for reverse parameter ramping: $p(t) = p_0 - (\epsilon t)^P$, where $p_0 > p_c$.

ACKNOWLEDGMENT

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant No. NRF-2013R1A1A2010067). Y.-C.L. was supported by AFOSR under Grant No. FA9550-12-1-0095. G.W.L. was supported by the Korea Institute of Construction Technology (project name: Development of a Micro Raingauge Using Electromagnetic Wave).

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