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Origin of chaotic transients in excitatory pulse-coupled networks

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We develop an approach to understanding long chaotic transients in networks of excitatory pulse-coupled oscillators. Our idea is to identify a class of attractors, sequentially active firing (SAF) attractors, in terms of the temporal event structure of firing and receipt of pulses. Then all attractors can be classified into two groups: SAF attractors and non-SAF attractors. We establish that long transients typically arise in the transitional region of the parameter space where the SAF attractors are collectively destabilized. Bifurcation behavior of the SAF attractors is analyzed to provide a detailed understanding of the long irregular transients. Although demonstrated using pulse-coupled oscillator networks, our general methodology may be useful in understanding the origin of transient chaos in other types of networked systems, an extremely challenging problem in nonlinear dynamics and complex systems.

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I. INTRODUCTION

Irregular or chaotic transients are ubiquitous in nonlinear systems, which physically lead to interesting phenomena such as fractal basin boundaries, chaotic scattering, noise-induced chaos, etc. [1]. Current theory of transient chaos is mostly built upon low-dimensional dynamical systems, where the research focus has been on various dynamical properties of the underlying nonattracting chaotic sets, for example, chaotic saddles. Chaotic transient dynamics in high-dimensional dynamical systems [2–6], especially in networked systems of significant physical and biological interest [7–12], has remained to be an active area of research [1].

The occurrence of chaotic transients can be explained by the existence of nonattracting chaotic sets. The system can stay in the vicinity of such a chaotic set for a long time before settling into an attractor. A nonattracting chaotic set can be created through the destabilization of a chaotic attractor, as in a crisis [13]. In networks of pulse-coupled oscillators with excitatory coupling, chaotic transients were reported [14]. In this type of systems, an oscillator fires when reaching a threshold and its state is reset to a lower value. During this process, a pulse is sent out. The pulse is received by some other oscillators if they have incoming links from the oscillator. The receipt of the pulse makes the states of the corresponding oscillators closer to the threshold for excitatory couplings. Such networks have been used to study and understand the collective dynamics of a host of biological systems, such as neural systems, flashing fireflies, and cardiac pacemaker cells [15–17]. The irregular transients observed in the excitatory pulse-coupled networks are chaotic in the sense that they are sensitive to perturbations, which are mainly due to the excitatory couplings giving rise to the local expanding dynamics [14]. To understand the origin of these irregular transients in networked dynamical systems is extremely challenging, due to nonlinear nodal dynamics and complex interactions among the nodes. In this regard, the theory of low-dimensional dynamical systems has established that the origin of chaotic transients is due to nonattracting chaotic sets in the phase space, and special numerical techniques have been developed to calculate such chaotic sets [1]. However, it is difficult to apply these methods to networked dynamical systems that are typically very high-dimensional. This difficulty to explore the origin of chaotic transients is hindered by the fact that such a system can typically possess a large number of coexisting attractors. Nonetheless, for networks of excitatory pulse-coupled oscillators, there are dynamical properties that can be exploited to probe chaotic transients.

A key step in our analysis is a scheme that we articulate to characterize the attractors of the network system by using temporal events. In particular, we uncover the existence of a certain type of attractors: *sequentially active firing* (SAF, to be defined below) attractors. Generally, all possible attractors of the system can then be classified into two types: SAF and non-SAF attractors. The central result of this paper is that long chaotic transients occur in the parameter region of transition between the situations where the SAF and non-SAF attractors dominate, respectively. The chaotic transients are associated with the collective destabilization of SAF attractors. In the transitional region, the basins of SAF and non-SAF attractors are highly mixed. We show that the destabilization of SAF attractors tends to induce long irregular transients because the states of many oscillators are close to the critical value. To our knowledge, collective destabilization is a new mechanism for generating long and chaotic irregular transients in networked dynamical systems.

The organization of this paper is as follows. In Sec. II, we describe the network model and introduce our scheme to classify attractors in terms of temporal sequence of events. In Sec. III, we demonstrate that chaotic transients typically occur in the transitional region. In Sec. IV, we analyze the detailed bifurcation processes for SAF attractors and their contribution to long chaotic transients. Conclusions are drawn in Sec. V.

II. CLASSIFICATION OF ATTRACTORS IN EXCITATORY PULSE-COUPLED NETWORKS

A. Model description

We consider a system of oscillators interacting with each other by sending and receiving pulses. Specifically, we consider random directed networks of size N with M directed links, where self-links and multiple connections are excluded. The maximum number of links is $N(N-1)$ for directed networks without self-links. The density of links $p = M/[N(N-1)]$ can be used to characterize the network. The dynamics of an individual oscillator is given by the Mirollo-Strogatz model, which is equivalent to some analytically solvable, one-dimensional neuron models such as the leaky integrate-and-fire model [15]. The state of oscillator i is described by a phase variable $\phi_i(t)$ that satisfies

$$d\phi_i/dt = 1. \tag{1}$$

Upon reaching the threshold $\Theta = 1$, the oscillator fires a pulse and its phase $\phi_i(t)$ is reset to zero: $\phi_i(t^+) = 0$. The pulse will be received by the oscillators that have incoming links from i after a time delay τ. The associated coupling strength for the pulse from i to j is $\varepsilon_{ji} \equiv \varepsilon/k_j$, which is normalized according to the number of incoming links of node j (or in-degree of node j). The excitatory couplings $\varepsilon_{ji} > 0$ can drive the state of the corresponding oscillator close to the threshold. Specifically, a pulse from oscillator i received by oscillator j at time t will induce a phase jump in $\phi_j(t)$ according to the following rule:

$$\phi_j(t^+) = \min[U^{-1}(U(\phi_j(t)) + \varepsilon_{ji}),1], \tag{2}$$

where the function U mediating the phase jump is twice continuously differentiable, monotonically increasing, concave, and normalized [$U(0) = 0$ and $U(1) = 1$]. To be concrete, we follow Ref. [15] and choose $U(\phi) = b^{-1}\ln[1 + (e^b - 1)\phi]$, where $b > 0$. Throughout this paper, b is fixed to be 1.0. It is useful to define the transfer function H,

$$H(\phi_t,\varepsilon') = U^{-1}(U(\phi_t) + \varepsilon'), \tag{3}$$

which characterizes the response of an oscillator of phase ϕ_t to the incoming pulses of total strength ε' in the case of subthreshold input: $U(\phi_t) + \varepsilon' < 1$. We note that the so-described pulse-initiated interaction mechanism with time delay models a range of real-world biological phenomena [15,16].

B. Classification of attractors

The interactions among oscillators are mediated by the events of sending and receiving pulses. These events are key to the emergence and evolution of collective dynamics on the network. Our idea is then to analyze these events to understand transients. For convenience, we introduce some notations. Let S_i denote the spiking or firing of the oscillator i from which a pulse is generated and R_j denote the event that a pulse from oscillator j is received by others. Events occurring at two different times are separated by the minus sign "$-$". Therefore, a sequence of events can be recorded for a given time interval. For example, a segment of an event sequence is "$\cdots - R_6 - R_1S_3S_5 - S_2 - \cdots$" showing the events at three different times. The first event R_6 denotes that a pulse from oscillator 6 is received. The next events $R_1S_3S_5$ indicate that oscillators 3 and 5 are firing due to the receipt of pulse from oscillator 1 directly. This also implies that each of oscillators 3 and 5 has an incoming link from oscillator 1. The third event S_2 denotes the firing of oscillator 2.

The firing events can be further classified into two types: *passive* and *active* [18]. Specifically, for any given oscillator, if all the incoming pulses *directly* drive its phase to the threshold, the firing is passive; otherwise, it is active. The main difference between active and passive firings is their different responses to perturbations. Perturbations on the phase of an oscillator immediately preceding passive firing will disappear after the firing. In contrast, a small perturbation introduced right before an active-firing event will change the firing time, albeit slightly. In this sense, the effect of the perturbation survives through the firing event, so active firing is capable of spreading fast perturbations. Due to this dynamical behavior, we highlight the active firing events by capital letter $\mathbf{S}$.

A key to understanding the origin of the chaotic transients is to explore the behaviors and organization of the attractors of the coupled dynamics on networks. However, the large number of attractors makes it intractable to analyze them individually. Our idea is to classify the attractors according to the underlying temporal structures of the firing events. There are two situations when a firing event occurs, for example, S_i for oscillator i. The first one is that all the generated pulses have been received before the occurrence of S_i. The second is that some generated pulses have not been received yet. Generally, the firings (including passive and active firings), which occur after the receipt of all the generated pulses, are *sequential firings*. Of particular importance are active firing events, which are capable of spreading perturbations. Similarly, an active firing is called *sequential active firing* (SAF) if it occurs after all the generated pulses have been received. If all the active firing events associated with an attractor are SAF events, it is called an *SAF attractor*. Otherwise, it is a non-SAF attractor.

Based on the definition of SAF attractors, we have developed a numerical procedure to verify whether an attractor is an SAF or a non-SAF attractor. First we locate the attractor with a random initial condition. Then we analyze the firing events associated with this attractor. Special attention is paid to the firings that are not caused by the receipt of pulses directly, that is, active firings. If all the generated pulses have been received just before each of the active firings, the attractor will be an SAF attractor. In other cases where some generated pulses arrive later after an active firing or there are no active firings, the attractor is of the non-SAF type.

For SAF attractors, the minimal time difference between two successive receiving events is the delay time τ; for non-SAF attractors, this constrain does not apply. To give a concrete example, we consider a network of six oscillators. The attractors usually are period-one attractors in the return map; that is, each oscillator reaches the threshold once during one period of time. Here the return map governs the system evolution between two successive resettings of the reference oscillator, for example, oscillator 1. Thus, a period-one SAF attractor contains only one SAF event. For parameters $p = 0.6$ (link density), $\varepsilon = 0.1$, and $\tau = 0.15$, we observe the sequence of events associated with an SAF attractor:

$$R_1S_4S_5 - R_4R_5S_3S_6 - R_3R_6S_2 - R_2 - \mathbf{S}_1. \tag{4}$$

Immediately preceding the active firing ($\mathbf{S}_1$) of oscillator 1, all the generated pulses have been received. Hence, the corresponding attractor is an SAF attractor. An event sequence associated with a non-SAF attractor is

$$R_2 - R_1S_5 - R_5S_3 - R_3S_4 - R_4S_6 - R_6S_2 - \mathbf{S}_1. \quad (5)$$

Note that the generated pulse from oscillator 2 is not received before the active firing of oscillator 1.

In the parameter region where τ and ε are small, SAF attractors dominate in the sense that a random initial condition leads to one such attractor with high probability. This can be qualitatively understood, as follows. Consider period-one attractors in the return map, whose basins typically dominate in the phase space in a wide parameter region. The events associated with period-one attractors typically occur at a small number of different times. The total phase change is due to (1) phase jump caused by receiving pulses, that is, $\Delta\phi = H(\phi_t,\varepsilon') - \phi_t$, and (2) free evolution according to $d\phi_i/dt = 1$. The first term is small for small coupling strength. For non-SAF attractors, the time delay τ is the upper bound of the time duration of two successive receiving events, so the amount of phase change due to free evolution is relatively small. Therefore period-one attractors are typically not non-SAF attractors when τ and ε are small. The time duration between an SAF event and the last receiving event is $1 - \phi$, where ϕ is the phase of the SAF oscillator at that receiving event. Thus, this time duration can be relatively large, so SAF attractors tend to dominate the phase space for small τ and ε values. As ε and τ are increased through some critical region, SAF attractors become rare. In fact, in the parameter plane (ε,τ), there exists a *transitional region* where the fraction of SAF attractors decreases from near unity to near zero, which is shown in detail in Sec. III.

We can observe long chaotic transients preceding the network's settling into some periodic attractors, as shown in Fig. 1(a) for a network of size $N = 18$ and link density $p = 0.6$. The firing times when oscillators reach the threshold are recorded, as shown in Fig. 1(b). The active and passive firings are denoted in green and red, respectively. During the transient, oscillators undergo rapid switches between passive and active firings.

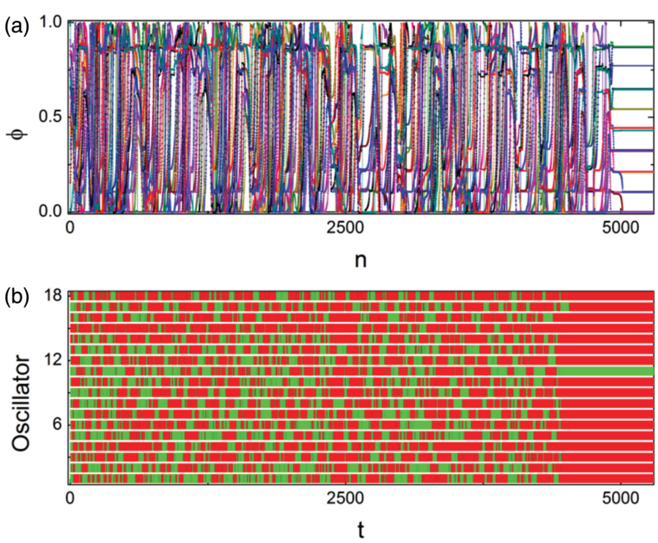

FIG. 1. (Color online) (a) A typical long chaotic transient trajectory recorded at the nth resetting of the reference oscillator 1 for a network of $N = 18$ oscillators. Simulation parameters are $\varepsilon = 0.1$, $\tau = 0.105$, and link density $p = 0.6$. (b) Corresponding times for active and passive firings for each oscillator, shown in green and red, respectively.

III. CHAOTIC IRREGULAR TRANSIENTS IN THE TRANSITIONAL REGION

Our goal is to find where chaotic transients occur in the two-dimensional parameter space defined by (τ,ε). For a period-one attractor, due to the excitatory couplings, the period is always smaller than the intrinsic period without couplings, which is unity. We focus on three quantities: the fraction of basins occupied by period-one attractors f_{p1}, the average transient time $\langle T\rangle$, and the fraction f_{SAF} of initial points leading to SAF attractors.

Figures 2(a)–2(c) show, for $N = 18$ and $p = 0.6$, the dependence of f_{p1}, f_{SAF}, and $\langle T\rangle$ on the parameters, respectively. In Fig. 2(a), f_{p1} is near unity in a wide parameter region, implying that the phase space is mostly occupied by the basins of period-one attractors. A sharp transition where f_{SAF} decrease from 1 to 0 abruptly can be seen from Fig. 2(b), where the transition appears to occur in the parameter region where period-one attractors are dominant. On both sides of the transitional region, the attractors are period-one attractors, but with different types of event structures. Figure 2(c) shows that the long chaotic transients occur in the transitional region. Similar results have been obtained for higher link density, for example, $p = 0.8$, as shown in Figs. 2(d)–2(f), where unstable attractors can appear [18,19]. We find that the unstable attractors mostly appear in the SAF parameter region and hence are mostly SAF attractors. In both cases, there is a strong coincidence between the region with long transients and the transitional region. The occurrence of this correspondence does not appear to depend on the link density or the network size, as shown in Figs. 3(a)–3(c) for $p = 0.4$ and $N = 24$, 30, and 36, respectively, where the results are obtained by averaging over 2×10^4 random initial conditions.

Can the appearance of long chaotic transients be explained by the variation in the number of attractors? Reference [14] conjectures that the appearance of chaotic transients is probably due to the small number attractors left in the phase space after the destabilization of many attractors, for example, unstable attractors. Generally, there are a large number of attractors, as shown in Fig.4. For $N = 24$ and $N = 30$, we can observe several significant changes in the number of attractors near some parameters. However, the average transient time does not change appreciably. In addition, near the parameter region where long chaotic transients occur, the number of attractors does not change appreciably either. For $N = 36$, the number of attractors becomes so large that the number of random initial conditions used is not sufficient to detect any significant changes. Based on these findings, we conclude that variation in the number of attractors cannot explain the appearance of long transients. As a matter of fact, near the transitional region, many SAF attractors are destroyed but many non-SAF attractors are created.

We then focus on the dynamical properties of the phase space near the transitional region, where significant numbers of SAF and non-SAF attractors coexist. Naturally, one can divide the phase space into two parts: the basins of SAF and of

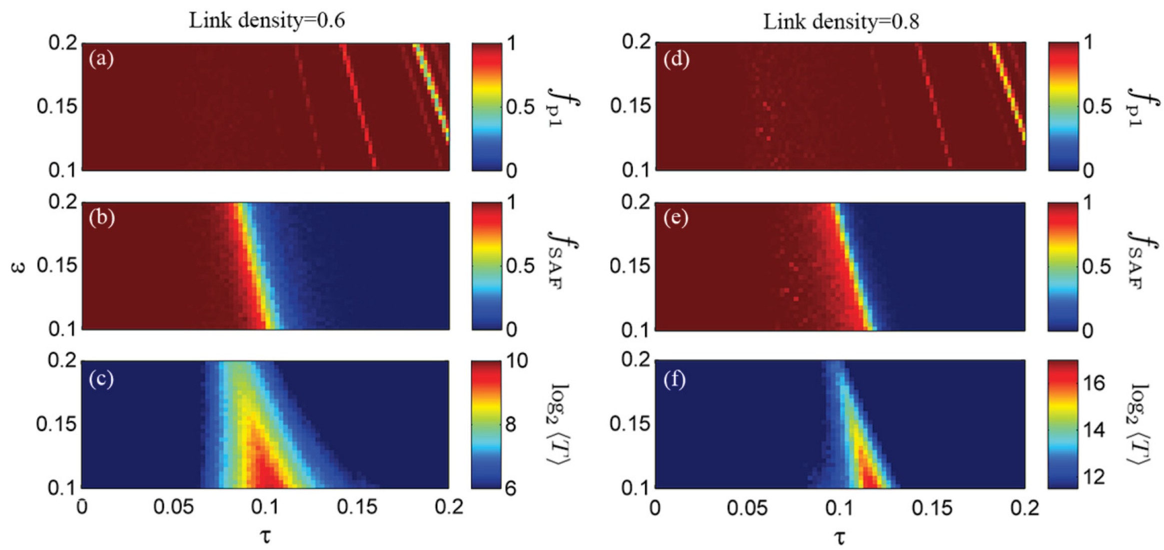

FIG. 2. (Color online) Dynamical behaviors in the parameter plane (τ,ε) for two networks of size $N = 18$ and link density $p = 0.6$ and 0.8, respectively. Panels (a)–(c) are for $p = 0.6$, and (d)–(f) are for $p = 0.8$. Panels (a) and (d) show the dependence of the fraction of period-one attractors f_{p1} on parameters, panels (b) and (e) show the fraction of SAF attractors, f_{SAF}, and panels (c) and (f) show the dependence of average transient time $\langle T\rangle$ (on a logarithmic scale) on parameters. Long transients occur near the transitional region where f_{SAF} changes from near unity to near zero, but occur in the region where the phase space is dominated by periodic attractors of low periods. The results are obtained through ensemble average of 500 initial conditions.

non-SAF attractors. Long chaotic transients are likely when the basins are mixed in a complicated manner, because a trajectory will undergo many "zigzag" type of paths before settling into the final attractor. The degree of basin mixing can be conveniently characterized by the uncertainty exponent [20]. Specifically, consider two nearby initial conditions of phase-space distance ϵ apart. If the two resulting trajectories approach the same type of attractors (i.e., SAF or non-SAF), the initial conditions are called "certain" with respect to perturbation ϵ; otherwise, they are "uncertain." The fraction of uncertain initial conditions $f(\epsilon)$ scales with ϵ algebraically: $f(\epsilon) \sim \epsilon^{\alpha}$, where $0 \leqslant \alpha \leqslant 1$ is the uncertainty exponent [20]. The closer the value of α is to zero, the stronger the degree of mixing of the two types of basins. An example of the scaling law of $f(\epsilon)$ is shown in Fig. 5(a) for the parameter setting of $N = 18$, $p = 0.6$, $\varepsilon = 0.1$, and $\tau = 0.105$, where 5×10^4 random initial conditions are used to calculate $f(\epsilon)$ for each value of ϵ. We obtain $\alpha \approx 0.0707$, indicating an extremely interwoven structure of the two types of basins. A visual illustration of the basin structure in a two-dimensional cross section (ϕ_2,ϕ_3) is shown in Fig. 5(b).

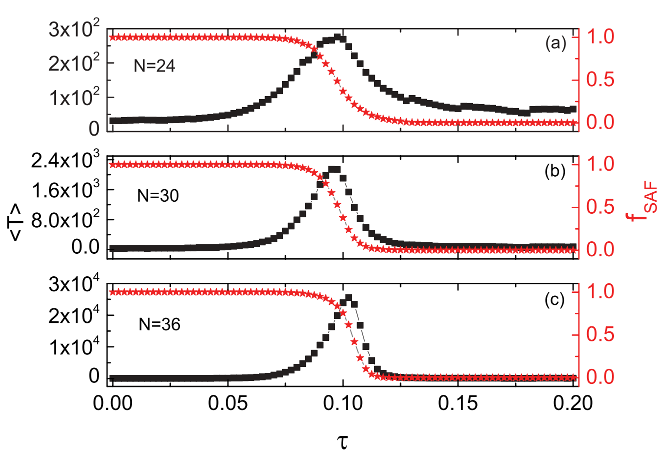

FIG. 3. (Color online) (a)–(c) Dependence of average transient time $\langle T\rangle$ and fraction of SAF attractors, f_{SAF}, on τ for three networks with size, $N = 24,30,36$, respectively ($p = 0.4$ and $\varepsilon = 0.1$). The common phenomenon is that long transients occur in the transitional region.

IV. DESTABILIZATION OF SAF ATTRACTORS

A large number of SAF attractors destabilize near the transitional region. As long chaotic transients accompany this collective destabilization, it is important to study the bifurcation process in detail. Our analysis mainly focuses on

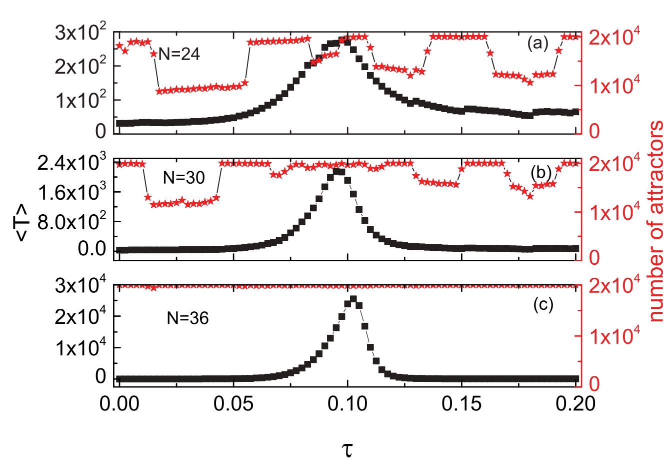

FIG. 4. (Color online) (a)–(c) Dependence of average transient time and number of attractors on τ for $N = 24,30,36$, respectively ($p = 0.4$ and $\varepsilon = 0.1$).

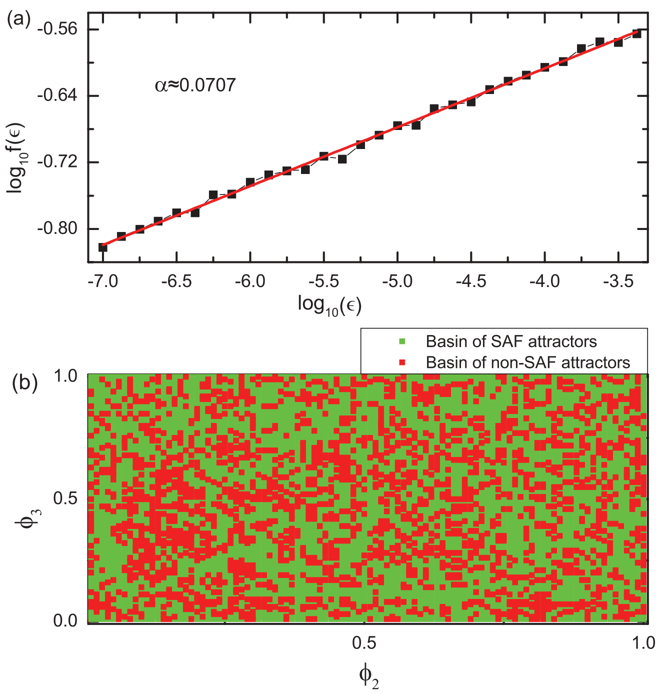

FIG. 5. (Color online) (a) Numerically obtained algebraic scaling of $f(\epsilon) \sim \epsilon^{\alpha}$ (see text for parameter setting). The uncertainty exponent is estimated to be $\alpha \approx 0.07$, indicating an extremely strong degree of mixing of basins of SAF and non-SAF attractors. (b) An example of mixed basins in an arbitrary two-dimensional cross section (ϕ_2,ϕ_3) in the phase space.

period-one attractors, as they are the dominant attractors, as shown in Fig. 2(a). The event structures of any period-one SAF attractor consist of one sequential active firing event and a number of passive firings induced by the receipt of pulses. We find that the bifurcation of SAF attractors is typically caused by the conversion of passive firing into active firing. Specifically, a passive-firing event of oscillator i occurs when it receives pulses of suprathreshold strength ϵ', that is, $h_p^i = H[\phi_i(t),\epsilon'] > 1$, at time t. Immediately after the firing, the phase is reset to zero. We can then use the quantity h_p^i to conveniently denote the hidden state of oscillator i right before the resetting due to passive firing. If h_p^i crosses 1 when varying parameters, the passive firing will become active after the receiving event. In the other case, if the phase of oscillator i is less than 1 just before the receiving event, the oscillator i will become active before the receiving event. Thus, h_p^i should be smaller than $H[1,\epsilon']$ for a passive firing. To be concrete but without loss of generality, we derive the dependence of h_p^i on parameters τ and ε.

Suppose there are m receiving events and an active firing for the reference oscillator j so that the active firing is the $(m+1)$th event. Let i denote a passive-firing oscillator and suppose it fires just after the Lth receiving event, where $1 \leqslant L \leqslant m$. For the kth receiving event, the numbers of pulses received by j and i are n_k^j and n_k^i, respectively. The quantity h_p^i depends on both the number of pulses received by oscillator i and the waiting time W of oscillator j, that is, the time needed to reach the threshold just after the mth receiving event. Thus, $W = 1 - \phi_j^m$, where ϕ_j^m denotes the phase of oscillator j just after the mth receiving event, which can be obtained recursively by

$$\phi_j^{s+1} = H\left(\phi_j^s + \tau,\varepsilon n_s^j/k_j\right),$$

for $s = 2,\ldots,m-1$ with $\phi_j^1 = H(\tau,\varepsilon n_1^j/k_j)$. We thus have

$$W = 1 - \left[\exp\left(\varepsilon b\sum_{h=1}^{m} n_h^j/k_j\right) + \exp\left(\varepsilon b\sum_{h=2}^{m} n_h^j/k_j\right) + \cdots + \exp\left(\varepsilon b n_m^j/k_j\right)\right]\tau - \frac{\exp\left(\varepsilon b\sum_{h=1}^{m} n_h^j/k_j\right)-1}{\exp(b)-1},$$

which can be simplified by using $\sum_{h=1}^{m} n_h^j = k_j$ for period-one attractors. We obtain

$$W = 1 - A_j\tau - B, \tag{6}$$

where the coefficients A_j and B are defined as

$$A_j = \exp\left(\epsilon b\sum_{h=1}^{m} n_h^j/k_j\right) + \exp\left(\epsilon b\sum_{h=2}^{m} n_h^j/k_j\right) + \cdots + \exp\left(\epsilon b n_m^j/k_j\right),$$
$$B = [\exp(\varepsilon b) - 1]/[\exp(b) - 1].$$

We can now derive a formula for h_p^i. Immediately after the Lth firing event, the phase of oscillator i is zero. Similar to the derivation of W, we can obtain ϕ_i^m by sequentially considering the effect of pulses received at the $L,L+1,\ldots,m$th events:

$$\phi_i^m = \left[\exp\left(\varepsilon b\sum_{h=L+1}^{m} n_h^i/k_i\right) + \exp\left(\varepsilon b\sum_{h=L+2}^{m} n_h^i/k_i\right) + \cdots + \exp\left(\varepsilon b n_m^i/k_i\right)\right]\tau + \frac{\exp\left(\varepsilon b\sum_{h=L+1}^{m} n_h^i/k_i\right)-1}{\exp(b)-1}.$$

Free evolution due to W gives rise to $\phi_i^{m+1} = \phi_i^m + W$. After considering the effect of the 1st, $\ldots$, and Lth receiving events, we have

$$h_p^i = \left[A_i - A_j\exp\left(\varepsilon b\sum_{h=1}^{L} n_h^i/k_i\right)\right]\tau + C, \tag{7}$$

where the coefficients A_i and C are given by

$$A_i = \exp\left(\varepsilon b\sum_{h=1}^{m} n_h^i/k_i\right) + \exp\left(\varepsilon b\sum_{h=2}^{m} n_h^i/k_i\right) + \cdots + \exp\left(\varepsilon b n_m^i/k_i\right),$$
$$C = \exp\left(\varepsilon b\sum_{h=1}^{L} n_h^i/k_i\right) - B\exp\left(\varepsilon b\sum_{h=1}^{L} n_h^i/k_i\right) + B.$$

The terms A_i and A_j have the same structure with respect to the corresponding numbers of pulses received by oscillators i and j. The extra multiplication factor $\exp(\varepsilon b \sum_{h=1}^{L} n_h^i / k_i)$ for A_j can make the whole term negative, in particular for large L. In the case of $L = m$, we have $A_i - e^{\varepsilon b} A_j$, which is always negative because each term in A_i or A_j is not larger than $e^{\varepsilon b}$. Hence, passive firing oscillators tend to be closer to the critical value 1 as τ is increased.

When the system moves closer to the transitional region from the region of SAF attractors, which can be realized through increasing the delay time τ, the passive firings associated with SAF attractors can readily be converted into active firings. In this sense, the SAF attractors become more vulnerable near the transitional region. Because of this, perturbation induced by the new active firing can convert more passive firing events into active firing ones. Consequently, the bifurcation can make the system undergo irregular transients before settling into a new attractor. As an example, Fig. 6 demonstrates a detailed bifurcation behavior of an SAF attractor caused by the conversion of passive firing of an oscillator (No. 4 in this case) to active firing, when h_p^4 crosses the critical value 1. Just before the bifurcation, we see that all h_p^i values for the passive firing oscillators decrease as τ is increased and hence are closer to the critical value 1 [Fig. 6(a)]. Just after the bifurcation, other oscillators, for example, 3, 17, 13, 15, 10, 16, 12, and 7, can be converted into active firings subsequently. The system undergoes irregular transient before settling on a new attractor, as shown in Fig. 6(b).

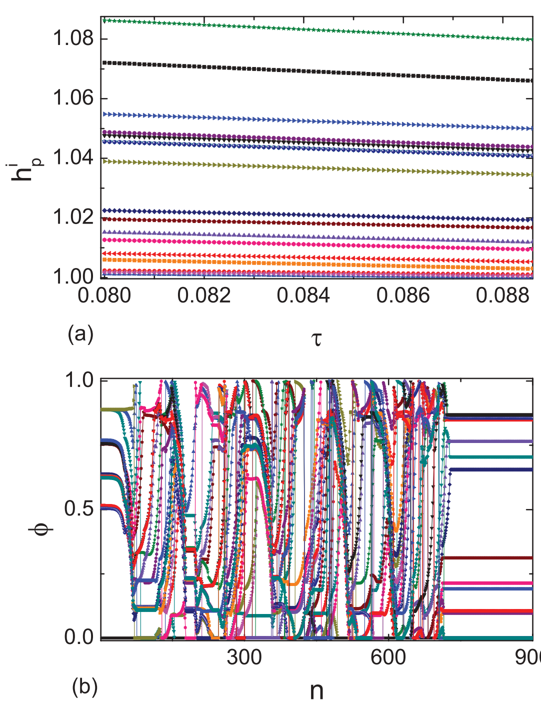

FIG. 6. (Color online) (a) Bifurcation of an SAF attractor induced by the conversion of passive firing of oscillator 4 to active firing. The bifurcation value $\tau \approx 0.0885$ is determined by h_p^4's approaching the critical value 1.0. All values of h_p^i for passive-firing oscillators decrease as τ is increased. (b) The bifurcation further causes other passive-firing oscillators to be converted into active-firing ones, generating long chaotic transients, before settling into a new attractor. Here n represents the nth resetting of the reference oscillator 1.

The irregular transients are easily observed for SAF attractors destabilized near the transitional region. For non-SAF attractors with typical event structures as described by Eq. (5), the time difference between two successive receiving events is not constrained by τ. As a result, the bifurcation of non-SAF attractors is usually induced by the merge of two receiving events, caused by the time difference between two successive receiving events going to zero. The event structure for the new attractor is approximately invariant, so the system typically spends a short time after bifurcations of non-SAF attractors before settling into the new attractors.

Generally, the destabilization of an SAF attractor near the transitional region tends to leave a trace in the phase space, where the system can irregularly be redirected to some remote regions. As a large number of SAF attractors collectively become destabilized, long chaotic transients arise near the transitional region.

V. CONCLUSION

In summary, we have investigated the underlying mechanism for the occurrence of long chaotic irregular transients in networks of excitatory pulse-coupled oscillators. Exploring the dynamical origin of transient dynamics in networked systems is extremely difficult [1,14]. We have developed a framework for understanding long chaotic transients based on analyzing temporal events of pulse firing and receiving. We define a special type of attractors, SAF attractors, which allows us to classify all attractors of the system into this and the complementary type, making it possible to analyze the mechanism of transients. Our finding shows that long chaotic transients occur in the transitional region where a large number of SAF attractors are collectively destroyed. The bifurcation of an SAF attractor tends to induce irregular transients. In addition, the basins of SAF attractors and non-SAF attractors are highly mixed in the transitional region. We emphasize that the source of chaotic transients is collective destabilization of SAF attractors.

Our work demonstrates that extremely complex dynamical phenomena in networked dynamical systems may be understood through the structures of attractors. Although our technique is illustrated using networks of pulse-coupled oscillators, it can be generalized to other networked systems following the basic principle of finding some suitable schemes to classify coexisting attractors of the system into a few groups.

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