Onset of chaotic phase synchronization in complex networks of coupled heterogeneous oscillators

Francesco Ricci,[1] Roberto Tonelli,[2] Liang Huang,[3,4] and Ying-Cheng Lai[4,5]
[1]*Department of Physics, University of Cagliari, I-09042 Monserrato, Italy*
[2]*Dept. of Electrical and Electronical Engineering, University of Cagliari, piazza d'Armi, I-09123 Cagliari, Italy*
[3]*Institute of Computational Physics and Complex Systems, Lanzhou University, Lanzhou, Gansu 730000, China*
[4]*School of Electrical, Computer and Energy Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[5]*Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*
(Received 25 February 2012; published 14 August 2012)

Existing studies on network synchronization focused on complex networks possessing either identical or nonidentical but simple nodal dynamics. We consider networks of both complex topologies and heterogeneous but chaotic oscillators, and investigate the onset of global phase synchronization. Based on a heuristic analysis and by developing an efficient numerical procedure to detect the onset of phase synchronization, we uncover a general scaling law, revealing that chaotic phase synchronization can be facilitated by making the network more densely connected. Our methodology can find applications in probing the fundamental network dynamics in realistic situations, where both complex topology and complicated, heterogeneous nodal dynamics are expected.

DOI: 10.1103/PhysRevE.86.027201 PACS number(s): 05.45.Xt, 89.75.Hc

Synchronization in complex networks has been a problem of continuous interest [1–10]. For networks having identical nodal dynamics, e.g., a network of coupled identical oscillators, the question of whether the network is synchronizable can be answered through the approach of master-stability function (MSF) [11,12]. In particular, due to the fact that the nodal dynamics are all identical, synchronization among all nodes in the network is a mathematical solution of the system, defining a synchronization manifold in the phase space of the whole system. If the solution is stable with respect to perturbations in all subeigenspaces that are transverse to the manifold, synchronization is physically observable or realizable. The MSF, an invariant property determined solely by the nodal dynamics, provides a computationally feasible way to determine the transverse stabilities of the synchronization manifold in terms of the network structure. This approach has been applied to analyzing the synchronizability of various complex networks, such as small-world networks [2–4], scale-free networks [5,6], weighted complex networks [7], adaptive complex networks [8], complex clustered networks [9], and complex gradient networks [10]. Particularly, in Ref. [6] the authors found that on the example of coupled Rossler systems, that small network size N and dense network connection favor synchronization. The approach has also been extended to situations where the nodal dynamics are slightly nonidentical [13].

In real-world networks heterogeneity in the nodal dynamics is expected. In such a case, an exact synchronization manifold cannot be defined. To probe into the fundamental synchronization dynamics for heterogeneous networks, a viable approach is to reduce the complexity of the nodal dynamics. In this regard, the Kuramoto model [14,15] has been studied extensively in which the nodal dynamics is given by that of a uniform rotation: $\dot{\theta} = \omega$, where θ is a phase variable and ω is the frequency. Heterogeneity can be modeled by assuming that the frequency for each node is distinct and can be drawn from a random distribution. For complex networks hosting the Kuramoto phase dynamics, transition to synchronization can be understood fairly comprehensively [16,17]. For example, say K is a general coupling parameter. As K is increased through a critical value K_c, partial synchronization in the network in the form of synchronous clusters can arise, where K_c is determined by the network topology and the frequency distribution of the oscillators. Transition to complete or global synchronization, where all nodes in the network are synchronized, has also been investigated [18]. In other systems of heterogenous dynamics, like two-dimensional nonidentical neuron-like maps, phase synchronization has been investigated [19].

Despite vast literature on complete synchronization of identical chaotic oscillators in complex dynamical systems, the problem of phase synchronization in complex networks with *heterogeneous* nodal dynamics that are more complicated than uniform rotation has remained to be outstanding. The specific question that we ask is, if the nonidentical individual nodal dynamics are chaotic, what determines the transition to synchronization? Since the nodal dynamics are not identical, chaotic phase synchronization is expected to arise [20], especially in the weakly coupling regime. To be concrete and realistic, we shall then focus on the onset of chaotic phase synchronization in complex networks, with the goal to uncover scaling relation of the critical coupling strength K_c required for this type of synchronization to occur in the entire network. A heuristic analysis reveals an algebraic scaling relation between K_c and a parameter characterizing the link density of the network, indicating that as the network becomes more densely connected, the threshold coupling value required for chaotic phase synchronization decreases. To verify the scaling relation, we develop a computational procedure to accurately determine K_c for large networks in an extremely efficient manner. Our result has the following significance. In complex dynamical systems complete synchronization is difficult to be realized, as enormous coupling may be required. However, phase synchronization, a weaker type of synchronization, can be typical and finds applications in many areas of significant interest such as epileptic seizures [21]. The scaling relation uncovered in this paper can be used to assess, for complex networks of given topology, size, and linkage, when phase synchronization can be expected.

We consider networks of coupled, nonidentical oscillators, mathematically described by

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}_i(\mathbf{x}_i) - \epsilon \sum_{j=1}^{N} G_{ij}\mathbf{H}(\mathbf{x}_j), \tag{1}$$

where N is the number of oscillators (nodes) in the network, $\mathbf{x}_i$ is a d-dimensional vector of the dynamical variables of node i, $\mathbf{F}_i(\mathbf{x}_i)$ is the vector field of node i, ϵ is a coupling parameter, and $\mathbf{G}$ is a coupling matrix determined by the connection topology. The elements of $\mathbf{G}$ are $\mathbf{G}_{ij} = -1, i \neq j$ if oscillators i,j are coupled and $\mathbf{G}_{ij} = 0$ if they are not. The diagonal elements are given by $\mathbf{G}_{ii} = -\sum_{j\neq i} G_{ij}$ in order to satisfy the condition $\sum_{j=1}^{N} G_{ij} = 0$ for any i, where N is the network size. When all the oscillators are identical, a complete synchronized state defined by $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N = \mathbf{s}$ is an exact solution of Eq. (1). The coupling matrix $\mathbf{G}$, as determined by the network topology, can be diagonalized with a set of real eigenvalues $\{\lambda_i, i = 1,\ldots,N\}$ and the corresponding set of eigenvectors $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_N$. Full connectivity of the network ensures that there is one zero eigenvalue and the eigenvalues can be sorted as $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_N$. The variational equations governing the time evolution of the set of infinitesimal vectors transverse to the synchronization manifold, $\delta\mathbf{x}_i(t) \equiv \mathbf{x}_i(t) - \mathbf{s}(t)$, are $d\delta\mathbf{x}_i/dt = \mathbf{DF}(\mathbf{s}) \cdot \delta\mathbf{x}_i - \epsilon \sum_{j=1}^{N} G_{ij}\mathbf{DH}(\mathbf{s}) \cdot \delta\mathbf{x}_j$, where $\mathbf{DF}(\mathbf{s})$ and $\mathbf{DH}(\mathbf{s})$ are the $d \times d$ Jacobian matrices of the corresponding vector functions evaluated at $\mathbf{s}(t)$. The transform $\delta\mathbf{y} = \mathbf{Q}^{-1} \cdot \delta\mathbf{x}$, where $\mathbf{Q}$ is a matrix whose columns are the set of eigenvectors of $\mathbf{G}$, leads to the block-diagonally decoupled form of the variational equation: $d\delta\mathbf{y}_i/dt = [\mathbf{DF}(\mathbf{s}) - \epsilon\lambda_i\mathbf{DH}(\mathbf{s})] \cdot \delta\mathbf{y}_i$. Let $K_i = \epsilon\lambda_i (i = 2,\ldots,N)$ be a specific set of values of a normalized coupling parameter K. All blocks of the decoupled equation are structurally the same with only the factor of K_i being different, leading to the following generic form for all the decoupled blocks: $d\delta\mathbf{y}/dt = [\mathbf{DF}(\mathbf{s}) - K\mathbf{DH}(\mathbf{s})] \cdot \delta\mathbf{y}$.

The largest Lyapunov exponent of the above block-diagonal variational equation is the MSF $\Psi(K)$. If $\Psi(K)$ is negative, a small disturbance from the synchronization state will diminish exponentially so that the synchronous solution is stable, at least when the oscillators are initialized in its vicinity. The synchronous solution is unstable and cannot be realized physically if $\Psi(K)$ is positive because small perturbations from the synchronous state will lead to trajectories that diverge from the state. For the coupled oscillator network Eq. (1), a necessary condition for synchronization is then that all normalized coupling parameters $K_i (i = 2,\ldots,N)$ fall in an interval on the K axis where $\Psi(K)$ is negative. The network is more synchronizable if the spread in the set of K_i values, or equivalently, the spread in the eigenvalue spectrum λ_i, is smaller. The MSF allows the synchronization interval to be determined, which depends only on the coupling function ($\mathbf{H}$) but is independent of the network topology. In particular, the condition for complete synchronization is given by $\epsilon\lambda_2 \simeq K_1$, and the critical parameter for the onset of synchronization is $K_c \simeq K_1/\lambda_2$, where K_1 is the value of K at which $\Psi(K)$ becomes negative from the positive side.

In Ref. [13], a stability analysis for synchronization of nearly identical oscillators was carried out, which was based on the following extended master-stability function (eMSF):

$$\dot{\xi} = [D_\omega f - K\mathbf{DH}] \cdot \xi + D_\mu f \cdot \psi. \tag{2}$$

The term in the square parentheses is the same as that in the block-diagonal form of the variational equation in the case of identical nodal dynamics. In the second term, D_μ is the Jacobian matrix with respect to the parameter vector μ and $\psi = \sum_{j=1}^{N} \mathbf{u}_{ij}\,\delta\mu_j$, where $\delta\mu_i = \mu_i - \bar{\mu}$ is the parameter mismatch with respect to mean of μ_i over all oscillators, and $\mathbf{u}_i$ is the ith eigenvector of $\mathbf{G}$. The stability of Eq. (2) can then be determined as a function of the two parameters K and ψ, and we can decompose the problem into two separate parts: one that depends only on the nodal dynamics and the coupling function $\mathbf{H}$, and another determined by the parameter mismatch among the oscillators.

Let $\Psi(K,\psi)$ be the largest Lyapunov exponent of Eq. (2). In general, the value of K_1 depends on ψ. To be specific, we fix the number of oscillators and variation in the parameter mismatch, and focus on the scaling relation between K_c and the link density. In this case, $\delta\mu_i$ obeys the same statistic as that for ψ in Eq. (2). For different link density, since the mismatch is bounded in the same range, the value of K_1 can be regarded as a constant, which is slightly larger than $K_1|_{\psi=0}$. As the amount of mismatch is increased, i.e., with a larger standard deviation σ as in our case, K_1 also increases, which has been verified numerically. Since the oscillators are not identical, K_1 is not the critical coupling strength for complete synchronization, but the value for which the variation for the dynamical variables are bounded [13], so it is essentially the onset point of phase synchronization. Approximately, we can use $K_1|_{\psi=0}$ to represent K_1 with parameter mismatches. In fact, we find numerically that the product $K_c\lambda_2$ for different networks is nearly constant and the values are comparable with those of identical oscillators. We thus have $K_c\lambda_2 \simeq K_1$.

For complex topologies we consider random and scale-free networks. For the former, the link density is determined by p, the probability that an arbitrary pair of oscillators in the network are coupled. For the latter, we consider those generated by the preferential-attachment mechanism [22]: starting from a small number m_0 of completely connected oscillators, a new oscillator is introduced into the network with m links according to the preferential-attachment rule. The parameter m thus determines the link density. For both types of networks, estimates of the value of λ_2 are available [23]. In particular, for random networks, we have $\lambda_2 \sim Np - 2\sqrt{Np(1-p)}$, while for scale-free networks, we have $\lambda_2 \sim Cm$. The relationship $K_c\lambda_2 \simeq K_1$ gives, for random and scale-free networks, respectively, the following scaling law governing the onset of phase synchronization:

$$K_c \sim \begin{cases} \frac{K_1}{Np-2\sqrt{Np(1-p)}}, & \text{random networks,} \\ \frac{K_1}{Cm}, & \text{scale-free networks.} \end{cases} \tag{3}$$

The scaling laws (3) are the main result of this paper. We note that, for large random networks that satisfy $Np \gg 1$, the scaling law becomes $K_c \sim p^{-1}$.

To provide numerical verification of Eq. (3), we consider networks of coupled Rössler chaotic oscillators. The vector field of the ith oscillator (node i) is given by $\mathbf{F}_i(\mathbf{x}) = [-(\omega_i y + z), \omega_i x + 0.165y, 0.2 + z(x - 10)]$, where the parameter ω_i is

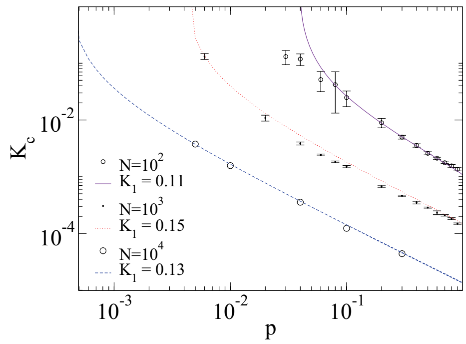

FIG. 1. (Color online) For random network of $N = 10^2$, 10^3, and 10^4 chaotic Rössler oscillators, nonlinear fit $K_c \sim A/[Np - 2\sqrt{Np(1-p)}]$, as predicted theoretically, where A is a fitting parameter.

different for each oscillator and is taken from some random distribution. For oscillator i, a phase variable $\phi_i(t)$ can be calculated [20]. For a pair of oscillators i and j, phase synchronization is defined by $\Delta\phi_{ij} = |\phi_i(t) - \phi_j(t)| < 2\pi$. The average phase-synchronization time τ_{ij} is the average time interval during which the phase difference is bounded within 2π. As the coupling parameter K is increased toward K_c, τ_{ij} increases and obeys the scaling law [24]: $\tau_{ij} \sim \exp[C(K_c - K)^{-\gamma}]$, where C and γ are positive constants and τ_{ij} diverges for $K \geqslant K_c$. For a network of N oscillators, as K_c is reached, all $N(N-1)/2$ values of τ_{ij} (one for each distinct pair) diverge. Computationally the behavior of τ_{ij} can thus be used to determine the onset of phase synchronization on the network. In particular, τ_{ij} can be regarded as an element of the $N \times N$ symmetric matrix, Γ. For finite time series measured from oscillators in the network, when the diagonal elements of Γ are chosen properly, the determinant of the matrix provides an effective way to determine the onset of chaotic phase synchronization globally [21].

To calculate the determinant is computationally costly. To remedy this difficulty, we propose the following quantity:

$$P = \prod_{i=1}^{N-1} \sqrt[N]{\tau_{i,i+1}}, \tag{4}$$

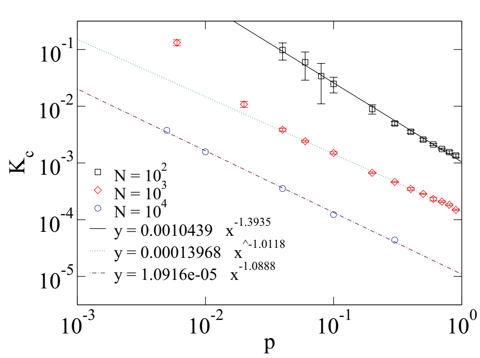

FIG. 2. (Color online) For random networks of $N = 10^2$, 10^3, and 10^4 nodes, numerically obtained scaling of K_c with the linking probability p. The scaling is algebraic in the regime of large Np values, as predicted by our theory.

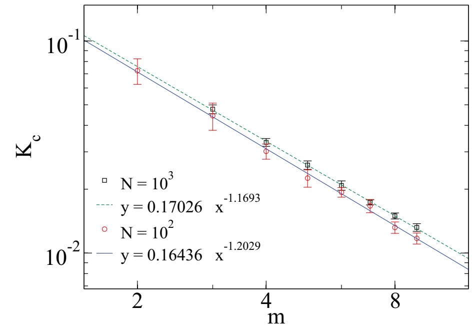

FIG. 3. (Color online) For scale-free networks of $N = 10^2$ and 10^3 oscillators, numerically obtained scaling of K_c with the network-generating parameter m.

which can be computed extremely efficiently. As phase synchronization is approached, P tends to diverge. The computational load is only proportional to N, the network size. We have verified that the values of critical coupling parameter K_c computed using P and the previous method of determinant coincide with high accuracy.

Figure 1 shows for random networks of $N = 10^2$, 10^3, and 10^4 nodes, the relationship between K_c and p, fitted according to the theoretical prediction in Eq. (3). We observe a good agreement between numerics and theory. For the regime of large Np values, the scaling becomes algebraic, as shown in Fig. 2 for networks of three different sizes ($N = 10^2$, 10^3, and 10^4). Figure 3 shows for scale-free networks of $N = 10^2$ and 10^3 oscillators, numerically obtained scaling of K_c with the network-generating parameter m. The scaling is algebraic, as

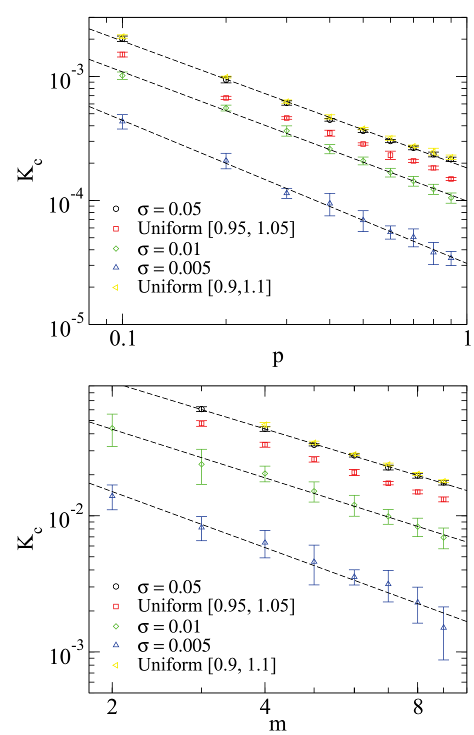

FIG. 4. (Color online) For random (upper panel) and scale-free (lower panel) networks, scaling of K_c with p and m, respectively, for different distributions of the frequency parameter ω_i of the chaotic oscillators. The network size is $N = 10^3$.

predicted. The scaling laws (3) are independent of the intrinsic properties of the oscillators, as shown in Fig. 4.

In summary, we have demonstrated, by using a heuristic analysis and numerical computations, that the onset of phase synchronization in complex networks of coupled, heterogeneous chaotic oscillators can be facilitated by increasing the link density. This is substantiated by a scaling law relating the critical coupling parameter required for phase synchronization among all oscillators in the network to a parameter characterizing the network linkage. Computational detection of the onset of global chaotic phase synchronization is made possible by an efficient numerical method to calculate the pairwise average phase-synchronization time. Our work treats both complex network topology and complicated heterogeneous nodal dynamics at the same time, versus existing works, e.g., on complex networks of simple Kuramoto phase oscillators. The complexity of our problem renders infeasible any analytic treatment at a comprehensive level, but nonetheless we are able to obtain quantitative results on chaotic phase synchronization in the network. Networked systems in real applications are often heterogeneous, complicated in both topology and nodal dynamical processes. The methodology developed in our work can be useful and further developed to probe into the fundamentals of the network dynamics with significant applications.

This work was partly supported by a grant from R.A.S. (Regione Autonoma della Sardegna) awarded to F. Ricci, PO Sardegna FSE 2007-2013, L.R.7/2007 "Promotion of the scientific research and technological innovation in Sardinia". Y.C.L. and L.H. were supported by AFOSR under Grant No. FA9550-10-1-0083. L.H. was also supported by NSF of China under Grants No. 11005053 and No. 11135001.

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization - A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
[2] L. F. Lago-Fernandez, R. Huerta, F. Corbacho, and J. A. Siguenza, Phys. Rev. Lett. **84**, 2758 (2000).
[3] M. B. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[4] X. F. Wang and G. Chen, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 187 (2002); IEEE Trans. Circuits Syst. I **49**, 54 (2002).
[5] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003); P. G. Lind, J. A. C. Gallas, and H. J. Herrmann, Phys. Rev. E **70**, 056207 (2004); Y.-C. Hung, Y.-T. Huang, M.-C. Ho, and C.-K. Hu, *ibid.* **77**, 016202 (2008).
[6] S.-H. Yook and H. Meyer-Ortmanns, Physica A **371**(2), 781 (2006).
[7] M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005).
[8] C. Zhou and J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006).
[9] L. Huang, K. Park, Y.-C. Lai, L. Yang, and K. Yang, Phys. Rev. Lett. **97**, 164101 (2006).
[10] X. G. Wang, L. Huang, Y.-C. Lai, and C. H. Lai, Phys. Rev. E **76**, 056113 (2007).
[11] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[12] L. Huang, Q.-F. Chen, Y.-C. Lai, and L. M. Pecora, Phys. Rev. E **80**, 036204 (2009).
[13] J. Sun, E. M. Bollt, and T. Nishikawa, Europhys. Lett. **85**, 60011 (2009).
[14] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, Berlin, 1984).
[15] S. H. Strogatz, Physica D **143**, 1 (2000).
[16] J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. E **71**, 036151 (2005); Chaos **16**, 015107 (2005).
[17] W. S. Lee, E. Ott, and T. M. Antonsen, Phys. Rev. Lett. **103**, 044101 (2009); E. A. Martens, E. Barreto, S. H. Strogatz, E. Ott, P. So, and T. M. Antonsen, Phys. Rev. E **79**, 026204 (2009).
[18] X.-G. Wang, L. Huang, S.-G. Guan, Y.-C. Lai, and C. H. Lai, Chaos **18**, 037117 (2008).
[19] C. A. S. Batista, A. M. Batista, J. A. C. de Pontes, R. L. Viana, and S. R. Lopes, Phys. Rev. E **76**, 016218 (2007).
[20] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[21] Y.-C. Lai, M. G. Frei, I. Osorio, and L. Huang, Phys. Rev. Lett. **98**, 108102 (2007).
[22] A.-L. Barabasi and R. Albert, Science **286**, 509 (1999).
[23] L. Huang, Y.-C. Lai, and R. A. Gatenby, Phys. Rev. E **78**, 045102(R) (2008).
[24] V. Andrade, R. L. Davidchack, and Y.-C. Lai, Phys. Rev. E **61**, 3230 (2000).