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Characterization of nonstationary chaotic systems

Ruth Serquina,[1] Ying-Cheng Lai,[2,3] and Qingfei Chen[2]
[1]*Department of Mathematics, MSU-Iligan Institute of Technology, the Philippines*
[2]*Department of Electrical Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Department of Physics and Astronomy, Arizona State University, Tempe, Arizona 85287, USA*
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Nonstationary dynamical systems arise in applications, but little has been done in terms of the characterization of such systems, as most standard notions in nonlinear dynamics such as the Lyapunov exponents and fractal dimensions are developed for stationary dynamical systems. We propose a framework to characterize nonstationary dynamical systems. A natural way is to generate and examine ensemble snapshots using a large number of trajectories, which are capable of revealing the underlying fractal properties of the system. By defining the Lyapunov exponents and the fractal dimension based on a proper probability measure from the ensemble snapshots, we show that the Kaplan-Yorke formula, which is fundamental in nonlinear dynamics, remains valid most of the time even for nonstationary dynamical systems.

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I. INTRODUCTION

In many previous studies of nonlinear dynamical systems, stationarity is assumed. That is, the underlying system equations and parameters are assumed to be fixed in time. One can then define asymptotic invariant sets such as unstable periodic orbits, attractors, chaotic saddles (nonattracting invariant sets), study their properties such as the spectra of Lyapunov exponents and of fractal dimensions, and search for various bifurcations that concern how the time-asymptotic behaviors of the system vary with parameters [1]. There are, however, practical situations where the assumption of stationarity does not hold. For a nonstationary dynamical system, many notions that are fundamental to the development of nonlinear dynamics such as periodic orbits and attractors, are no longer meaningful. The purpose of this paper is to develop a systematic and physically meaningful way to characterize nonstationary dynamical systems.

We shall be concerned with typical nonlinear systems which, when being stationary, can have both chaos and periodic motions depending on the parameters. Without loss of generality we will focus on the relatively simple situation where a single parameter of the system varies with time. In discrete time, our model system can be represented by

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t, p_t), \tag{1}$$

where $\mathbf{x}$ is a d-dimensional dynamical variable, $\mathbf{f}$ is a nonlinear mapping function, and p_t is a time-dependent parameter. In a time interval of interest, the parameter can vary in a range, say $[p_a, p_b]$ $(p_a < p_b)$ where for any $p \in [p_a, p_b]$, the corresponding stationary dynamical system $\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t, p)$ can possess a chaotic attractor, or a periodic attractor, or even multiple coexisting attractors. Because of the time variation of the parameter, a long trajectory originated from a single initial condition typically appears random and exhibits no fractal structure. To reveal the intrinsic fractal structure associated with the deterministic but nonstationary chaotic system, a viable approach is to examine *simultaneously* the evolution of a large number of trajectories from an ensemble of initial conditions. At a given instant of time, the trajectories tend to form a pattern that can be apparently fractal. Such patterns are called *snapshot attractors* in the context of random dynamical systems that have proven effective to reveal the underlying fractal structure [2–7]. For a nonstationary system, the notion of "attractor" is no longer meaningful as a trajectory will in general not have sufficient time to settle down to any asymptotic state of the system. We shall call the phase-space images of an ensemble of trajectories at a given time *ensemble snapshots*. The question to be addressed in this paper concerns the dynamical properties of such ensemble snapshots. In particular, we will focus on their Lyapunov exponents and the fractal dimensions.

Due to nonstationarity, we are restricted to examining the dynamical evolution of an ensemble of trajectories in short time intervals, during which the system can be regarded as "stationary." The lengths of these time intervals depend on the rate of change of the system parameter: a slower rate would give a relatively longer interval and vice versa. For convenience, we call them adiabatic time intervals. Since the rate of parameter change is in general time-dependent and can even be random, in a long experimental time the adiabatic time intervals are not necessarily uniform. Nonetheless, assuming adiabatic time intervals allows the Lyapunov exponents of an ensemble snapshot to be defined as the ensemble-averaged values of the corresponding short-time Lyapunov exponents from all trajectories comprising the snapshot. Due to nonstationarity, the exponents exhibit random fluctuations with time. If for any given time all trajectories in the ensemble snapshot are contained in a single basin of attraction for the "frozen" dynamical system at that time, the variance of the fluctuations of the exponents is independent of time. However, if the trajectories can be in different basins of attraction for the frozen system, the magnitude of the fluctuations of the exponents will depend on the value of the instantaneous Lyapunov exponents [8] and therefore can vary with time.

For a stationary dynamical system, the Kaplan-Yorke formula holds, which relates the information dimension of an attractor to its Lyapunov spectrum [9]. Thus, for our nonstationary system, after the Lyapunov exponents of an ensemble snapshot have been calculated, one may wonder whether the information dimension of the snapshot can be defined and related to the exponents. We shall argue that it is possible to define a dimension spectrum for an ensemble snapshot. The main result of this paper is that, if all trajectories constituting the ensemble snapshot are contained in a single basin of the underlying temporarily stationary dynamical system, the Kaplan-Yorke formula still holds in the sense that the information dimension obtained by a straightforward box-counting procedure can be approximated by the value determined by the Lyapunov exponents.

In Sec. II, we propose a proper natural measure for nonstationary dynamical systems, based on which the Lyapunov exponents and fractal-dimension spectrum can be defined. In Sec. III, we discuss the Kaplan-Yorke formula in the context of nonstationary systems. Numerical examples, one from discrete-time maps and another from continuous-time flows, are presented in Sec. IV. A brief summary is given in Sec. V.

II. DEFINITION OF LYAPUNOV EXPONENTS AND FRACTAL DIMENSIONS FOR NONSTATIONARY DYNAMICAL SYSTEMS

For a stationary dynamical system, asymptotic trajectories of infinite lengths can be obtained. Given a grid of cells that cover the asymptotic invariant set (e.g., a chaotic attractor), the natural measure of a cell is defined as the frequency of visit of a trajectory from a random initial condition (a typical trajectory) to the cell. The Lyapunov exponents and the fractal-dimension spectrum can then be defined [1]. For a nonstationary system, long trajectories with stationary dynamical properties are not available. To define a probability measure, a remedy is to use a large number of trajectories from an ensemble of initial conditions and to generate ensemble snapshots at different instants of time.

Let T be a long experimental or measurement time interval, which defines the largest time scale of the system, during which the system equations and/or the parameters of system can change significantly. To be concrete but without loss of generality, we assume that one of the parameters, say p, increases from p_a for $t=0$ to p_b for $t=T$, where $p_b > p_a$. The average rate of parameter change is thus $\Delta p=(p_b-p_a)/T$, which is small for large T. The parameter change can thus be regarded as *adiabatic*. To facilitate the definition of Lyapunov exponents and dimension spectrum, we divide T into K epochs of time: $T_1, T_2, \ldots, T_K$, where $\Sigma_{i=1}^{K} T_k = T$ and $T_k \ll T$ for $k=1,\ldots,K$. The parameter assumes constant value p_k in epoch k. In the next epoch, the parameter value is changed to $p_k+\Delta p$. The subintervals of time T_k need not be uniform, enabling modeling of an arbitrary form of the parameter variation $p(t)$. The special case where all T_k's are identical corresponds to a uniform rate of change of the parameter, and a random set of T_k's models stochastic parameter changes.

Say we choose N initial conditions at $t=0$ and evolve them simultaneously under Eq. (1). For a fixed epoch of time T_k, the system can be regarded as stationary with parameter $p=p_k$. To define a measure, we cover a proper phase-space region that contains all the trajectories by a grid of cells, each of size ε. Let N_i be the number of trajectory points in the ith cell. A measure can be defined as the probability for a trajectory point to be in the cell: $\mu_i = \lim_{N\to\infty} N_i/N$. In a strict sense, μ_i is time-dependent since T_k is finite. However, since T_k is short, μ_i will not change significantly during the epoch. Following the definition of dimension spectrum of an invariant set in stationary dynamical systems [1], we define the dimension spectrum of the ensemble snapshot for any epoch of time as

$$D_q = \frac{1}{1-q} \lim_{\varepsilon\to 0} \frac{\ln I(q,\varepsilon)}{\ln 1/\varepsilon}, \tag{2}$$

where $I(q,\varepsilon)=\Sigma_{i=1}^{N(\varepsilon)} \mu_i^q$ is a sum over all $N(\varepsilon)$ nonempty cells. The information dimension D_1 is given by

$$D_1 = \lim_{\varepsilon\to 0} \frac{I_1(\varepsilon)}{\ln \varepsilon}, \tag{3}$$

where $I_1(\varepsilon) \equiv \Sigma_{i=1}^{N(\varepsilon)} \mu_i \ln \mu_i$ is the information sum. To define the Lyapunov exponents for epoch T_k, we first fix an individual trajectory and choose an orthonormal set of infinitesimal vectors: $\delta \mathbf{x}_i^{(j)}(0)$ at the beginning of the epoch, where $j=1,\ldots,d$. We next calculate the evolutions of these tangent vectors for $t=1,\ldots,T_k$ according to $\delta\mathbf{x}_i^{(j)}(t+1) = \mathbf{DF}[\mathbf{x}_i(t)] \cdot \delta\mathbf{x}_i^{(j)}(t)$, where $\mathbf{DF}[\mathbf{x}_i(t)]$ is the Jacobian matrix of the map function. We can then define, for this trajectory, the following set of finite-time Lyapunov exponents:

$$\lambda_i^{(j)} = \frac{1}{T_k} \ln \left| \frac{\delta\mathbf{x}_i^{(j)}(T_k)}{\delta\mathbf{x}_i^{(j)}(0)} \right|, \tag{4}$$

for $j=1,\ldots,d$. Finally, we define the spectrum of Lyapunov exponents for the ensemble snapshot in this epoch of time as the ensemble average of $\lambda_i^{(j)}$:

$$\lambda^{(j)} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \lambda_i^{(j)}. \tag{5}$$

III. KAPLAN-YORKE FORMULA

Having defined the spectra of fractal dimension and of Lyapunov exponents for ensemble snapshots, we now ask whether a Kaplan-Yorke type of formula exists that relates the information dimension to the exponents. The following heuristic argument suggests a positive answer. For simplicity we focus on a two-dimensional phase space. Consider the probability measure constituted by an ensemble of trajectory points at any instant of time. We can use a grid of size ε to cover a finite fraction α of the measure, where $0<\alpha<1$. Assume the number of cells required is $N(\varepsilon,\alpha)$. A result in nonlinear dynamics is that the box-counting dimension given by the algebraic scaling of $N(\varepsilon,\alpha)$ with ε is in fact the information dimension of the underlying set, insofar as $\alpha \neq 1$. Let $\lambda_2 < 0 < \lambda_1$ be the two Lyapunov exponents in the kth epoch of duration T_k. Suppose we lay the grid of cells at the beginning of the epoch and consider one nonempty, square cell. At the end of the epoch, the cell will be stretched by the positive exponent along one direction and compressed by the negative exponent along another direction. That is, the cell will become a thin, elongated parallelogram with sizes of the order of $\varepsilon \exp(T_k\lambda_1)$ and $\varepsilon \exp(T_k\lambda_2)$, respectively. Using the smaller size $\varepsilon \exp(T_k\lambda_2)$ to cover the same α fraction of the probability measure, we need $[\varepsilon \exp(T_k\lambda_1)/\varepsilon \exp(T_k\lambda_2)]N(\varepsilon,\alpha)$ cells. That is

$$N[\varepsilon \exp(T_k\lambda_2),\alpha] \sim \exp[T_k(\lambda_1+|\lambda_2|)]N(\varepsilon,\alpha). \quad (6)$$

Since $N(\varepsilon,\alpha)\sim\varepsilon^{-D_1}$, Eq. (6) becomes

$$[\varepsilon \exp(T_k\lambda_2)]^{-D_1} \sim \exp[T_k(\lambda_1+|\lambda_2|)]\varepsilon^{-D_1}, \quad (7)$$

which suggests

$$D_1 \approx 1+\frac{\lambda_1}{|\lambda_2|} \equiv D_L, \quad (8)$$

where D_L is the Lyapunov dimension. For a stationary dynamical system, the time involved in Eqs. (6) and (7) can be arbitrarily long so that the Kaplan-Yorke formula Eq. (8) can be expected to hold [10]. For a nonstationary dynamical system, the luxury of taking the infinite-time limit is lost and we are restricted to studying the dynamics in finite (small) time epochs. Thus it is questionable whether the Kaplan-Yorke formula Eq. (8) would still hold. Numerical verifications are necessary.

IV. NUMERICAL EXAMPLES

A. Optical-cavity map

We consider the following two-dimensional Ikeda-Hammel-Jones-Moloney (IHJM) map [11] that models the dynamics of a nonlinear optical cavity, which has been a paradigmatic model in nonlinear dynamics:

$$z_{n+1} = A + Bz_n \exp\left[ik - \frac{ip_n}{1+|z_n|^2}\right], \quad (9)$$

where $z=x+iy$ is a complex dynamical variable and A, B, k, and p_n are parameters. The time dependence of the parameter p_n stipulates nonstationarity of the system. We choose $A=0.85$, $B=0.9$, $k=0.4$, and allow p_n to vary in the range $[p_a,p_b]=[4.0,20.0]$, We focus on the situation where the parameter changes at a constant rate from p_a to p_b. In particular, we choose an experimental interval of 1000 epochs, where each epoch corresponds to a time duration of $T_k=10$ iterations. Several examples of the ensemble snapshots are shown in Fig. 1, where the number of initial conditions used is 50000. The snapshots are apparently fractals. Figure 2 shows, for $t=800T_k$ on a logarithmic scale, the scalings of $N(\varepsilon)$ (the number of boxes of size ε required to cover the ensemble snapshot) and of the information sum $I_1(\varepsilon)$ with ε. We obtain, for this time epoch, $D_0\approx 1.78$ and $D_1\approx 1.14$. We then compute the Lyapunov exponents. The time evolution of the largest Lyapunov exponent is shown in Fig. 3(a), where we observe a significant amount of fluctuations, the origin of which can be attributed to multiple coexisting attractors and the nonstationary nature of the system. In particular, due to nonstationarity (short time duration of each epoch), the system is not able to settle into any attractor. When there are more than one attractor in the frozen system,

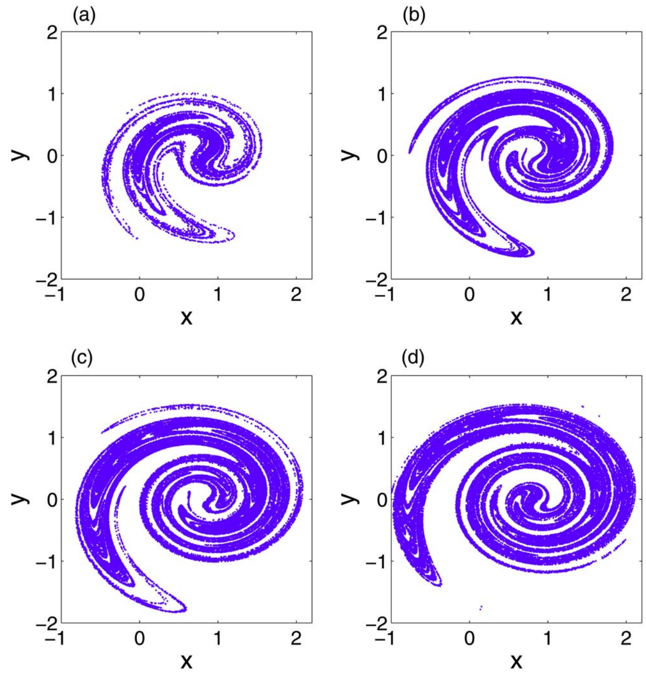

FIG. 1. (Color online) For the nonstationary IHJM map Eq. (9), (a)–(d) four ensemble snapshots observed for the 200th, the 400th, the 600th, and the 800th epoch. All trajectories are initiated randomly in the small phase-space region defined by $0\leq(x,y)\leq 0.1$. The fractal geometry of the snapshots is apparent.

at the end of any epoch, a random number of trajectories can be found near each attractor, and this number varies from epoch to epoch. This implies that increasing the number of trajectories will do little to reduce the fluctuations, as we have observed numerically. Signature of multiple attractors can be seen from the distributions of the finite-time Lyapunov exponent (say λ_1) at different times, as shown in Figs. 4(a)–4(d) for epoch number $k=200$, 400, 600, and 800.

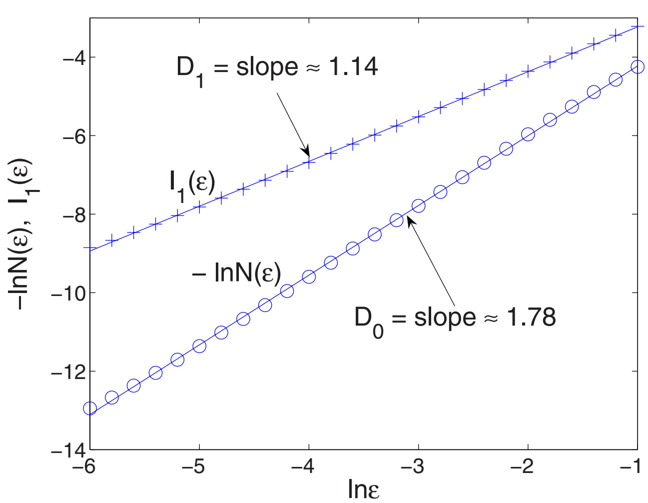

FIG. 2. (Color online) For the nonstationary IHJM map Eq. (9), estimates of the box-counting and the information dimension of the ensemble snapshot for $t=800T_k$. The number of trajectories used is 2×10^6.

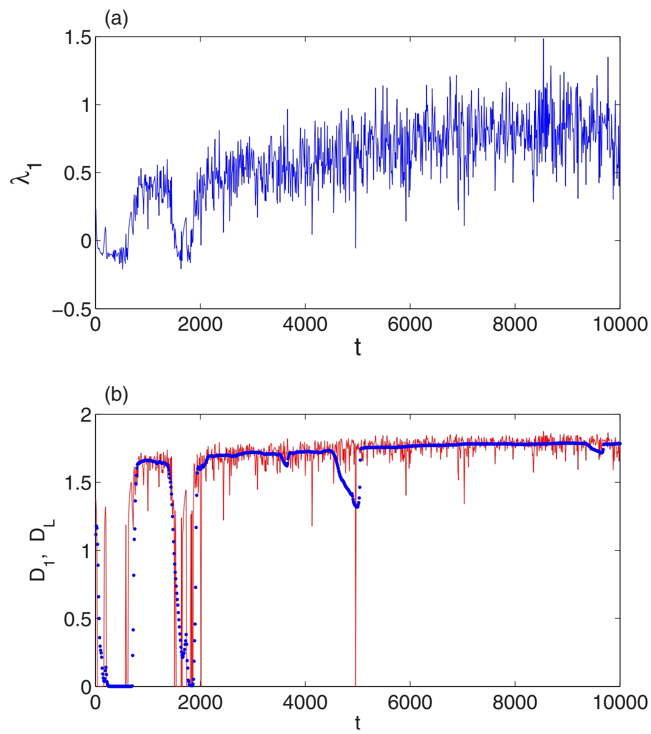

FIG. 3. (Color online) For the nonstationary IHJM map Eq. (9), (a) evolution of the larger Lyapunov exponent and (b) evolutions of the Lyapunov dimension D_L and of the information dimension D_1. We observe that D_L fluctuates about D_1 (estimated using a box-counting procedure), indicating the validity of the Kaplan-Yorke formula.

Appearance of distinct peaks in such a distribution during a specific epoch indicates coexisting attractors in the underlying frozen system at that time. We observe that, at different times, due to the parameter variation, the locations of the peaks are different, causing significant fluctuations of the exponent as in Fig. 3(a). The fluctuations are also reflected in the evolution of the Lyapunov dimension of the ensemble snapshot, as shown in Fig. 3(b). The remarkable phenomenon is that, for most of the epochs, the information dimension calculated using the box-counting procedure lies about the middle of the fluctuating Lyapunov dimension, as indicated by the dots in Fig. 3(b). This suggests that the Kaplan-Yorke formula is meaningful for nonstationary chaotic systems [12].

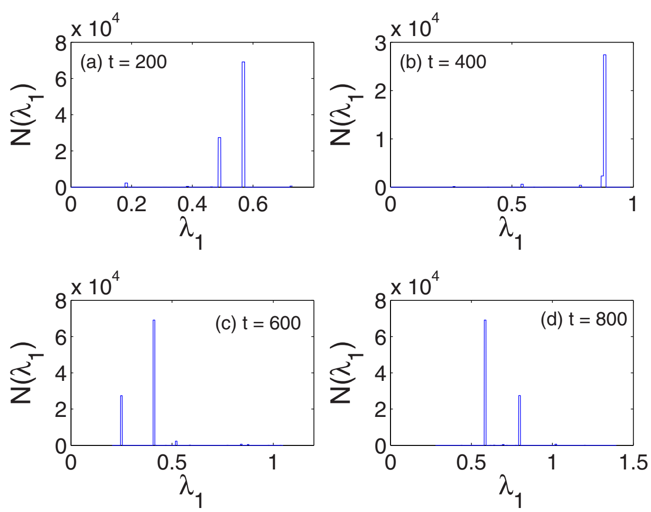

FIG. 4. (Color online) For the nonstationary IHJM map Eq. (9), histograms of λ_1 for (a) $k=200$, (b) $k=400$, (c) $k=600$, and (d) $k=800$. There is indication of multiple coexisting attractors in the underlying frozen system, and the Lyapunov exponents of these attractors vary with time.

B. Forced Duffing oscillator

We now present an example from continuous-time flows. We consider the forced Duffing equation which models the mechanical oscillations of a cantilever beam [13]

$$\dot{x}=y, \quad \dot{y}=x-x^3-0.25y+0.3\cos(\omega\theta), \quad \text{and } \dot{\theta}=1. \tag{10}$$

The driving angular frequency ω is chosen to vary with time in the interval [1.05, 1.1] to model nonstationarity. The total number of epochs is set to be 50 and the time duration of each epoch is 20. It is convenient to use the box-counting procedure to obtain the fractal dimension of the attractor in the (x,y) plane, say D_1'. Because of $\dot{\theta}=1$ the dimension of the attractor in the full phase space is $D_1=D_1'+1$. To obtain the Lyapunov dimension, the following procedure has been employed. We first calculate the Lyapunov dimension of the two-dimensional Poincaré map on the plane $\Sigma:\{(x,y,\theta)|_{\cos(\omega\theta)=0}\}$ by the formula

$$D_L' = 1 + \frac{\lambda_1}{|\lambda_2|}, \tag{11}$$

where λ_1 and λ_2 are the largest and the smallest Lyapunov exponents of system (10), respectively, which satisfy $\lambda_2<0<\lambda_1$. By including the time dimension, the Lyapunov dimension of the chaotic attractor in system (10) can be obtained as $D_L=D_L'+1$.

In our numerical experiments, 10 000 initial conditions have been used to calculate D_1 and D_L. For these initial points, $\theta|_{t=0}=0$, x and y are chosen randomly in the small interval $[-1.5,1.5]$. Figures 5(a) and 5(b) are typical ensemble snapshots. These projections of the attractor on (x,y) planes are apparently fractal. Figure 6 illustrates the information dimension D_1 and the Lyapunov dimension D_L as a function of time. It can be seen that the two dimensions agree with each other reasonably well, indicating the validity of the Kaplan-Yorke formula for continuous-time nonstationary dynamical systems.

V. DISCUSSIONS

In summary, we have demonstrated that ensemble snapshots can be used to characterize nonstationary chaotic systems in terms of Lyapunov exponents and fractal dimensions. Indeed, the snapshot technique can reveal the fractal structure of the underlying chaotic system, despite nonstationarity. We have presented evidence for the validity of the

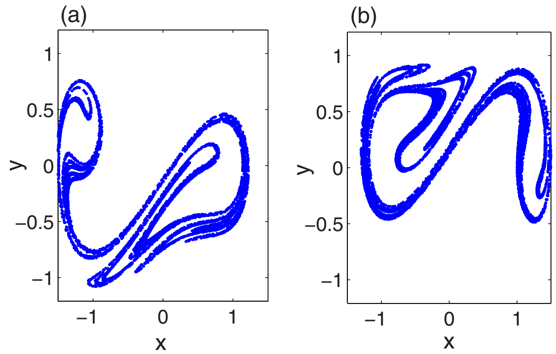

FIG. 5. (Color online) Ensemble snapshots observed for the 10th and the 30th epoch in the forced Duffing system given by Eq. (10). All trajectories are initiated randomly in the small phase-space region defined by $-1.5 \leq (x,y) \leq 1.5$.

Kaplan-Yorke formula, both for a discrete-time map and for a continuous-time flow.

The methodology developed in this paper is suitable for studying nonstationary chaotic systems numerically. In experimental systems where obtaining an ensemble of trajectories is difficult, the use of our method is limited. There are, however, experimental situations where the evolution of a large number of trajectories can be determined simultaneously, such as the dynamics of floaters on the surface of a chaotic flow, where previous experiments focused on the characterization of fractal attractors by using the Kaplan-Yorke formula under the assumption of random (but stationary) dynamical systems. Our results suggest that the same technique can be applied experimentally to studying the fractal patterns generated by chaotic dynamics even when the system is nonstationary.

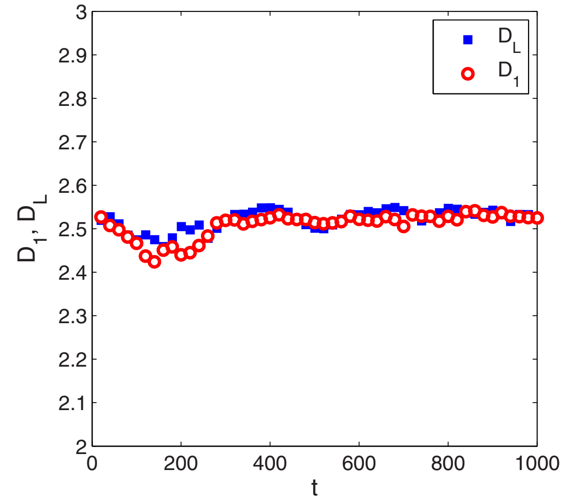

FIG. 6. (Color online) Time evolutions of the Lyapunov dimension D_L and the information dimension D_1.

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