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Dynamics-based scalability of complex networks

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We address the fundamental issue of network scalability in terms of dynamics and topology. In particular, we consider different network topologies and investigate, for every given topology, the dependence of certain dynamical properties on the network size. By focusing on network synchronizability, we find both analytically and numerically that globally coupled networks and random networks are scalable, but locally coupled regular networks are not. Scale-free networks are scalable for certain types of node dynamics. We expect our findings to provide insights into the ubiquity and workings of networks arising in nature and to be potentially useful for designing technological networks as well.

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Scalability is an important issue in many branches of science and engineering. For example, in biology, synchronization can occur in systems of different sizes, ranging from neuronal and cellular networks to population dynamics in natural habitats of vast distances. In computer science, whether a particular program can work in systems containing orders-of-magnitude different numbers of components is always a pressing issue. Similar scalability issues arise in large-scale circuit designs. Our interest here is in *dynamics-based scalability* of complex networks. In particular, we ask, if a dynamical phenomenon of interest occurs in networks of size N_1, can the same phenomenon be anticipated in networks of size N_2, where N_2 is substantially larger than N_1? More importantly, does the scalability so defined depend on the network topology? Addressing these questions can provide insights into fundamental issues such as the ubiquity of certain types of networks in nature with respect to specific dynamical functions. A good understanding of the scalabilities of networks of different topologies can also be important for practical design of various technological networks. Despite extensive research on complex networks in recent years, the issue of network scalability has not been considered.

To address the issue of network scalability, we focus on synchronization, a fundamental type of collective dynamics in natural systems [1], and investigate the interplay between synchronization-based scalability and network topology. The distinct type of network topologies included in our pursuit are globally connected, locally coupled regular, random [2,3], and scale-free [4]. We assume an identical nonlinear dynamical process on every node. The associated master-stability function (MSF) [5–7] $\Psi(K)$ can then be determined, where K is a generalized coupling parameter. Let $0=\lambda_1<\lambda_2\leqslant\cdots\leqslant\lambda_N$ be the eigenvalue spectrum of the coupling (Laplacian) matrix $\mathbf{L}$ for a given network. The system allows a stable synchronization state if, for all $i=2,\ldots,N$, $\Psi(K_i)$ is negative [6], where $K_i=\varepsilon\lambda_i$ and ε is the actual coupling strength. There are three typical classes of node dynamics under which synchronization can occur [8,9]: (class I) $\Psi(K)<0$ in a single finite interval (K_a,K_b); (class II) $K_b\to\infty$; and (class III) $\Psi(K)<0$ in several distinct intervals $(K_{a1},K_{b1}),(K_{a2},K_{b2}),\ldots,(K_{af},K_{bf})$, where K_{bf} can be either finite or infinite. Consider, for example, class-I node dynamics. The stability condition becomes $K_a<\varepsilon\lambda_2\leqslant\varepsilon\lambda_N<K_b$. As a result, we shall analyze the dependence of λ_2 and λ_N on parameters N so that regions in the two-dimensional parameter plane (N,ε), where the underlying network is synchronizable, can be determined analytically. Since the scalability results for class-III node dynamics can be inferred from those from class-I and class-II dynamics, and class II is actually a special case of class I (a synchronizable system under class-I node dynamics is also synchronizable under class-II dynamics), it is convenient to focus on class-I dynamics and discuss situations of class-II dynamics where the network is unsynchronizable for class-I dynamics.

The main results of this Rapid Communication are as follows. For globally connected and random networks, for any system size N, there exists a nonzero coupling-parameter interval $(\varepsilon_a,\varepsilon_b)$ for which the network is synchronizable [6]. However, for locally coupled regular and scale-free networks, no such interval exists for sufficiently large system size if K_b is finite. That is, these networks cannot be synchronized if their sizes are too large when the node dynamics belong to class I. For class-II node dynamics, scale-free networks can be synchronized, but locally coupled regular networks require arbitrarily large coupling to be synchronized so that they are practically not scalable. Our findings can provide insights into some fundamental issues in sciences and engineering. For example, in biology, synchronization can occur on networks of various sizes [1]. However, large scale-free networks can be unsynchronizable and, hence, the scale-free topology may not be important, or less ubiquitous, in situations where synchronization is key to system functions. From the standpoint of network design to achieve some desired synchronization-dependent performance, random networks are advantageous.

(1) Globally coupled networks. For such a network, every node is coupled to all other nodes in the network ($L_{ii}=N-1$, $L_{ij}=-1$ if $i\neq j$) and we have $\lambda_1=0$ and $\lambda_2=\cdots=\lambda_N=N$. The network is synchronizable if $K_2=\lambda_2\varepsilon>K_a$ and $K_N=\lambda_N\varepsilon<K_b$. Synchronization is stable if $K_a/\lambda_2<\varepsilon<K_b/\lambda_N$, and we thus have $\varepsilon_a=K_a/N$ and $\varepsilon_b=K_b/N$ and, hence, $\Delta\varepsilon=(K_b-K_a)/N\sim N^{-1}$. That is, for any physical network whose size is finite, there exists a finite interval of the coupling parameter for which synchronization can be achieved. The behavior is shown in Fig. 1(a), where the shaded strip in the

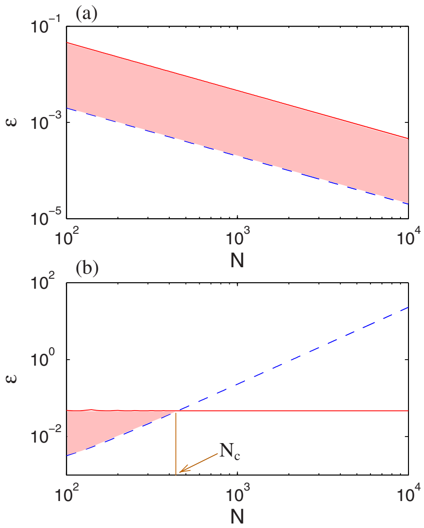

FIG. 1. (Color online) Synchronizable region (shaded) in the parameter plane (N,ε) for (a) globally coupled networks, and (b) locally coupled regular networks with fixed average degree $\langle k\rangle=80$. The node dynamics is described by the chaotic Rössler oscillator: $d\mathbf{x}/dt=\mathbf{F}(\mathbf{x})\equiv[-(y+z),x+0.2y,0.2+z(x-9)]^T$, for which we find $K_a\approx 0.2$ and $K_b\approx 4.6$.

parameter plane (N,ε) (on a logarithmic scale) indicates the synchronizable region. For any fixed system size, as the coupling parameter is increased, the network can undergo transitions from desynchronization to synchronization, and to desynchronization again. A remarkable feature is that, for a reasonably fixed coupling parameter, as its size is increased, a network can go from being desynchronized to being synchronized and then to being desynchronized again. This means that a globally coupled network can be synchronized if its size is neither too small nor too large. There exists an optimal range of the system size for which synchronization can be achieved. This is basically a system-size resonance phenomenon [10].

(2) Locally coupled regular networks. In such a network, every node is connected to m nearest neighbors, i.e., $\langle k\rangle=m$. We assume periodic boundary conditions. The elements of the Laplacian matrix $\mathbf{L}$ are then $L_{ii}=m$, $L_{ij}=-1$ for $j=i\pm 1,\ldots,i\pm m/2$, and $L_{ij}=0$ otherwise. The eigenvector associated with λ_2 is

$$\mathbf{e}_2=\sqrt{2/N}[\sin(2\pi/N),\sin(4\pi/N),\ldots,\sin(2\pi)]^T,$$

where $(\cdot)^T$ denotes the transpose. The key eigenvalue λ_2 can then be expressed as $\lambda_2=\mathbf{e}_2^T\cdot\mathbf{L}\cdot\mathbf{e}_2=\Sigma_{i,j=1}^N L_{ij}e_{2i}e_{2j}$, where $e_{2i}=\sqrt{2/N}\sin(2\pi i/N)$. After lengthy algebra, we obtain

$$\lambda_2=m+2-2\cos\left(\frac{m\pi}{2N}\right)\frac{\sin(\pi/N+m\pi/2N)}{\sin(\pi/N)}.$$

The largest eigenvalue can be obtained by manipulating

$$\lambda_N=\mathbf{e}_N^T\cdot\mathbf{L}\cdot\mathbf{e}_N=\Sigma_{i,j=1}^N L_{ij}e_{Ni}e_{Nj},$$

where $e_{Ni}=\sqrt{2/N}\sin[(2\pi j/(N/f)]$ and f is the basic spatial Fourier frequency, an integer between 1 and $N/2$. A similar calculation gives

$$\lambda_N=m+2-2\cos\left(\frac{m\pi}{2N/f}\right)\frac{\sin(\pi/N/f+m\pi/2N/f)}{\sin(\pi N/f)}.$$

Because of the frequency dependence of λ_N, the upper bound of the synchronizable parameter interval is given by $\varepsilon_b=K_b/\lambda_{N,\max}$, where $\lambda_{N,\max}=\lambda_N(f_c)$ and f_c is given by $f_c=\min\{f|d\lambda_N(f)/df=0\}$. For a sparse network, we have $\langle k\rangle=m\ll N$. In this case, the expression for $\lambda_2(N)$ and $\lambda_{N,\max}(N)$ can be further simplified by proper Taylor expansions. We obtain $\lambda_2\approx\pi^2(m+2)(m^2+m+1)/6N^2$ and $\lambda_N\approx m$. That is, $\varepsilon_a=\alpha N^2$ and $\varepsilon_b=K_b/m$ (independent of N), where $\alpha\equiv 6K_a/[\pi^2(m+2)(m^2+m+1)]$. We then have $\Delta\varepsilon=K_b/m-\alpha N^2$. The key feature that distinguishes a locally coupled network from a globally coupled network is the existence of a critical system size, above which the network is unsynchronizable, regardless of the coupling. Our analysis gives the following formula for the critical system size: $N_c=\sqrt{K_b/(m\alpha)}$. In principle, knowing the specific node dynamics (which gives specific values of K_a and K_b), we can predict N_c. A typical behavior of the network synchronizability in the parameter plane (N,ε) is shown in Fig. 1(b). We see that locally coupled regular networks are unscalable for class-I node dynamics. Physically, this could be understood that for globally coupled networks, the number of links per node increases with network size, while for locally coupled regular networks the number of links per node is a constant. Thus as network size increases, the network distance becomes larger and it is more difficult for a node to communicate with its diametrical counterparts, leading to degraded synchronizability. For class-II dynamics, $K_b\to\infty$, the stability condition becomes $\varepsilon>K_a/\lambda_2$, which can be satisfied in principle. A practical issue is that, since λ_2 can be small for large N, the coupling parameter needs to be unreasonably large, e.g., $\varepsilon>\alpha N^2$, for synchronization to occur. If there exists a limit of the coupling parameter, say ε_u, the critical network size is given by $N_c=\sqrt{\varepsilon_u/\alpha}$. In this sense, locally coupled regular networks are not scalable.

(3) Random networks. Let p be the probability for a pair of nodes to be connected. The average degree of the network is $\langle k\rangle=pN$. For the adjacency matrix $\mathbf{A}$ ($A_{ij}=-1$ if nodes i and j are connected, $A_{ij}=0$, otherwise, and $A_{ii}=0$), the distribution of the eigenvalues $\lambda_i^{(A)}$ follows the Wigner semicircle law [11], where the center of the semicircle is at zero. In particular, we have $\lambda_1^{(A)}\approx -Np$, $\lambda_2^{(A)}\approx -2\sqrt{Np(1-p)}$, $\lambda_N^{(A)}\approx 2\sqrt{Np(1-p)}$, and $\Sigma_i\lambda_i^{(A)}=0$. For the Laplacian matrix $\mathbf{L}$, where $L_{ij}=A_{ij}$ for $i\neq j$ and $L_{ii}=k_i$, we have $\lambda_1=0$ and $\mathrm{Tr}(\mathbf{L})=\Sigma_i k_i=N^2p$. The nontrivial eigenvalues are still distributed according to the semicircle law except that the center of the semicircle is now at Np. We thus have $\lambda_2\approx Np-2\sqrt{Np(1-p)}$ and $\lambda_N\approx Np+2\sqrt{Np(1-p)}$, which give $\varepsilon_a=K_a/[Np-2\sqrt{Np(1-p)}]$ and $\varepsilon_b=K_b/[Np+2\sqrt{Np(1-p)}]$.

Random networks arising in nature are typically sparse [3]. For a sparse random network, the average degree satisfies $\langle k\rangle\ll N$ or $p\ll 1$. We thus obtain $\varepsilon_a\approx K_a/(\langle k\rangle-2\sqrt{\langle k\rangle})$ and $\varepsilon_b\approx K_b/(\langle k\rangle+2\sqrt{\langle k\rangle})$. A remarkable consequence is that, if $\langle k\rangle$ is fixed, both ε_a and ε_b are independent of the network size. As a result, arbitrarily large networks can be synchronized, insofar as the network becomes increasingly sparse

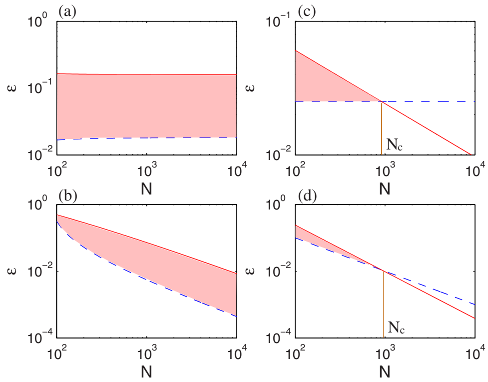

FIG. 2. (Color online) For the same node dynamics as in Fig. 1, synchronizable region (shaded) in the parameter plane (N, ε) for (a) random networks with fixed average degree $\langle k \rangle = 20$, (b) random networks with $\langle k \rangle = 0.05N$, (c) scale-free networks with degree exponent $\gamma = 3.5$ and fixed average degree $\langle k \rangle = 20$, and (d) scale-free networks with $\langle k \rangle = 0.05N$.

and the coupling strength falls in a constant interval [12]. The size of this interval does not decrease as N is increased, as exemplified in Fig. 2(a) for $\langle k \rangle = 20$. In this sense, random networks are more synchronizable than globally connected networks, as for the latter, the synchronizable parameter interval $\Delta\varepsilon$ decreases inversely with the increase of the system size [Fig. 1(a)]. Note, however, if $\langle k \rangle$ increases with N (e.g., $\langle k \rangle = pN$ and p is fixed), then for large N we have $\langle k \rangle \gg \sqrt{\langle k \rangle}$ and, hence, $\Delta\varepsilon \approx (K_b - K_a)/\langle k \rangle \sim 1/N$, as shown in Fig. 2(b). This is similar to the synchronization behavior of a globally connected network. Thus random networks are scalable for all three classes of node dynamics.

(4) Scale-free networks. For a scale-free network, the degree distribution follows a power law [4]: $P(k) = ak^{-\gamma}$ for $k \geqslant m_0$, where $\gamma > 0$ is the degree exponent and a is a constant. The minimum degree is $k_{\min} = m_0$. The constant a can be determined by $\int_{m_0}^{\infty} P(k)dk = 1$. For a scale-free network of infinite size, the maximum degree $k_{\max}$ diverges. However, for any physical network, its size is finite. One can consider the average number of nodes that have degrees larger than $k_{\max}$, which is $N\int_{k_{\max}}^{\infty} P(k)dk$. If this number is less than 1, $k_{\max}$ is the largest degree. This condition yields $\int_{k_{\max}}^{\infty} P(k)dk \approx 1/N$, which gives $k_{\max} \approx m_0 N^{1/(\gamma-1)}$. For scale-free networks, we have $\lambda_2 \approx Ck_{\min}$, where the constant C is of the order of unity, and $\lambda_N \approx k_{\max}$ [13]. Thus λ_2 is independent of the system size but λ_N increases with N as a power law. We then have $\varepsilon_a \approx K_a/(Cm_0)$ and $\varepsilon_b \approx (K_b/m_0)N^{-1/(\gamma-1)}$ and, consequently, $\Delta\varepsilon \approx (K_b/m_0)N^{-1/(\gamma-1)} - K_a/(Cm_0)$. The point is that there exists a critical system size $N_c \approx [K_a/(CK_b)]^{-(\gamma-1)}$, above which synchronization is impossible. The synchronizable region in the (N, ε) plane is shown in Fig. 2(c) for scale-free networks of fixed average degree. A qualitatively similar behavior occurs when the average degree increases with the system size, as shown in Fig. 2(d) for $\langle k \rangle = 0.05N$. Thus large scale-free networks are not synchronizable if the node dynamics belongs to class I. For class-II dynamics, since λ_2 does not decrease with network size N, synchronization can occur when the coupling parameter is in a proper range, regardless of the system size. Therefore, scale-free networks are not scalable for class-I node dynamics but are scalable for class-II dynamics. An implication is that, if synchronization is important for the functions of some complex networked systems, the scale-free topology should not be the choice if the node dynamics has a finite K_b. Likewise, in biological situations where synchronization can occur in systems of all kinds of sizes, we expect the random-network topology to be more pervasive since it is scalable for all cases.

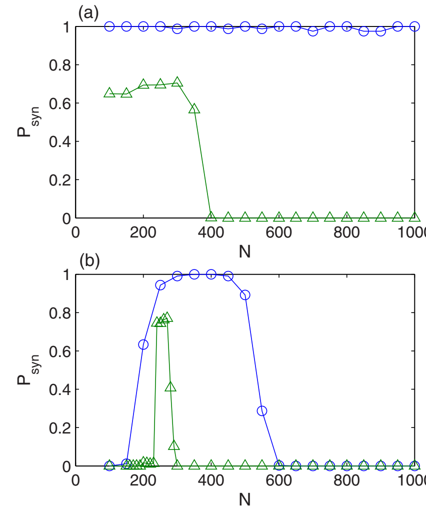

FIG. 3. (Color online) Synchronization probability versus network size. (a) Fixed average degree $\langle k \rangle = 20$ for random (circles, $\varepsilon = 0.05$) and scale-free (triangles, $\varepsilon = 0.035$, $\gamma = 3.5$) networks. (b) Average degree proportional to network size: $\langle k \rangle = 0.05N$, for random (circles, $\varepsilon = 0.1$) and scale-free (triangles, $\varepsilon = 0.06$, $\gamma = 3.5$) networks. Simulation parameters are $T_0 = 3000$ and $\delta = 0.01$. Each data point is from 1000 network realizations.

We now provide direct numerical support for our analysis. To compare with eigenvalue analysis we again use class-I node dynamics. The oscillatory networked system is described by $d\mathbf{x}_i/dt = \mathbf{F}(\mathbf{x}_i) - \varepsilon\sum_{j=1}^{N} L_{ij}\mathbf{H}(\mathbf{x}_j)$, where $\mathbf{F}(\mathbf{x}) = [-(y+z), x+0.2y, 0.2+z(x-9)]^T$ and $\mathbf{H}(\mathbf{x}) = [x, 0, 0]^T$ is a coupling function. Because of the complexity of the system dynamics, the degree of synchronization can be characterized only statistically. In particular, we define the synchronization probability P_{syn} as the probability that the fluctuation width $W(t)$ of the system is smaller than a small number δ (chosen somewhat arbitrarily) at all time steps during an interval T_0 in the steady state, where $W(t) = \langle |x(t) - \langle x(t) \rangle| \rangle$, and $\langle \cdot \rangle$ denotes the average over nodes of the network. In computation, P_{syn} can be calculated by the ensemble average, i.e., the ratio of the number of synchronized cases over the number of all random network realizations. Figure 3 shows P_{syn} versus the system size for both random and scale-free networks, where Fig. 3(a) corresponds to the situation where $\langle k \rangle = 20$ and Fig. 3(b) is for $\langle k \rangle = 0.05N$. Indeed, for random networks of fixed average degree, synchronization can occur for all system sizes tested [open circles, Fig. 3(a)]. However, a scale-free network with fixed average degree cannot be synchronized if its size becomes too large [open triangles, Fig. 3(a)]. When the average degree of the network is proportional to its size, for both random and scale-free networks, for a fixed coupling parameter, there exists a range of system sizes with which synchronization can occur [Fig. 3(b)]. These results agree with those from our spectral analysis.

In summary, we have addressed the fundamental issue of scalability in both complex and regular networks, by focusing on their synchronizabilities. Our analysis indicates that random networks are scalable in the sense that they are synchronizable, regardless of their sizes, insofar as the coupling parameter is chosen properly. However, scale-free networks are scalable only for certain types of node dynamics. For the regular topology, globally coupled networks are scalable but locally coupled networks are not. Investigating network scalability not only can provide a better understanding of the workings of networks in nature, but also is important for designing technological networks, notably computer networks in information infrastructure.

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[7] We address network scalability by focusing on the network's ability to synchronize, not on actual synchronization. If one were to consider actual synchronization, it would be difficult to obtain general results as the synchronization would depend on many specific details such as initial conditions. Thus, when we say that a type of network is scalable in the sense of synchronization, we mean only that a network can be synchronized, regardless of its size, if the coupling parameter and initial conditions can be adjusted. In contrast, if a class of networks is not scalable, a network absolutely cannot be synchronized if its size exceeds a critical value, regardless of how coupling or initial conditions are adjusted. It is in this sense of scalability which makes the MSF formalism meaningful.

[8] We have carried out a comprehensive study to calculate the MSFs for all known typical nonlinear oscillators under all commonly used coupling configurations. For instance, for a three-dimensional system (x_1,x_2,x_3), there are nine commonly used, linear coupling configurations: 11, 12, 13, 21, 22, 23, 31, 32, and 33, and we have calculated the MSFs for all these configurations. [Here the notation ij $(i,j=1,2,3)$ stands for the coupling scheme $\delta_{lj}x_i$, where δ_{lj} is the Kronecker delta.] The dynamical systems that we have tested include the Rössler oscillator, Lorenz oscillator, Chua's circuit, Chen's oscillator, Duffing's oscillator, and the Van der Pol system. The results are the following. The Rössler oscillator with 11 coupling, Lorenz oscillator with 21 coupling, Chua's circuit with 31 and 33 couplings, and Chen's system with 33 coupling all belong to class I. The Rössler oscillator with 22 and 31 coupling, Lorenz oscillator with 11, 22, and 12 couplings, Chua's circuit with 11, 22, 12, 23, and 21 couplings, Chen's oscillator with 22 and 12 coupling, driven Duffing system with 11 and 22 couplings, and driven Van der Pol oscillator with 11, 22, and 12 couplings belong to class II. Lorenz oscillator with 33 coupling, Driven Duffing system with 12 and 21 couplings, and Van der Pol oscillator with 21 coupling belong to class III. All other coupling schemes do not give a negative $\Psi(K)$ region. Such coupled systems thus cannot be synchronized by adjusting coupling parameters. The most astonishing finding is that, regardless of the differences in the details of the oscillator dynamics, for most oscillators there exists a coupling configuration for which the MSF is negative in a finite parameter interval.

[9] In relation to biological systems, we have calculated MSFs for a Hindmarsh-Rose neuron using parameters and coupling functions from M. Dhamala, V. K. Jirsa, and M. Ding [Phys. Rev. Lett. **92**, 028101 (2004)], which also fits in our classification: 21 coupling belongs to class-I dynamics; 11, 12, and 22 couplings belong to class-II dynamics; and all other coupling schemes do not give a negative $\Psi(K)$ region.

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[12] An underlying assumption in our study, as in any previous studies on network synchronization, is that the network must remain connected. The connectedness of a network can be characterized by its eigenvalue spectrum. In particular, a network is connected if λ_2 of the associated coupling matrix is not zero. Thus, insofar as $\lambda_2 > 0$, a typical network is connected. This in turn provides a criterion for connectivity of large random networks: $p > 4/(N+4) \approx 4/N$, which is consistent with the result by Erdös and Rényi [2].

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