Effect of noise on generalized chaotic synchronization

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When two characteristically different chaotic oscillators are coupled, generalized synchronization can occur. Motivated by the phenomena that common noise can induce and enhance complete synchronization or phase synchronization in chaotic systems, we investigate the effect of noise on generalized chaotic synchronization. We develop a phase-space analysis, which suggests that the effect can be system dependent in that common noise can either induce/enhance or destroy generalized synchronization. A prototype model consisting of a Lorenz oscillator coupled with a dynamo system is used to illustrate these phenomena.

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One of the remarkable discoveries in nonlinear science is that random noise can induce order. For instance, in stochastic resonance [1], noise can enhance system's response to input signals in terms of measures such as the signal-to-noise ratio. In coherence resonance [2], noise can induce ordered oscillations or enhance the temporal regularity of system response. It has also been found that noise, when applied identically to a set of nonlinear oscillators, can induce or enhance synchronization among them, even in the absence of coupling [3,4]. This phenomenon has recently been extended to chaotic phase synchronization [5].

In this paper, we investigate the effect of common noise on generalized synchronization [6–10], which occurs in systems of characteristically different chaotic oscillators. In particular, say we consider two chaotic systems: A described by $\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n)$, where $\mathbf{x}\in\mathbf{R}^N$, and system B by $\mathbf{y}_{n+1}=\mathbf{G}(\mathbf{y}_n)$, where $\mathbf{y}\in\mathbf{R}^M$. When there is a coupling between the two systems, even when the map functions $\mathbf{F}$ and $\mathbf{G}$ are different, there can be a functional relation between the dynamical variables $\mathbf{x}$ and $\mathbf{y}$ when generalized synchronization occurs. Our question is under what conditions common noise can enhance or induce generalized synchronization. As this type of synchronization can be expected in quite general settings of dynamical systems, a clarification of whether noise can be constructive or destructive is an interesting issue with practical implications. For instance, in a biological network, individual oscillators are responsible for different functions, so dynamically they can be quite different. Nonetheless, some global function of the network may rely on some degree of coherence among the dynamics of the oscillators. Whether noise can be beneficial for the coherence can be an issue of interest. The main point of this paper is that noise can either enhance and induce or weaken and destroy generalized synchronization, depending on systems details, such as the driving-driven configuration. That is, common noise may not always be beneficial for chaotic synchronization. We shall develop a theoretical criterion, based on a phase-space analysis of the conditional Lyapunov exponents (to be defined below), for pinning down the role of noise in generalized synchronization. To numerically illustrate both the constructive and the destructive role of noise, we will use the chaotic Lorenz oscillator [11] and a dynamo system [12] with different driving-driven configurations.

A convenient method to detect generalized synchronization is the auxiliary-system approach [7]. In particular, suppose there is a unidirectional coupling from system A to B. That is, A is the driving and B is the driven system, so we write $\mathbf{y}_{n+1}=\mathbf{G}(\mathbf{y}_n,\mathbf{x}_n)$, where now the map function depends also on the driving variable $\mathbf{x}$. One can imagine a replica B' of system B, which is also driven by A. Synchronization between B and B' under the identical driving from A implies generalized synchronization between A and B. This approach is similar to the subsystem approach suggested by Pecora and Carroll in their original work on chaos synchronization [13], where A and B are two constituting subsystems in a single dynamical system, and B' is a replica of B. In both cases, whether B and B' are synchronized is determined by the sign of the largest conditional Lyapunov exponent of B under the driving from A. Given a trajectory $\{\mathbf{y}_n\}_{n=0}^{N-1}$ on B, the following matrix product determines this exponent:

$$\mathbf{Q}\equiv\prod_{n=0}^{N-1}\mathbf{DG}(\mathbf{y}_n,\mathbf{x}_n), \qquad (1)$$

where $\mathbf{DG}(\mathbf{y}_n,\mathbf{x}_n)=\partial\mathbf{G}/\partial\mathbf{y}|_{\mathbf{y}_n,\mathbf{x}_n}$ is the Jacobian matrix of the driven system evaluated at the trajectory point $(\mathbf{y}_n,\mathbf{x}_n)$. The largest conditional exponent is given by

$$\lambda_1=\lim_{N\to\infty}\frac{1}{N}\ln|\mathbf{Qu}|, \qquad (2)$$

where $\mathbf{u}$ is a randomly chosen unit vector at the initial point $\mathbf{y}_0$ in the driven system. When a trajectory of the driven system moves in its phase space, due to driving, there can be regions where an infinitesimal tangent vector expands or contracts. Let Σ_E and Σ_C denote the unions of the expanding and the contracting regions, respectively. We can classify the

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Jacobian matrices into two types: One for trajectory points in Σ_E and another for points in Σ_C. Due to chaos, $\mathbf{Q}$ in Eq. (1) can be regarded as a product of random matrices, one type drawn from the collection of all possible matrices in Σ_E and another from the collection in Σ_C. The time average in Eq. (2) can then be replaced by a microcanonical ensemblelike average (for fixed N) [14]

$$\lambda_1 = \langle \ln|\mathbf{Qu}|\rangle_m. \tag{3}$$

Since the matrices in the product are drawn from two types of collection in the ensemble, the ensemble average is effectively the weighted average with respect to them. In particular, let $r_e > 0$ and $-r_c < 0$ be the values of the ensemble averages as in Eq. (3) but with matrices in the products all drawn from the collection in Σ_E and in Σ_C, and let p_e and p_c be the respective probabilities of selecting a matrix from the collection in Σ_E and in Σ_C, respectively. We can write

$$\lambda_1 = p_e r_e - p_c r_c, \tag{4}$$

where p_e and p_c can be interpreted as the *weights* of Σ_E and Σ_C for a continuous trajectory, and r_e and r_c are the average expanding and contracting rates of Σ_E and Σ_C, respectively. In the computation, one can take a sufficiently long trajectory in the phase space, and calculate the local expansion or contraction rates by using the set of individual Jacobian matrices along the trajectory. The probabilities in Eq. (4) can be approximated by the frequencies of visits to distinct phase-space regions. The largest conditional Lyapunov exponent λ_1 can then be obtained. Under common noise, all four quantities depend on the noise amplitude. Suppose, in the absence of noise, the combination of the four quantities is such that $\lambda_1 \gtrsim 0$. Then, common noise of appropriate amplitude can cause changes in them so that λ_1 becomes negative. In this case, noise induces generalized synchronization. Noise can also enhance the synchronization if it makes an originally negative exponent more negative, through changes in these average quantities. There is also the possibility that the noise-induced changes in the quantities are such that an originally negative exponent becomes positive. In this case, common noise can destroy generalized synchronization.

A few remarks are in order. (1) While the random-matrix-based arguments leading to Eq. (4) and their interpretations are for the setting of maps (for notational convenience), they apply equally to flows. (2) A result similar to Eq. (4) has been used in other contexts, such as the bifurcation from strange nonchaotic to chaotic attractors [15] and the transition to chaotic attractors in random dynamical systems [16], but the arguments here are more formal. (3) The idea of identifying expanding and contracting regions in the phase space has been used to understand the effect of common noise on (phase) synchronization [5], but here we emphasize the division of the *full* phase space into these regions. As we will demonstrate numerically, division in terms of subspaces, such as phase-space projections, may not capture the variations of the quantities in Eq. (4) with the noise amplitude.

To provide numerical support, we use the classical chaotic Lorenz oscillator under noise, given by

$$\frac{dx}{dt} = 10(y - x),$$

$$\frac{dy}{dt} = 35x - y - xz + D\xi(t),$$

$$\frac{dz}{dt} = xy - (8/3)z, \tag{5}$$

and the chaotic dynamo oscillator in Ref. [12],

$$\frac{du}{dt} = vw - 1.7u,$$

$$\frac{dv}{dt} = (w - 0.5)u - 1.7v + D\xi(t),$$

$$\frac{dw}{dt} = 1 - uv, \tag{6}$$

where D is the noise amplitude and $\xi(t)$ is a Gaussian random process of zero mean and unit variance, identical for both oscillators. For illustrative purposes, we shall consider two cases where: (i) the Loretz oscillator drives the dynamo system, which can be modeled by adding the coupling term $-\epsilon(u-x)$ in the u-equation in Eq. (6), and (ii) the opposite situation where the coupling term $-\epsilon(x-u)$ is added to the x equation in Eq. (5). Here, ϵ is a parameter characterizing the "strength" of the coupling. We will demonstrate that common noise can induce or enhance generalized synchronization for case (i), but it does the opposite for case (ii).

• *Case (i) Noise induced/enhanced generalized synchronization*. Figure 1(a) shows the largest conditional Lyapunov exponent λ_1 of the dynamo oscillator versus the coupling parameter for four values of common-noise amplitude D. We see that for $D=0$, λ_1 becomes negative for $\epsilon > \epsilon_c \approx 0.16$, indicating generalized synchronization. As D is increased from zero, the transition point ϵ_c decreases toward zero. For a fixed value of ϵ, the value of λ_1 decreases as D is increased, suggesting that noise can *enhance* the synchronization. A verification of the noise-enhanced generalized synchronization for $\epsilon=0.05$ is shown in Fig. 1(b), where we see that for $D=0$, the variable w from the driven dynamo oscillator and the same variable w_a from an auxiliary system are not synchronized but they do for $D=1.5$. For relatively large noise amplitude (e.g., $D=1.5$ and $D=2.0$), λ_1 is negative for all values of ϵ considered, even for $\epsilon=0$. That is, even without coupling, generalized synchronization can occur if the common noise is suitably strong—the phenomenon of noise-induced generalized synchronization. In this case, although there is no direct coupling between the driving and the driven systems, a degree of indirect interaction between them can still be established by the common noisy forcing.

To demonstrate the necessity of using *full* phase-space division into expanding and contracting regions to understand the interplay between noise and generalized synchronization, we map out these regions in two different planes of phase-space projection, as shown in Figs. 2(a)–2(f). In par-

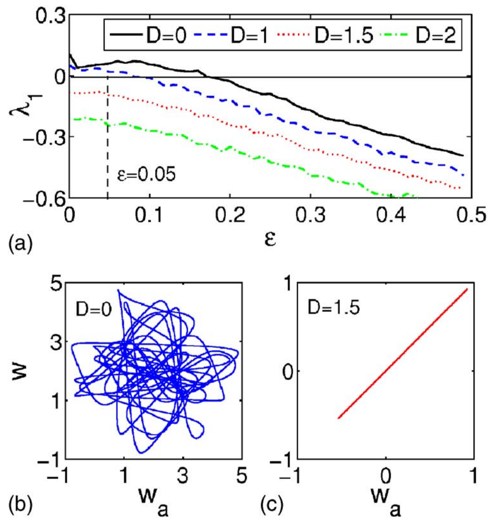

FIG. 1. (Color online) For case (i), (a) the largest conditional Lyapunov exponent of the driven dynamo oscillator versus the coupling parameter for four values of the common-noise amplitude, demonstrating both noise-enhanced and noise-induced generalized synchronization. (b) Verification of noise-enhanced generalized synchronization for $\epsilon=0.05$: for $D=0$ there is no synchronization between the dynamical variable w and its counterpart from an auxiliary system but the two are synchronized for $D=1.5$.

ticular, Fig. 2(a) shows, for $D=0$ and $\epsilon=0.5$, projection of the chaotic trajectory of the dynamo oscillator in the (u,w) plane and the contracting region (shadowed). A similar plot for $D=3$ is shown in Fig. 2(b), where we see that the trajectory appears to locate relatively more in the contracting region. Indeed, the probability of a visit to the contracting region, when calculated from the (u,w) subspace, increases with the noise amplitude, as shown in Fig. 2(c). However, when a different subspace is examined, the behavior can be

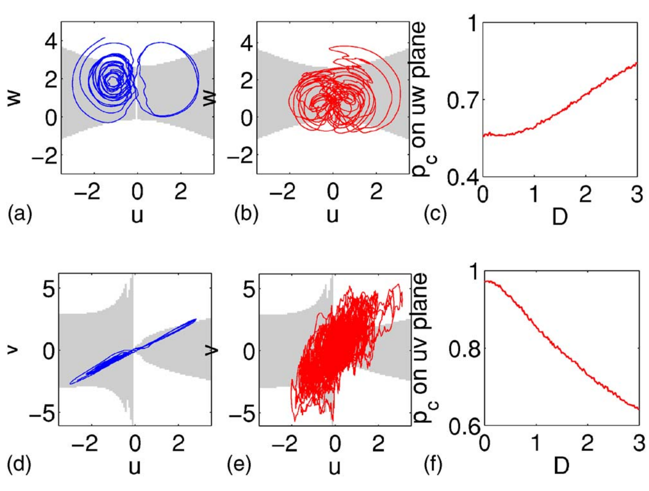

FIG. 2. (Color online) For case (i) for $\epsilon=0.5$, a chaotic trajectory of the driven dynamo oscillator and the contracting region (shadowed) in the (u,w) subspace for $D=0$ (a) and $D=3.0$ (b). In the (u,w) subspace, the probability of visit to the contracting region increases with the noise amplitude (c). (d,e) Similar plots to (a,b), respectively, but for the (u,v) subspace. $D=0$ for (d) and $D=3$ for (e). In this case, the probability decreases with D (f).

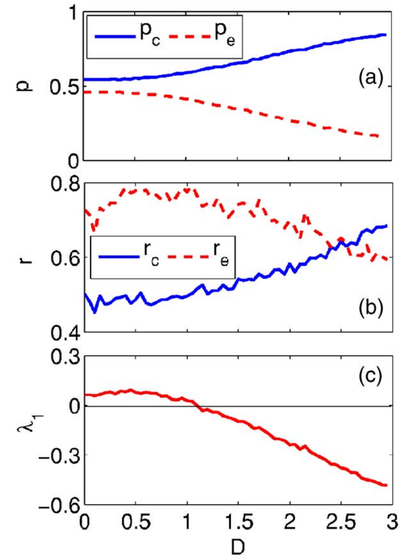

FIG. 3. (Color online) For case (i), an example of phase-space analysis for understanding noise-induced generalized synchronization: The frequencies of visit to the expanding and the contracting region (a), the average expanding and contracting rates (b), and the largest conditional Lyapunov exponent (c), versus the noise amplitude for $\epsilon=0.05$. The exponent is calculated by using Eq. (4).

completely the opposite, as shown in the corresponding plots in Figs. 2(d)–2(f). These results indicate that information for generalized synchronization obtained from some subspaces in the phase space can be incomplete and may lead to contradictory conclusions. To understand generalized synchronization based on the expanding and contracting dynamics, it is necessary to examine trajectories in the *full* phase space.

To exemplify the use of the phase-space analysis for understanding the effect of common noise on generalized synchronization, we calculate the frequencies of visit to the expanding and contracting regions, and the expanding and the contracting rates, as a function of the noise amplitude, as shown in Figs. 3(a) and 3(b), respectively, for $\epsilon=0.5$. We see that *both* p_e and r_e decrease with D, but *both* p_c and r_c increase with D. These lead to a decreasing behavior of the largest conditional exponent as D is increased, as shown in Fig. 3(c). We see that noise-induced generalized synchronization occurs for $D \gtrsim 1.1$ in this case.

Case (ii) Destruction of generalized synchronization by common noise. Comparing with case (i), here the roles of the driving and the driven system are exchanged in that now the Lorenz oscillator is driven by the dynamo system. To see how common noise can destroy generalized synchronization, in Figs. 4(a) and 4(b) we plot the probabilities for a trajectory in the expanding and contracting regions, and the average rates, respectively, as a function of the noise amplitude for $\epsilon=4.5$. Comparing Figs. 4(a) and 4(b) with Figs. 3(a) and 3(b), respectively, we see that, while the probabilities of visits show a similar behavior, the average expanding rate versus D exhibits a completely opposite behavior: It increases with D relatively more rapidly as compared with the increase of the average contracting rate. This large increase in r_e weighs over the tendency to decrease the largest conditional exponent by the other three quantities (p_e, p_c, and r_c). As a

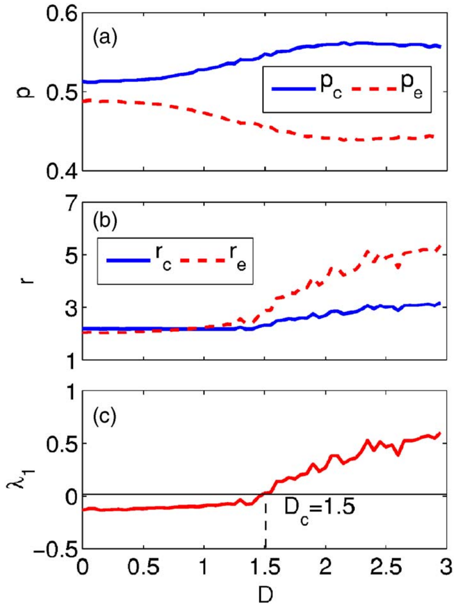

FIG. 4. (Color online) For case (ii), an example of destruction of generalized synchronization by common noise: The frequencies of visit to the expanding and the contracting region (a), the average expanding and contracting rates (b), and the largest conditional Lyapunov exponent (c), versus the noise amplitude for $\epsilon=4.5$. The main factor that causes the exponent to become positive is the relatively large increase of the average expanding rate with the noise amplitude. The exponent is calculated by using Eq. (4).

result, the exponent increases with D, as shown in Fig. 4(c). We see that for $D=0$, the exponent is negative but it becomes positive for $D>D_c\approx 1.5$, signifying the loss of generalized synchronization.

The destruction of the generalized synchronization by

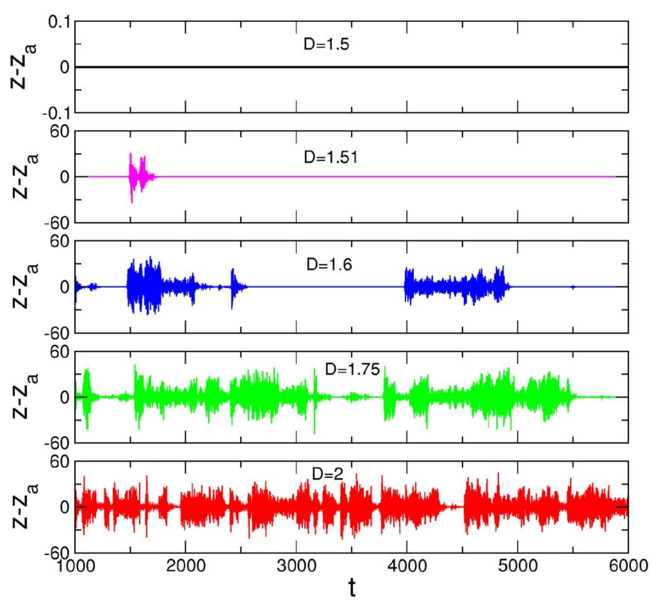

FIG. 5. (Color online) For case (ii), on-off intermittency associated with the destruction of generalized synchronization by common noise. Shown are the evolutions of the difference $z-z_a$ in the dynamical variable for five values of the noise amplitude.

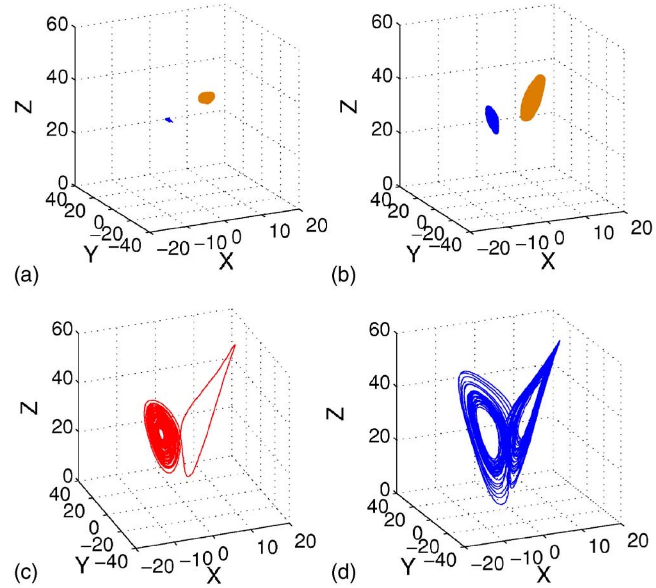

FIG. 6. (Color online) For the Lorenz oscillator under noise, (a,b) two isolated attractors for $D=0$ and $D=1.2$, respectively. A noise-induced interior crisis occurs for $D=D_c\approx 1.5$, after which there is a single attractor in the phase space. Shown in (c) is a segment of a trajectory on part of the chaotic saddle that is responsible for the crisis, for $D=1.5$. In (d), an attractor after the crisis is shown for $D=2.0$, which contains the original small attractors and the chaotic saddle.

common noise manifests itself as an on-off intermittent behavior when a dynamical variable of the driven system is compared with its counterpart in the auxiliary system. To demonstrate this behavior, we plot in Fig. 5 the evolutions of the difference $z-z_a$ in the dynamical variable z of the Lorenz oscillator between the driven and the auxiliary system for five values of the noise amplitude. For D slightly above D_c, generalized synchronization can still be achieved in relatively long time intervals, but as D is increased further, bursts from zero in $z-z_a$ becomes increasingly frequent. Desynchronization through on-off intermittency is typical in chaotic systems [17], and Fig. 5 indicates that the mechanism applies to generalized synchronization as well.

What is the dynamical mechanism for the large increase of the average expanding rate in the driven Lorenz oscillator? To answer this question, we examine the phase-space structure of the Lorenz oscillator more closely. Our recent work on generalized synchronization in the Lorenz oscillator [18] has indicated a bistable behavior: Depending on the initial conditions, the Lorenz oscillator can have two attractors that both can be synchronized with some characteristically different chaotic driving in the generalized sense. An example is shown in Fig. 6(a), the three-dimensional phase space of the Lorenz oscillator. Under noise, the ranges that the attractors extend expand, as shown in Fig. 6(b). As the noise becomes stronger, the attractors collide with a chaotic saddle, triggering an interior crisis [19] that generates a single and larger attractor, as shown in Figs. 6(c) and 6(d). The key point is that the chaotic saddle contains the unstable steady state of the Lorenz oscillator, which is near the origin of the phase space and has a strongly expanding local unstable manifold. After the interior crisis, the attractor contains this steady-state point. When a typical trajectory falls in its neighborhood, a large contribution to the average expanding rate is picked up. We thus expect a large increase in the rate after the crisis occurs. Numerically, we find that the critical noise amplitude for crisis is $D_c \approx 1.5$, which coincides with the value of D required for the destruction of the generalized synchronization.

In summary, we have addressed the role of common noise in chaotic generalized synchronization by using a phase-space analysis. Depending on the driving-driven configuration of the chaotic oscillators, under noise the dynamical properties of trajectories in the driven system can show characteristically different behaviors in terms of their frequencies to experience expansion and contraction and the corresponding rates. As a result, common noise can either induce/enhance or destroy generalized synchronization. For the specific example of the Lorenz-dynamo system, we are able to understand the observed noise-induced destruction of generalized synchronization in terms of the phenomenon of noise-induced interior crisis. As generalized synchronization can be expected in complex dynamical systems, our results are useful for assessing whether the role of inevitable environmental noise can be constructive or destructive for the synchronization.

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