Self-organized scale-free networks

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Growth and preferential attachments have been coined as the two fundamental mechanisms responsible for the scale-free feature in complex networks, as characterized by an algebraic degree distribution. There are situations, particularly in biological networks, where growth is absent or not important, yet some of these networks still exhibit the scale-free feature with a small degree exponent. Here we propose two classes of models to account for this phenomenon. We show analytically and numerically that, in the first model, a spectrum of algebraic degree distributions with a small exponent can be generated. The second model incorporates weights for nodes, and it is able to generate robust scale-free degree distribution with larger algebraic exponents. Our results imply that it is natural for a complex network to *self-organize* itself into a scale-free state without growth.

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Complex networks have become an area of tremendous recent interest [1–4] since the discoveries of the small-world [5] and scale-free [6] properties in many natural and man-made networks. A small-world network is characterized by a short network distance and a high clustering coefficient. Watts and Strogatz demonstrated that the two small-world characteristics can be obtained from a regular network by rewiring or adding a few long-range links (shortcuts), which connect otherwise distant nodes [5]. Indeed, since a regular network intrinsically already has a high clustering coefficient but has a large network distance, a few shortcuts can significantly reduce the network distance, while having little effect on the clustering coefficient. The scale-free property is characterized by an algebraic degree distribution: $P(k) \sim k^{-\gamma}$, where the degree variable k measures the number of links of node in the network, and $\gamma > 0$ is the algebraic scaling exponent. Barabási and Albert discovered the scale-free property and also proposed *growth* and *preferential attachment* as the two basic mechanisms responsible for the scale-free property [6,7]. Here, growth requires that the numbers of nodes and links increase with time and preferential attachment means that when a new node is added to the network, the probability that it connects to an existing node is proportional to the number of links that this node has already had.

Although growth and preferential attachment can indeed account for the scale-free property in many real-world complex networks, there are networks that are apparently scale-free but for which the growth mechanism seems to be lacking or not particularly relevant. Typical examples are biological networks [8–10]. For instance, the basic building blocks of a cell and the energy required for its survival come from a sequence of a large number of intracellular biochemical reactions that decompose complex molecules from food. These biochemical reactions can be regarded as links in a network, where the nodes are various chemicals participating in the reactions. This is the so-called metabolic network, which was shown to exhibit both scale-free [8] and small-world features [9]. It is hard to imagine that these networks, which are fundamental to all living cells, are constantly growing. It is only reasonable to assume that evolution and natural selection are the mechanisms responsible for the formation and function of these networks. That is to say, the networks and their scale-free property arise only because the links, or the various biochemical reactions, have been changing and optimizing themselves continuously in time. Consider also the various networks of neurons in the brain. The numbers of neurons, in general, do not increase during the lifetime. The evolution and development of the brain are accomplished by connections among neurons, which vary constantly in time. Although the scale-free feature seems to be universal, it is also important to note that in many complex networks, there is always an exponential component in the degree distribution [11,12]. In particular, the distribution typically contains an algebraic component for small degree and an exponential decay component for large degree. That is, the scale-free characterization is valid only to certain extent. Many realistic networks show the scale-free feature with algebraic scaling exponent between 2 and 3 and thus most studies thus far have focused on network structure with $\gamma \geq 2$. However, some biological networks with the scale-free feature have their algebraic exponents close to unity [10]. To be able to account for such scale-free networks with small algebraic exponent without the need to incorporate the growth mechanism is of great interest.

In this paper, we propose two classes of nongrowing network models that can naturally generate the scale-free feature in a *self-organized* manner. For the first model, starting from a regular network of a large number (fixed) of nodes, the links among various nodes can be removed and reestablished constantly in time. In particular, links are detached from nodes randomly in time. Once such an event occurs, the link can be rewired following a rule that contains a component of preferential attachment, which is controlled by a parameter $0 \leq \alpha < 1$. We will show that a network evolving

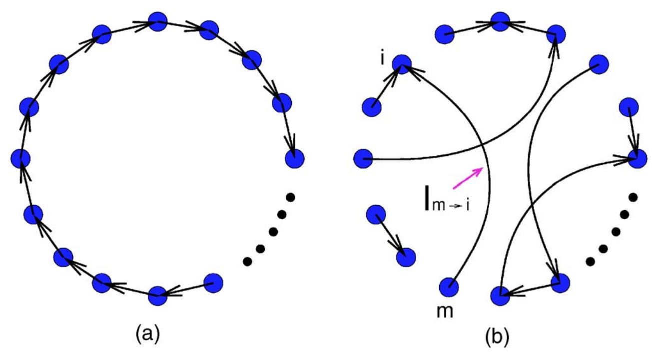

FIG. 1. (Color online) (a) Initial configuration: a one-dimensional regular network on a ring. Each node has one link that can be rewired. (b) The resulting network after a number of rewiring steps.

following such a simple rule generally reaches an *equilibrium* configuration with both algebraic and exponential degree distributions

$$P(k) \sim k^{-\gamma}e^{-\zeta k}, \tag{1}$$

where the algebraic exponent γ and the exponential rate ζ depend on the model parameter α. For the attachment rule that we used, analytic expressions of γ and ζ can be derived. Extensive numerical experiments using network size of as large as 10^5 nodes yield scaling results that are in good agreement with the theory. Considering that nodes and links can play different roles in terms of physical functions, we also introduce a weighted, nongrowing network model and show that it can generate robust scale-free behavior with value of the exponent γ in the range that fits many realistic networks (between 2 and 3). As most previous works emphasize growth as one of the fundamental, necessary mechanisms for the scale-free property, our results suggest that this may not always be true. In fact, a large, nongrowing network can evolve by itself into a scale-free state. As we have elaborated, such *self-organized scale-free* networks may find applications, particularly in biological systems. Since self-organized criticality has been speculated to be a universal mechanism for generating algebraic (power-law) behaviors in natural systems [13–15], it is reasonable that the scale-free feature in complex networks can arise through a self-organizing process.

In what follows, for clarity we focus our theoretical analysis on the first model (nonweighted). To obtain a self-organized scale-free network in the simplest possible way, we begin by generating a one-dimensional regular network of N (large) nodes on a ring, as shown in Fig. 1(a). Initially each node has one link that can be rewired (denoted by a solid line with arrow). Let $l_{m\rightarrow i}$ be the link between nodes m and i. The link is assumed to originate from the node m, i.e., the link cannot be detached from m, but it can be detached from node i (only incoming links can be detached from nodes) and reconnected to a third node in the network. At each time step, link $l_{m\rightarrow i}$ is chosen randomly and it is detached from node i and reconnected to node j according the following *rewiring probability*:

$$\Pi(k_j) = \frac{k_j-\alpha}{\sum_r (k_r-\alpha)} = \frac{k_j-\alpha}{N(2-\alpha)}, \tag{2}$$

where k_j is the number of links that node j has, $\alpha \geqslant 0$ is the control parameter and $\Sigma_j \Pi(k_j)=1$. Since the minimum number of rewirable links a node has initially is one, we have $\alpha < 1$. This rewiring process continues in time. One question is whether an equilibrium state can be reached in the sense that the degree distribution converges asymptotically.

To address this question, we use the master-equation approach. Let $P(k,t)$ be the degree distribution function at time t, i.e., the probability that a randomly selected node has exactly k links at time t. We focus on $P(k,t+1)$ and enumerate all possible contributions to it. There are four possible processes that can change $P(k,t)$. (i) At time t, a randomly selected link is detached from a node with $k+1$ rewirable links. Since there are N rewirable links in the network, the probability for this to occur is k/N. At time $t+1$ there is thus an increase of $(k/N)P(k+1,t)$ to $P(k,t)$. (ii) At time t, a randomly chosen link is detached from a node with k rewirable links. This causes a decrease in the amount of $[(k-1)/N]P(k,t)$ to $P(k,t)$. (iii) For nodes with $k-1$ links at time t, there is a probability that an additional link can be attached to one of them. This is the rewiring probability for nodes with $k-1$ links. There is thus an increase of $\Pi(k-1)P(k-1,t)$ to $P(k,t)$. (iv) Similarly, a node with k links at time t can lose one link with probability $\Pi(k)$, giving rise to a decrease of $\Pi(k)P(k,t)$ to $P(k,t)$. Summarizing all these contributions, we obtain the master equation for $P(k,t)$

$$\begin{aligned} P(k,t+1) &= P(k,t) + \frac{k}{N}P(k+1,t) - \frac{k-1}{N}P(k,t) \\ &\quad + \frac{(k-1)-\alpha}{N(2-\alpha)}P(k-1,t) - \frac{k-\alpha}{N(2-\alpha)}P(k,t). \end{aligned} \tag{3}$$

In the steady state, we have $P(k,t+1)=P(k,t)$ and, hence,

$$\begin{aligned} 0 &= kP(k+1) - (k-1)P(k) + \frac{(k-1)-\alpha}{(2-\alpha)} \\ &\quad \times P(k-1) - \frac{k-\alpha}{(2-\alpha)}P(k), \end{aligned} \tag{4}$$

where $P(k) \equiv P(k,t\rightarrow\infty)$. For network with large number of nodes, we can use the continuum limit $N\rightarrow\infty$ and treat k as a continuous variable. To obtain a differential equation for $P(k)$, we rewrite Eq. (4) as

$$\begin{aligned} 0 &= (k+1)P(k+1) - 2kP(k) + (k-1)P(k-1) - \frac{k}{2-\alpha}P(k) \\ &\quad + \frac{k-1}{2-\alpha}P(k-1) + kP(k) - (k-1)P(k-1) + \frac{\alpha}{2-\alpha}P(k) \\ &\quad - \frac{\alpha}{2-\alpha}P(k-1) - P(k+1) + P(k), \end{aligned} \tag{5}$$

which is equivalent to

$$\frac{d^2}{dk^2}\{kP(k)\} + \frac{d}{dk}\{(\zeta k + \gamma - 1)P(k)\} = 0, \tag{6}$$

where

$$\gamma = \frac{\alpha}{2-\alpha},$$

$$\zeta = 1 - 1/(2-\alpha). \tag{7}$$

Solution to Eq. (6) is the scaling law (1), with the algebraic exponent and exponential rate given by Eq. (7). We see that the degree distribution can indeed reach an equilibrium that contains both an algebraic and an exponential component. For $\alpha \approx 1$, we have $\zeta \approx 0$ so that the dominant contribution to $P(k)$ is algebraic. For $0 < \alpha < 1$, the degree distribution is algebraic for small k but has an exponential tail for large k. For $\alpha = 0$, we have $\gamma = 0$, so the degree distribution is entirely exponential. A prediction is then that the scale-free feature can exist in large, nongrowing networks through the mechanism of self-organization.

Intuitively, to see why a network can self-organize into a scale-free form for $\alpha \approx 1$, we note that both probabilities for a link to be detached from a node i with degree k and to be attached to it are $(k-1)/N$. This makes it possible that some nodes can capture a relatively large number of links. Let us consider n nodes with degree l and a node j with degree m, where $m = nl (\gg 1)$ and n is a small integer. The probabilities of attachment of a link to and detachment of it from node j are approximately the same as the corresponding probabilities for the n nodes. In general, in a steady state, $kNP(k)$ is distributed equally over large k. This is consistent with our theory $P(k) \sim 1/k$. When $\alpha = 0$, the probability of detachment of a link from a node with degree k is $(k-1)/N$, but the probability of attachment of a link to it is about $k/2N$. Thus the probability for a node to have large degree is relatively small. This tends to make the degree distribution uniform and, in fact, gives rise to an exponential distribution.

To provide numerical support for the scaling law (1), we generate a network of $N = 10^5$ nodes with initial configuration as shown in Fig. 1. We then perform the rewiring process according to the rule in (2). For network of this size, we find that the steady state can be reached typically after about $100N$ rewirings (or time steps). Figure 2 shows the time-averaged degree distributions defined by $\langle P(k,t)\rangle = \sum_{t=T_0}^{t=T_0+T} P(k,t)/T$, where $T \gg 1$ and $T_0 > \tau$ (to be defined below). Each data set in Fig. 2 is obtained for $\alpha \approx 1$. We observe deviations from the straight line for $k > k_{\max}$, where $k_{\max}$ increases as N does. Inset of Fig. 2 shows the asymptotic degree distribution for $\alpha = 0.997$ and $N = 10^5$. The distribution is apparently scale-free over two orders of magnitude of variation of k. Note that for this network of N nodes, the minimum value that $P(k)$ can have is $1/N$. For $N = 10^5$, $P(k)$ has the minimum value when $k = k_{\max} (\approx 200)$. Therefore, the distribution appears to be algebraic for k up to $k_{\max}$, but it dose not follow the algebraic behavior for $k > k_{\max}$. This is the reason why a cutoff in Fig. 2 occurs. The generated network has a peculiar sensitivity of the emergent structure to small deviations of the parameter α from 1, suggesting a narrow range for self-organizing scale-free structures. When α deviates from 1, the exponential component in the degree distribution becomes numerically observable, as shown in Figs. 3(a)–3(c), respectively, for $\alpha = 0.99$, 0.95, and 0.90, where the dashed lines indicate the theoretical prediction (1). The numerical data deviate from the theoretically estimated line when k is large, but the deviation can be reduced by increasing the network size. The agreement between numerics and theory is reasonable. The pure algebraic decay behavior in this nonweighted model can occur only when $\alpha \to 1$. For $\alpha = 0$, our theory predicts that the degree distribution is completely exponential, as shown in Fig. 3(d).

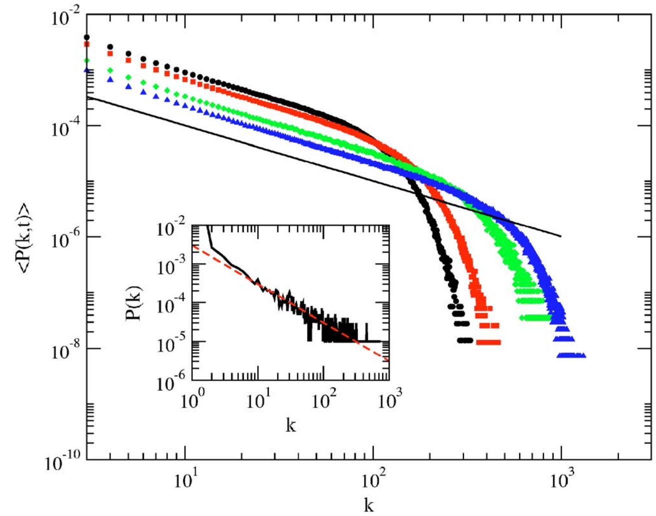

FIG. 2. (Color online) Time-averaged degree distribution with $\gamma \approx 1$ for $N = 5\times 10^3$, 10^4, 5×10^4, and 10^5 from top to bottom. Inset: Algebraic degree distribution with $\gamma \approx 1$ for $N = 10^5$ and $\alpha = 0.997$. The degree distribution has the minimum value $P(k) = 1/10^5$ when $k = k_{\max} (\approx 200)$.

An issue is how long it takes for an initially regular network to reach the equilibrium configuration through the re-

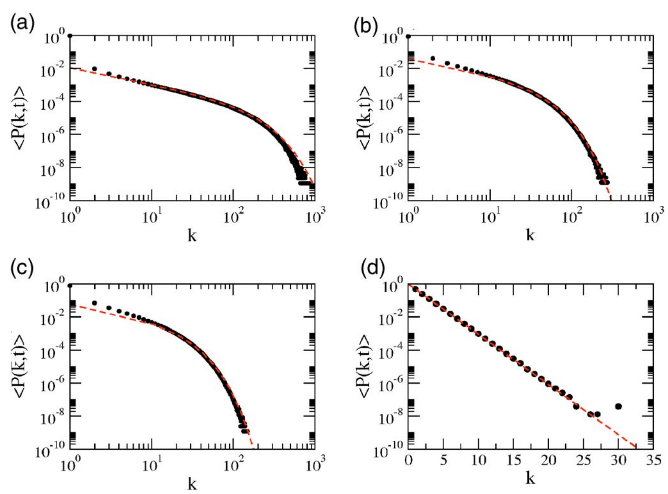

FIG. 3. (Color online) Time-averaged degree distribution for $N = 10^5$ and $\alpha = 0.99$ (a), 0.95 (b), 0.9 (c), and 0.0 (d). Degree distribution contains both an algebraic and an exponential component for $\alpha > 0$. The dashed lines correspond to $\gamma = 0.98$ (a), 0.91 (b), 0.82 (c), 0 (d) and $\zeta = 0.01$ (a), 0.05 (b), 0.09 (c), 0.7 (d).

wiring process. Here we can provide a lower and a upper bound for this time τ. For a network with N nodes and N rewirable links, at least N rewiring steps are necessary to guarantee that majority of the links are altered so that the network is no longer regular. Thus we have $\tau \gg N$. To obtain an upper bound for τ, we imagine a diffusionlike process that evolves a uniform distribution to a different one. This diffusion picture can be heuristically justified by noting that the master equation (3) can be written as

$$\frac{\partial Q(k,t)}{\partial t} = \frac{k}{N}\left[\frac{\partial^2 Q(k,t)}{\partial k^2} + \frac{\partial}{\partial k}\{(\zeta k + b)P(k,t)\}\right], \quad (8)$$

where $Q(k,t)=kP(k,t)$. For $\alpha \lesssim 1$, the parameters b and ζ are both near zero. Equation (8) thus models approximately a diffusionlike process. For a system of "spatial" size of N, an elementary scaling argument suggests that the typical time for a disturbance to diffuse through the whole system is proportional to kN. We thus have $\tau < N^2$. Combining both the lower and upper bounds, we obtain $N \ll \tau < N^2$. Numerical computations yield results that are consistent with these estimates. Although the above results are for the case where the number of links permanently attached to node (denoted by K_p) is 1, we find that the results remain essentially the same in the sense that the scaling law (1) can be generated but for $\alpha \approx K_p$.

We have numerically measured the clustering coefficient C [3],

$$C = \frac{3 \times (\text{number of triangles in the network})}{\text{number of connected triples of nodes}} \quad (9)$$

and found that it typically assumes small values (from 4×10^{-5} to 10^{-6} for α from 0 to 1). We have also calculated the degree-degree correlation coefficient r [16] for different values of α. In particular, to evaluate r, we measured the Person correlation coefficient of the degrees at both ends of the links, as follows:

$$r = \frac{N^{-1}\sum_i j_i k_i - \left[N^{-1}\sum_i \frac{1}{2}(j_i + k_i)\right]^2}{N^{-1}\sum_i \frac{1}{2}(j_i^2 + k_i^2) - \left[N^{-1}\sum_i \frac{1}{2}(j_i + k_i)\right]^2}, \quad (10)$$

where j_i and k_i are the degrees of the nodes at the ends of ith link, N is the total number of link, and the summation is respect to all links in the network. In our model, for $\alpha=0$ nodes with large degrees tend to connect to nodes with only one link and this tendency becomes stronger as α is increased from 0 to 1. The result is shown in Fig. 4, where we observe that the Person correlation coefficient is negative for $0 \leqslant \alpha \leqslant 1$. This indicates that our self-organized networks are *disassortative*. As biological networks are expected to be disassortative, Fig. 4 indicates that our model may be potentially relevant to biology [16].

We remark that the scale-free structure can originate from various processes, such as preferential rewiring of links [17], weighted linking [18], self-organized process [19], etc. The degree exponents from these models are typically $\gamma \geqslant 2$. However, there are scale-free biological networks with their degree exponents close to 1 [10]. Our study shows that this small exponent value can be explained by a simple rewiring process of links without growth.

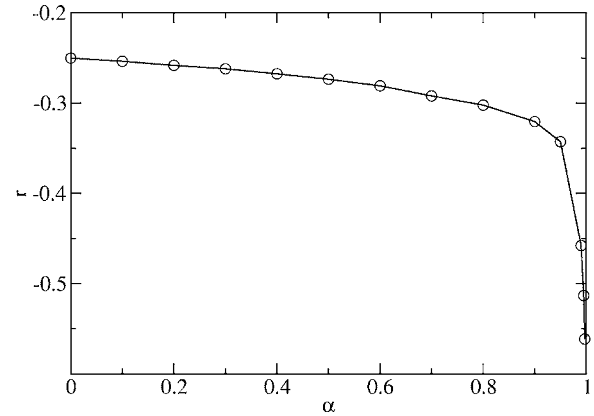

FIG. 4. Degree-degree correlation coefficient r for different values of α in the nonweighted network with $N=10\,000$.

We now discuss our weighted, nongrowing scale-free network model. In this model, random weight w_i between 0 and 1 is assigned to each node i in the regular network shown in Fig. 1, where w_i is drawn from uniform distribution. At each time step, link $l_{m\to i}$ and node j are chosen randomly and the link can be detached from node i and reconnected to node j with probability w_j. This process is then repeated. Note that in this model, the rewiring process is random. The distribution $P(k,t)$ in a steady state appears to be algebraic for k up to $k_{\max}$. The value of $k_{\max}$ is similar to that found in the nonweighted network model. When a steady state is reached, we find a robust scale-free behavior: $\langle P(k,t)\rangle \sim k^{-\gamma}$ with $\gamma \approx 2$, as shown representatively in Fig. 5 for a network of $N = 1.5 \times 10^5$ nodes. Again, similar scaling behavior arises for $K_p > 1$.

In summary, we have introduced two self-organized network models, where the total number of nodes is fixed, but a rewiring process of links occur continuously, to account for the scale-free property in nongrowing networks that can arise, for instance, in biological systems. Our nonweighted

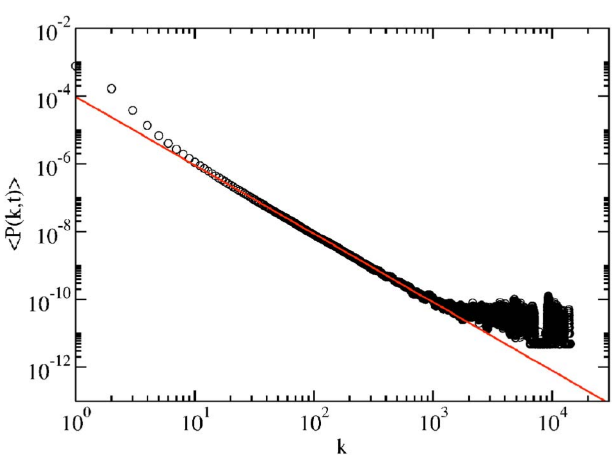

FIG. 5. (Color online) For a weighted network with $N=1.5 \times 10^5$, algebraic degree distribution with $\gamma \approx 2$. Data were averaged over 30 network realizations

network model can yield a spectrum of degree distributions ranging from algebraic to exponential, whereas the weighted network model is capable of generating robust scale-free behavior. In addition, our model generates networks that are apparently disassortative, suggesting its potential relevance to biological networks [16]. The simplicity of our models suggests that the scale-free feature, which has been demonstrated to be so pervasive in complex networks, can, in fact, arise via a self-organizing mechanism through natural evolution of links in the network, even without any growth mechanism [20]. Our finding extends to complex networks the speculation that many natural systems can self-organize themselves into criticality with power-law behaviors [13–15].

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