Characterization of weighted complex networks

Kwangho Park,[1] Ying-Cheng Lai,[1,2] and Nong Ye[3]
[1]*Department of Mathematics and Statistics, Arizona State University, Tempe, Arizona 85287, USA*
[2]*Departments of Electrical Engineering and Physics, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Department of Industrial Engineering, Department of Computer Science and Engineering, Arizona State University, Tempe, Arizona 85287, USA*
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To account for possible distinct functional roles played by different nodes and links in complex networks, we introduce and analyze a class of weighted scale-free networks. The weight of a node is assigned as a random number, based on which the weights of links are defined. We utilize the concept of *betweenness* to characterize the weighted networks and obtain the scaling laws governing the betweenness as functions both of the weight and of the degree. The scaling results may be useful for identifying influential nodes in terms of physical functions in complex networks.

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Complex networks arise in many natural systems of fundamental importance and they are also an essential part of modern society [1–6]: examples of the former are many biological networks while the latter include the Internet, electrical power grids, transportation networks, etc. Studies of complex networks have become a recent field of tremendous interest since the discoveries of the small-world [7] and the scale-free properties [8].

A basic assumption in most existing works on complex networks is that all links and nodes are identical in terms of their functional roles in the network [9]. This assumption may not be valid for a realistic network because different links and nodes can contribute differently to the overall performance of the network [10–16]. For instance, in a neural network, links, which are dendritic connections, can have very different capabilities in terms of transmitting electrical signals. Nodes, which are neurons, can also have different electrical and chemical properties and thus be very distinct in terms of their abilities to process information. In the Internet, the capabilities to process and transmit information of computers (nodes) can have a wide distribution. It is thus important to study *weighted* complex networks in which nodes and links are not treated on equal footing. Although the need to study these more realistic networks has been pointed out recently [4,13,14], so far there has not been much work in this direction.

In this paper, we introduce a class of weighted scale-free networks. In such a network, each node is assigned a random weight, which is the realization of random variable $\mathbf{W}$ in [0,1]. Given a pair of nodes with different weights, the weight of the link connecting them can be defined accordingly. Our interest is to develop proper characterizations of such weighted networks. For this purpose we use the concept of *betweenness* of a node, first proposed by Newman [13], which is the total number of optimal paths (to be defined below) between any pairs of nodes passing through this node. Given a random distribution of weights in the network, our question is how the betweenness scales with the weight w. A related issue concerns the scaling relation between the betweenness and k, where k is the realization of the degree variable $\mathbf{K}$ that measures the number of links of node in the network. In general, we write the betweenness as $B(w,k)$, with the corresponding *marginal* betweennesses: $B_W(w) = \int_1^{k_{\max}} B(w,k)dk$ and $B_K(k) = \int_0^1 B(w,k)dw$, where $k_{\max} < \infty$ is the maximum number of links of node in a finite but large network. Our main results are the following. (1) The weight-based betweenness obeys the following exponential scaling relation:

$$B_W(w) \sim Ne^{-\zeta w} \text{ for large } w, \tag{1}$$

where the exponential rate ζ scales with m, the average number of new links acquired by the network, as $\zeta \sim m^{\phi}$ (ϕ is a constant). (2) The link-based betweenness obeys the following algebraic scaling law:

$$B_K(k) \sim k^{\alpha}, \tag{2}$$

where $1 < \alpha < 2$. Since $\mathbf{W}$ and $\mathbf{K}$ are independent random variables, we have $B(w,k) \sim Ne^{-\zeta w} k^{\alpha}$ for large w and k. Suppose nodes with large values of betweenness are more influential, result (1) implies that, due to the natural process of evolution of the network, nodes with large values of weights may be less influential. On the other hand, nodes with large values of k are generally more influential, as can be expected intuitively.

Scale-free networks are characterized by algebraic behavior in the degree distribution $P(k)$. This property is *dynamic* because it is the consequence of the natural evolution of the network. The ground-breaking work by Barabási and Albert [8] demonstrates that the algebraic behavior is due to two basic mechanisms: growth and preferential attachment, where the latter means that the probability for a new node to be connected to an existing node is proportional to the number of links that this node already has.

Our weighted scale-free network is constructed as follows. We first generate a regular (nonweighted) scale-free network based on the Barabási-Albert (BA) model [8,17]. In particular, we start with a small number m of nodes and add a new node with m links at each time step following the preferential attachment rule, while allowing only one link between any pair of nodes. After t (large) time steps, we obtain a network with $N=t+m$ nodes and $N_l=mt$ links. To convert this nonweighted scale-free network into a weighted one, we choose a node i at random and assign the number "1" to it, then randomly choose another node j among those that do not yet have assigned numbers, and assign the number "2" to this node, and continue until all nodes are assigned a number (between 1 and N). The weight of a node is defined to be its assigned number divided by N, which is a fractional number between zero and unity. The weight d_{ij} of a link l_{ij} connecting a pair of nodes (i and j) is defined to be $d_{ij}=(w_i+w_j)/2$. The weight d_{ij} associated with the link l_{ij} may be interpreted, in a computer network, for instance, as the time required to transfer a data packet through this link.

Recall that the betweenness of a node i is the total number of *optimal* paths between all pairs of nodes that pass through the node i. Given a particular pair of nodes, say A and B, the optimal path is one that minimizes the sum $\Sigma_A^B d_{ij}$. The optimal path between a pair of nodes is in general different from the shortest path connecting them [18]. To obtain the scaling law (1), we assume that node i has weight w and let S denote the optimal path between A and B that passes through this node. Say an infinitesimal increase occurs in the weight of the node from w to $w+dw$, while the weights of all other nodes in the network remain unchanged. Then there is a probability that the path S is no longer an optimal path through i because the increase of the weight makes the optimal path length L_S larger by dw than before. If there exists at least one optimal path whose length is between L_S and L_S+dw which does not pass through node i before its weight is changed, then after dw increase in weight this new path can become the optimal path between A and B. Let ζ be the probability that an existing optimal path S *disappears* at the node i when its weight increases from w to $w+dw$. As this probability is determined by whether there is at least one optimal path of length between $L_S<L<L_S+dw$ in the weighted network, we see that ζ does not depend on the value of w and it can be regarded as a constant. Apparently, ζ will change if the network topology is altered. For instance, if the total number N of nodes in a weighted scale-free network is fixed, there are generally many more paths connecting two nodes with larger total number N_l of links than those with a smaller value N_l. Thus we expect ζ to increase with N_l, which can be verified numerically. Given ζ, the decrease in the betweenness of a node i is $dB_W(w)=-\zeta B_W(w)dw$, which gives the scaling law (1).

Numerically, optimal paths can be found by using the Dijkstra's algorithm [13,19]. Figure 1(a) shows the exponential relation between $B_W(w)$ and w for a weighted scale-free network of $N=4000$ nodes and for three values of m: 2, 10, and 20 (with decreasing slopes). Since ζ increases with $N_l \approx mN$, we expect it to increase with the basic network parameter m as well. This behavior is shown in Fig. 1(b), where we observe that $\zeta \sim m^{\phi}$ ($\phi=0.88\pm0.05$ for this particular network configuration). Numerical computations also indicate that $B_W(w)$ is proportional to N. We can thus write $B_W(w)=c_m N e^{-\zeta w}$ for large w, where $c_m>1$ is a constant depending on m. For a network with no closed loops ($m=1$), we expect $\zeta=0$ because an increase in the weight of a node does not change the optimal paths passing through it. This

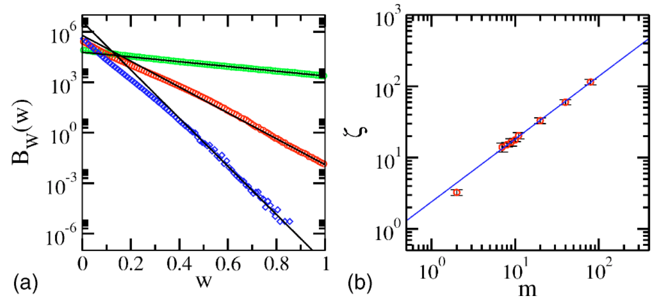

FIG. 1. (a) Scaling of the betweenness $B_W(w)$ with w for $m=2$, 10, and 20 for a weighted scale-free network of $N=4000$ nodes. For each value of m, the data were averaged over 5000 runs. The slopes of three lines are approximately −3.3, −17.7, and −33.0 for $m=2$, 10, and 20, respectively. (b) Dependence of ζ on m.

has been verified numerically. We then have $B_W(w)=c_1N=\text{const}>1$. In general, if $B_W(w)>1$, the probability that some optimal paths do pass through a node with weight w is close to one. Thus, for $m=1$ and $N \gg 1$, we have $B_W(w) \gg 1$, indicating that many optimal paths pass through every node in the corresponding weighted scale-free network. For $m \geqslant 2$ so that the resulting network possesses closed loops, optimal paths passing through a node i can disappear with nonzero probability. In this case, we expect $B_W(w)<1$ for large values of w. There is thus a nonzero probability that no optimal path exists through a node. Let w_ζ be the value of w for which $B(w)=1$, we have $w_\zeta=\ln(c_mN)/\zeta$. Let N_c be the value of N for which $c_m N e^{-\zeta}=1$. If $N>N_c(=e^{\zeta}/c_m)$, we have $B_W(w)>1$ for all values of w because $w_\zeta>1$. However, if $N<N_c$, $B_W(w)<1$ may happen for large w. For instance, for $m=2$ we find numerically that $c_2 \simeq 15$ and $\zeta \simeq 3.25$, which gives $N_c \simeq 2$. For $m=10$ and $N=4000$, we obtain $c_{10} \simeq 155$, $\zeta \simeq 17.7$, and $w_\zeta \simeq 0.75$. There must then be many nodes through which no optimal paths pass.

Note that Fig. 1 is obtained with uniform distribution of weights. Does the exponential scaling of the betweenness with weight depend on this distribution? To answer this question, we consider Gaussian weight distribution. In particular, for each node, a random number ξ is drawn from the standard Gaussian distribution of zero mean and unit variance. The weight of the node is chosen to be (arbitrarily) $w=(\xi+2)/4$. Simulations indicate that $B_W(w)$ again exhibits the exponential scaling behavior, as shown in Fig. 2(a) for three networks, all of $N=4000$ nodes but with different values of m (2, 7, and 12, corresponding to linear fits with decreasing slopes). We also observe that for large values of m, the exponential rate ζ increases according to an approximate algebraic relation: $\zeta \sim m^{\phi}$, as shown in Fig. 2(b), where $\phi=0.46\pm0.06$ for this particular network configuration. These results are similar to those in Fig. 1(a), suggesting that the exponential scaling of the betweenness with weight is general.

We now turn to the scaling law (2). To give a plausible argument for its validity, we first consider two analyzable, weighted scale-free networks, both having a treelike structure, as shown in Figs. 3(a) and 3(b), which are constructed using the BA model with $m=1$. In Fig. 3(a), all optimal paths between pairs of nodes pass through a node X regardless of

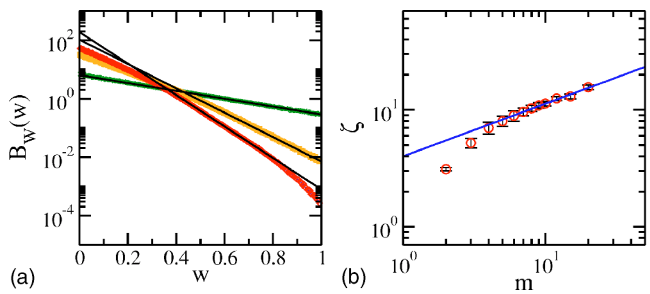

FIG. 2. (a) Scaling of the betweenness $B_W(w)$ with w for $m=2$, 7, and 12 for a weighted scale-free network of $N=4000$ nodes, where the distribution of weights in the network is Gaussian. For each value of m, the data were averaged over 5000 runs. The slopes of three lines are approximately −3.1, −9.6, and −12.5, for $m=2$, 7, and 12, respectively. (b) Dependence of ζ on m.

the value of its weight. The betweenness of a node X is thus $B(k)=k(k-1)/2$ $(k=N)$. We therefore obtain $B(k)\sim k^2$ for $k \gg 1$. On the other hand, Fig. 3(b) represents a scale-free tree with the smallest possible value of α. The betweenness of a node X is $B(k)=(k-1)(k-2)/2+(k-1)[N-(k-1)]$. For $N \gg k \gg 1$ we have $B(k)\sim k$. We thus see for the scale-free trees the scaling of $B_K(k)$ with k is algebraic and the value of the scaling exponent falls between 1 and 2.

The above argument can be extended to a general scale-free network with $m \geq 2$, where all nodes are connected to X as in Fig. 3(a). Many pairs of nodes excluding X may be connected as well. The largest possible value of the betweenness in this network can be obtained when all optimal paths between pairs of nodes pass through X, as in Fig. 3(a). We obtain $B(k)\sim k^2$ for X.

Figure 4 shows $B_K(k)$ versus k on a logarithmic scale for a weighted scale-free network with $m=2$ and $N=10\,000$. The values of the betweenness were averaged over 100 random realizations of the network. We observe a robust algebraic scaling with the exponent $\alpha \approx 1.5$ (indicated by the straight line). The inset shows a similar plot for the corresponding nonweighted network, the algebraic scaling exponent of which is $\alpha \approx 1.6$. In this case, for a given pair of nodes, if there are several optimal paths, we choose one at random.

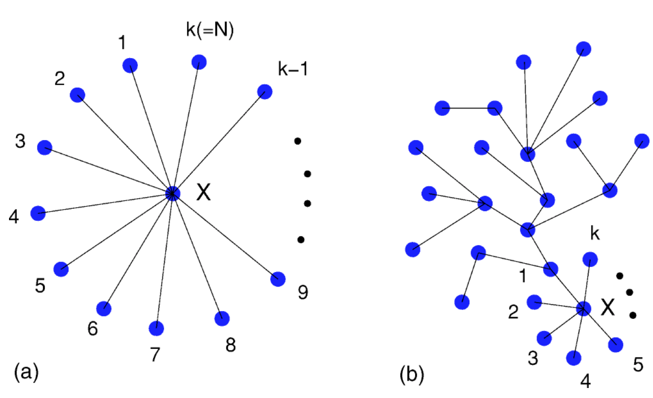

FIG. 3. Two treelike, scale-free networks with $m=1$, which can be used to derive the algebraic scaling law (2).

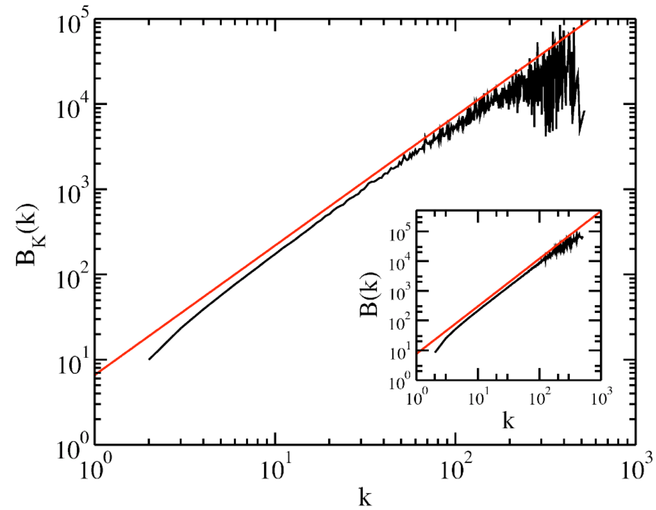

FIG. 4. Algebraic scaling between $B_K(k)$ and k for a weighted scale-free network with $m=2$ and $N=10\,000$. The value of the scaling exponent is $\alpha \approx 1.5$. A similar scaling relation is obtained for the corresponding nonweighted network, where $\alpha \approx 1.6$, as shown in the inset.

Intuitively, for two scale-free networks with identical parameters, one nonweighted and another weighted, we expect the value of α for the weighted network to be smaller than that for the nonweighted one. This can be seen as follows. In a nonweighted scale-free network, the optimal paths between pairs of nodes are exactly the same as the shortest paths between them. Shortest paths tend to pass through nodes having a relatively large number k of links, so we expect $B_K(k)$ to increase with k. For a weighted scale-free network, we expect the same to hold except with one complication: optimal paths tend to pass through nodes having small weight as well as nodes having a large number of links. That is, for nodes having a small number of links, some optimal paths tend to pass through them if they have small weight.

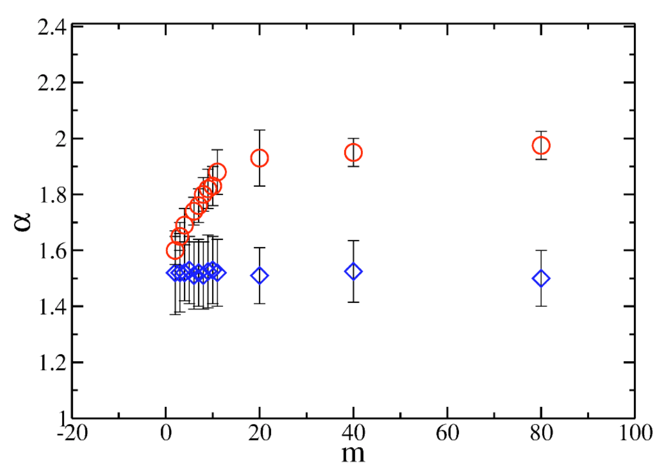

FIG. 5. Algebraic scaling exponent α vs m for a nonweighted scale-free network (upper trace) and the corresponding weighted one (lower trace). The data were obtained using 100 realizations of the network.

The value of $B_K(k)$ in a weighted scale-free network tends to be smaller for large k than that in the corresponding nonweighted scale-free network, but it can be a bit larger for small k. Thus the value of the scaling exponent α in weighted scale-free networks is generally less than that in nonweighted networks.

For a nonweighted scale-free network, as the network parameter m is increased, the betweenness can become significantly larger, particularly for large k values. Thus we expect the value of α to increase with m, at least for m not too large. For a weighted network, the compensating effect of the weight in reducing the value of the betweenness suggests that the scaling exponent α is likely to remain constant as m is increased. These behaviors are shown in Fig. 5, where α versus m is plotted for two networks of $N=10\,000$ nodes, one nonweighted (the upper trace) and another weighted (the lower trace). It is interesting to note that in all cases, the value of α is apparently bounded between 1 and 2, as predicted by our heuristic argument using scale-free trees.

In summary, we have introduced a class of weighted scale-free networks, motivated by the consideration that different nodes and links may play distinct functional roles in a realistic complex network. We use the quantity betweenness to characterize weighted networks and discovered exponential and algebraic scaling laws for the betweenness versus the weight and the degree, respectively. While our method to assign weights is straightforward, for a realistic network this can be done by using physical quantities of particular interest in terms of the functional role of the network. Our studies may provide insights as to how to identify the influential nodes in terms of not only the degree distribution but also the functional weight.

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