Detectability of dynamical coupling from delay-coordinate embedding of scalar time series

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We address under what conditions dynamical coupling between chaotic systems can be detected reliably from scalar time series. In particular, we study weakly coupled chaotic systems and focus on the detectability of the correlation dimension of the chaotic invariant set by utilizing the Grassberger-Procaccia algorithm. An algebraic scaling law is obtained, which relates the necessary length of the time series to a key parameter of the system: the coupling strength. The scaling law indicates that an extraordinarily long time series is required for detecting the coupling dynamics.

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The practically important area of chaotic time series analysis starts after the seminal work of Takens [1]. In many applications, the details of a nonlinear dynamical system are unknown and the only available information about the system is one or a few measured scalar time series. Assume the typical situation where the asymptotic invariant set of the system, such as a chaotic attractor, lies in a D-dimensional manifold $\mathcal{M}$. Given a measured scalar time series $u(t)$, an m-dimensional vector space of the following form: $\mathbf{x}(t)=\{u(t),u(t+\tau),\ldots,u(t+(m-1)\tau)\}$, can then be constructed to represent the original dynamical system, where τ is the *delay time* and m is the *embedding dimension*. Takens proved that for properly chosen τ, if the embedding dimension is at least more than twice the dimension of the manifold, i.e., $m\geqslant 2D+1$, then there exists a one-to-one correspondence between the reconstructed and the original phase spaces. As a practical matter, in many situations the dimension of the asymptotic invariant set d is of great interest, which can be much smaller than D. The work by Sauer, Yorke, and Casdagli [2] relaxes the requirement for the embedding dimension to $m\geqslant 2d+1$, where d is the box-counting dimension of the asymptotic invariant set. Milestone works in this area include that by Grassberger and Procaccia (GP) [3] who proposed a simple but efficient algorithm for estimating the correlation dimension of the underlying invariant set, and those by Wolf *et al.* and Eckmann *et al.* [4] on the computation of the Lyapunov exponents from time series. Chaotic time series analysis has since become one of the most active areas in nonlinear dynamics [5].

In this paper we address the following question: given a scalar time series measured from a chaotic system that consists of coupled subsystems, is the dynamical coupling *practically* detectable? This question is relevant to a variety of physical situations. One natural example is coupled chaotic oscillators. If a scalar time series is measured from one oscillator, can one detect the presence of other oscillators through the time series only? Another example is the experimental analyses of fluid turbulence where one typically embeds one or a few sensors in the fluid to measure the velocity field. Suppose the underlying dynamical invariant set is a high-dimensional chaotic attractor with multiple positive Lyapunov exponents. Our question concerns the detectability of the high-dimensional nature of the attractor.

Our approach will be to focus on the GP algorithm for estimating the correlation dimension. To formulate the problem conveniently, we consider a simple system of two coupled maps with coupling parameter K and suppose that a time series of length N is measured from one map (for continuous-time systems, this means that roughly, the time series contains N oscillations). Let d_2 be the correlation dimension computed from the time series by utilizing the GP algorithm, and let D_2 be the true correlation dimension of the chaotic attractor. In general, both d_2 and D_2 depend on K, so we write $d_2(K)$ and $D_2(K)$. To gain insight, we consider the two extreme situations.

(1) If K is large enough, there is a synchronization [6,7] or generalized synchronization [8] between the dynamics of the two maps. Because of synchronization, a reasonably long time series (to be made precise below), even if it is measured from one map, can yield the correct dimensionality of the attractor.

(2) If $K=0$, then time series from one map, no matter how long, will not reflect the dynamics of the whole system. For a given value of K (small), there then exists a minimum length N_{min} of the time series, where d_2 is expected to be a good approximation of D_2 only for $N>N_{min}$. Apparently, $N_{min}\to\infty$ as $K\to 0$. Equivalently, for a given length N, there exists a minimally detectable value of the coupling parameter K_{min}. The question of practical importance is how N scales with K_{min}. The principal result of this paper is the following algebraic scaling law between N and K_{min}:

$$N\sim K_{min}^{-\beta}, \tag{1}$$

where the scaling exponent is given by $\beta=D_2/2$. To appreciate the significance of the scaling, say we measure a coupled chaotic system with $D_2=100$ and suppose $K=0.1$. Then the required minimum length of the time series that can yieid the correct dimensionality of the attractor is on the order of 10^{50}, which is practically impossible to deal with. The implication is that, in a practical sense, coupled chaotic dynamics is undetectable from only a limited number of measurements [9].

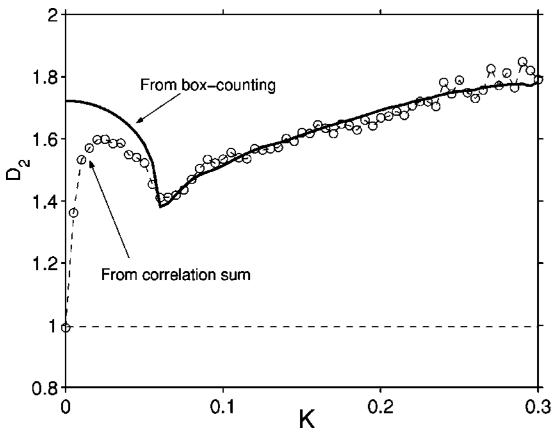

FIG. 1. For the modified Kaplan-Yorke map, Eq. (2), with $\alpha = 1/4$, the estimated correlation dimension versus the coupling parameter K obtained using time series of length $N = 2^{10}$ (open circles). Also shown is the correlation dimension D_2 computed from a box-counting procedure (solid curve).

To illustrate our result, we study an example for which both the true value of the correlation dimension and its estimated value from time series can be computed. In particular, we consider a coupled version of the two-dimensional Kaplan-Yorke map [10],

$$x_{n+1} = (3x_n + Ky_n) \mod (1),$$

$$y_{n+1} = \alpha y_n + 2\cos(2\pi x_n), \tag{2}$$

where $0 < \alpha < 1$ is a parameter and K is the coupling parameter. For $K = 0$, which corresponds to the original Kaplan-Yorke map with a skew-product structure, the two Lyapunov exponents are $\lambda_1 = \ln 3 > 0$ and $\lambda_2 = \ln \alpha < 0$, so there is a chaotic attractor. We fix $\alpha = 1/4$. The Lyapunov dimension (or equivalently the information dimension by the Kaplan-Yorke conjecture [10]) is $D_L = 1 + \ln 3/\ln 4 \approx 1.8$. As K is increased from zero, the "communication" between the x and y dynamics is also increased. Thus, intuitively, we expect D_L to decrease for $K \gtrsim 0$. The correlation dimension D_2 cannot be larger than the information dimension, so we expect $D_2 \leqslant D_L$. To compute the "true" value of D_2, we make use of the definition of dimension spectrum D_q [11] and perform a straightforward box-counting computation. The result is shown as the solid line in Fig. 1, which is D_2 vs the coupling parameter. To obtain d_2, the estimate of D_2 from data, we increase K systematically from zero and, for each value of K, we collect a scalar time series $\{x_i\}$, reconstruct a sequence of N delay-coordinate embedding vectors $\{\mathbf{x}_i\}_{i=1}^N$, and compute the following correlation sum in the GP algorithm [3]:

$$C(N,\epsilon) = \frac{2}{N(N-1)} \sum_{j=1}^{N} \sum_{i=j+1}^{N} \Theta(\epsilon - \|\mathbf{x}_i - \mathbf{x}_j\|), \tag{3}$$

where Θ is the Heaviside function [$\Theta(x) = 1$ if $x \geqslant 0$ and $\Theta(x) = 0$ otherwise], and $\|\cdot\|$ denotes a suitable vector norm, say $\|\mathbf{x}\| = \max\{\|\mathbf{x}_i\| : 1 \leqslant i \leqslant d\}$. Asymptotically, the correlation dimension d_2 is given by Ref. [3]: $d_2 = \lim_{\epsilon \to 0} \lim_{N \to \infty} \ln C(N,\epsilon)/\ln \epsilon$. In the actual computation, we fix the delay time to be $\tau = 1$ and choose the embedding dimension to be $m = 5$, which is at least two times larger than the actual value of D_2 (the solid line in Fig. 1). Empirically, these choices of the τ and m warrant that a sizable linear scaling regime in $\ln C(N,\epsilon)$ vs $\ln \epsilon$ exists [12], the slope of which gives d_2. Figure 1 also shows the value of d_2 (open circles) with $N = 2^{10}$. For $K = 0$, the measured time series is solely from the one-dimensional, piecewise linear chaotic map in x for which the correlation dimension is unity. Thus, the GP algorithm gives $d_2 \approx 1$, regardless of the length of the time series. As K is increased, the y dynamics begins to have an effect on the x dynamics, so the value of d_2 approaches that of D_2. For $N = 2^{10}$, we observe that for $K > K_N \approx 10^{-1.8}$, the values of d_2 and D_2 are the same within numerical errors.

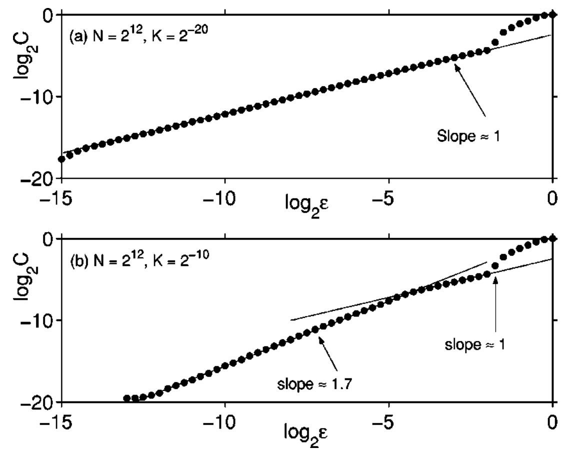

FIG. 2. Linear scaling behavior of the correlation sum for $N = 2^{12}$: (a) $K = 2^{-20}$ and (b) $K = 2^{-10}$. In (a) the interaction between the x and y dynamics is too weak to be detected. In (b) there are two distinct linear scaling regions, the slope of the region in the small distance scale is approximately the correct dimension of the chaotic attractor.

How do the coupling parameter and hence the true dimensionality of the chaotic attractor in the full phase space, manifest themselves in the plot of the correlation sum in Eq. (3)? To gain an intuition, we note that, because of the probablistic nature of the correlation sum, the longer the time series, the smaller the phase-space distance scale that the sum can resolve. For a fixed length N, let ϵ_{min} be the minimum distance scale that can be revealed in $C(N,\epsilon)$. In general, a variation in the phase-space distance is proportional to a variation in the system parameter. Thus, roughly, if $K < \epsilon_{min}$, the linear scaling region in the logarithmic plot of $C(N,\epsilon)$ cannot extend down to the distance scale that reflects the influence of the coupling. It is necessary for the coupling parameter to be large enough, say $K \gtrsim \epsilon_{min}$, for the full dimension of the chaotic attractor to be detected. The reasoning is supported by Figs. 2(a) and 2(b), where $\log_2 C$ is plotted vs $\log_2 \epsilon$ for $N = 2^{12}$ and $K = 2^{-20}$ (a) and $K = 2^{-10}$ (b), respectively. For the small value of K in Fig. 2(a), there appears to be a single linear scaling region with the slope of approximately one, indicating that the y dynamics cannot be detected using time series of this length. For the relatively large value of K in Fig. 2(b), there are two linear scaling

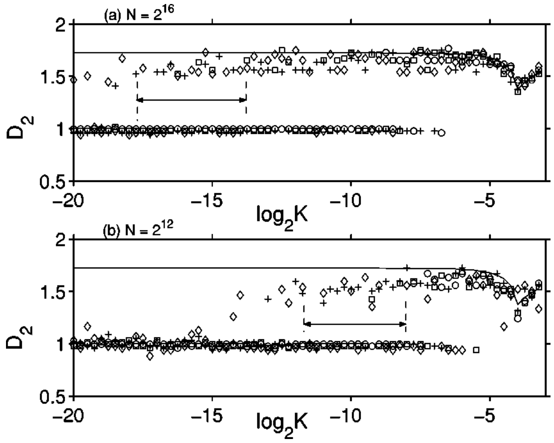

FIG. 3. Four most probable values of local slopes from the correlation sum for (a) $N=2^{16}$ and (b) $N=2^{12}$.

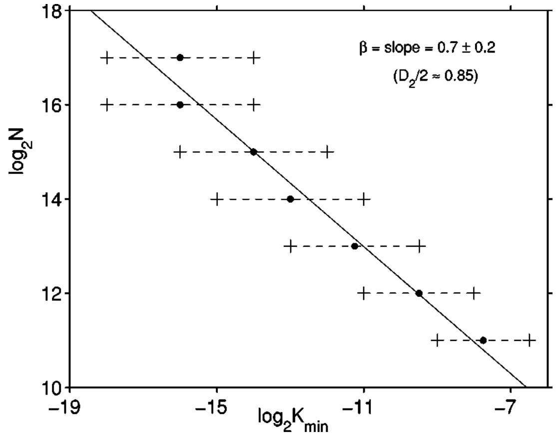

FIG. 4. Numerically obtained algebraic scaling relation between N and K_{min}.

regions, one in large distance scales which has the slope of about one and another in relatively small distance scales which yields the correct dimensionality of the chaotic attractor in the full phase space. Near $\epsilon_c \lesssim 2^{-5}$, there is a crossover from one linear scaling region to another. The crossover behavior is thus the key to assessing the correct dimensionality of the attractor. In general, the value of ϵ_c decreases as K is reduced and, hence, if N is finite, there exists a value of K below which the crossover behavior cannot be observed.

It is in fact, highly nontrivial to obtain the scaling relation between N and K_{min}, as it requires varying both K and N in a systematic way and analyzing the scaling behavior of the correlation sum for each combination of N and K. We have developed the following procedure. Since in the computation of $C(N,\epsilon)$, it is convenient to fix the length of the time series, we choose to determine the value of K_{min}. For a given value of N, we vary K systematically in the range $[2^{-20},2^{-3}]$. For each value of K, we compute the correlation sum $C(N,\epsilon)$, as in Figs. 2(a) and 2(b). Local slopes are then computed by utilizing, say a moving window of ten data points from the entire curve of $\log_2 C(N,\epsilon)$ vs $\log_2\epsilon$ and, their histogram is constructed. Such a histogram typically contains a number of distinct peaks. Figures 3(a) and 3(b) show, for $N=2^{16}$ and $N=2^{12}$, respectively, the first four most probable values of the local slopes vs the coupling parameter K, where the circles, squares, crosses, and diamonds denote the slopes of highest, second highest, third highest, and fourth highest probabilities, respectively, and the solid lines denote the correct values of the correlation dimension. There is a high probability for the local slopes to be one, which is the wrong dimension of the attractor for $K\neq 0$. For large N, the correct dimension starts to appear for smaller values of K, as expected. Letting $P(D_2\approx 1.7)$ and $P(D_2\approx 1.0)$ be the probabilities of observing the local slopes corresponding to the correct and incorrect dimensions, respectively, we use the following simple empirical criterion to estimate the minimally detectable value of K and its uncertainty: $K_{min}=(K_1+K_2)/2$ and $\Delta K=(K_2-K_1)$, where $K_1=K[P(D_2\approx 1.7)=\delta_1 P(D_2\approx 1.0)]$ and $K_2=K[P(D_2\approx 1.7)=\delta_2 P(D_2\approx 1.0)]$, and δ_1 and δ_2 are two constants satisfying $0<\delta_1<\delta_2<1$. The horizontal intervals specified in Figs. 3(a) and 3(b) indicate the corresponding ranges of the detectable values of the coupling parameter, for $\delta_1=0.1$ and $\delta_2=1.0$ (arbitrary). The choices of these constants will affect ΔK, but they will have a negligible effect on the scaling relation between N_{min} and K, which is shown in Fig. 4. Despite the appreciable amount of uncertainties in K_{min} for each fixed N, the algebraic scaling behavior appears to be reasonable. The algebraic scaling exponent is $\beta=0.7\pm 0.2$. In the scaling range of the coupling parameter, the correlation dimension of the attractor remains approximately constant: $D_2\approx 1.7$. Thus we see that the agreement between β and our predicted value $D_2/2$ is reasonable to within numerical uncertainty.

Theoretically, it is straightforward to argue for the validity of the scaling relation (1). For fixed delay time τ and embedding dimension m, the correlation sum can be written as [13]: $\log_2 C(N,\epsilon)=D_2\log_2\epsilon - m\tau H_2\log_2 e$, where H_2 is the order-two entropy. Equivalently, if J is the number of distinct pairs of points on the attractor within distance ϵ in the reconstructed phase space, then in the linear scaling region of $\log_2 C(N,\epsilon)$ vs $\log_2\epsilon$, we have: $J=(N^2/2)\epsilon^{D_2}e^{-m\tau H_2}$. For fixed N, in order for K_{min} to be detectable, at the corresponding distance scale $\epsilon\sim K_{min}$, it is necessary to have $J\gg 1$ to warrant a sizable scaling region for estimating the true dimensionality of chaotic attractor in the full phase space. We thus have: $N^2K_{min}^{D_2}\approx$const, which gives the scaling law (1).

We have also studied the following system of two coupled Rössler chaotic oscillators [14]:

$$\frac{dx_{1,2}}{dt}=y_{1,2}-z_{1,2}+\delta(x_{2,1}-x_{1,2}), \qquad (4)$$

$$\frac{dy_{1,2}}{dt}=x_{1,2}+0.165y_{1,2},$$

$$\frac{dz_{1,2}}{dt}=0.2+z_{1,2}(x_{1,2}-10.0),$$

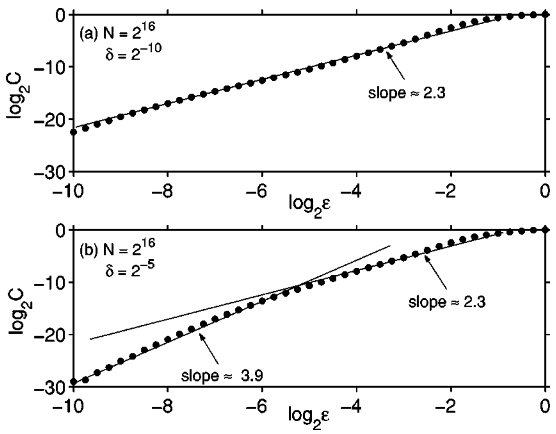

FIG. 5. For the system of coupled Rössler chaotic oscillators, scaling behavior of the correlation sum for $N=2^{16}$: (a) $\delta=2^{-10}$ and (b) $\delta=2^{-5}$. For (a) the coupling is too weak to be detected, as the slope of the correlation sum is approximately the correlation dimension of a single Rössler attractor. For (b) there is a crossover between two apparently linear scaling regions, indicating that coupling of this magnitude can be detected for the fixed number of data points (2^{16}).

where the coupling parameter δ is chosen from the range $[2^{-10},2^{-2}]$. The Lyapunov dimension of the chaotic attractor of the coupled system, estimated using the Kaplan-Yorke conjecture [10], is about 4.0, which is the upper bound of the correlation dimension. To obtain a scalar time series, we integrate Eq. (4) using time step $h=0.01$ and record $x_1(t)$ for $t=n(50h)(n=1,\ldots,2^{16})$. Since the average oscillating period of the Rössler system is about 5, there are roughly ten sampling points in each period. The delay time is chosen to be $\tau=300h$, which is about half of the oscillating period. For small coupling, for finite time series there is no crossover behavior in the plot of $\log_2 C(\epsilon)$ vs $\log_2\epsilon$, as shown in Fig. 5(a) for $\delta=2^{-10}$. There is apparently only one linear scaling region and the estimated slope from this region is about 2.3, approximately the correlation dimension of the single Rössler chaotic attractor. As the coupling is increased, two linear scaling regions appear with a clear crossover point, as shown in Fig. 5(b) for $\delta=2^{-5}$. The estimated slope in the large (relative) ϵ region is still 2.3, while the slope in the small ϵ region is about 3.9, the correct value of the correlation dimension of the coupled system Eq. (4). This behavior is completely similar to that observed for the two-dimensional map [Eq. (2)]. Because of the computational requirement, it is difficult to verify the scaling relation (1) for the six-dimensional system Eq. (4). (Note that it is already nontrivial to obtain the scaling for the two-dimensional Kaplan-Yorke map.)

In summary, we have obtained a scaling law for the required length of the time series in order to detect the influence of coupling in multidimensional chaotic systems. Our result suggests that, detecting the dynamical coupling from measured time series in such chaotic systems can be prohibitively difficult with limited data and computational power. We speculate that the popular delay-coordinate embedding technique, suitable for scalar time series, may not be effective for detecting coupled chaotic dynamics that arise in many natural situations. Perhaps, it is necessary to employ the method of spatiotemporal embedding [15] that typically requires many simultaneous measurements.

We remark that the requirement of a prohibitively long time series for dimension estimation in high-dimensional dynamical systems and thus the practical infeasibility of dimension computation in such situations have been well known. In particular, Eckmann and Ruelle [16] give the general estimate that the number of data points required for a correct estimation of dimension of value D is on the order of magnitude of 10^D. The focus of this paper is on the detectability of coupling, through dimension computation, for dynamical systems consisting of weakly interacting subsystems. Our scaling result (1), which is valid only for weak coupling, is therefore a consequence of the general Eckmann-Ruelle result. Indeed, our conclusion that weak coupling is practically not detectable is consistent with the result by Eckmann and Ruelle [16].

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[1] F. Takens, in *Dynamical Systems and Turbulence*, edited by D. Rand and L. S. Young, Lecture Notes in Mathematics Vol. 898 (Springer-Verlag, Berlin, 1981), p. 366.
[2] T. Sauer, J.A. Yorke, and M. Casdagli, J. Stat. Phys. **65**, 579 (1991).
[3] P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983); Physica D **9**, 189 (1983).
[4] A. Wolf, J.B. Swift, H.L. Swinney, and J.A. Vastano, Physica D **16**, 285 (1985); J.-P. Eckmann, S.O. Kamphorst, D. Ruelle, and S. Ciliberto, Phys. Rev. A **34**, 4971 (1986).
[5] H. Kantz and T. Schrwiber, *Nonlinear Time Series Analysis* (Cambridge University Press, Cambridge, UK, 1997).
[6] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); V.S. Afraimovich, N.N. Verichev, and M.I. Rabinovich, Radiophys. Quantum Electron. **29**, 747 (1986).
[7] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[8] N.F. Rulkov, M.M. Sushchik, L.S. Tsimring, and H.D.I. Abarbanel, Phys. Rev. E **51**, 980 (1995); L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).
[9] While we focus on the dimensionality of the chaotic attractor, we expect computations of other dynamical invariants such as the Lyapunov exponents to yield similar scaling relations, as the time-series requirements for computing, say the dimension and Lypunov exponents, are fundamentally the same [16].
[10] J.L. Kaplan and J.A. Yorke, in *Functional Differential Equations and Approximation of Fixed Points*, edited by H.-O. Peitgen and H.-O. Walter, Lecture Notes in Mathematics Vol. 730 (Springer, Berlin, 1979), pp. 204–277.

[11] P. Grassberger, Phys. Lett. **97A**, 227 (1983); H.G.E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).
[12] M. Ding, C. Grebogi, E. Ott, T. Sauer, and J.A. Yorke, Physica D **69**, 404 (1993); Y.-C. Lai and D. Lerner, *ibid.* **115**, 1 (1998).
[13] P. Grassberger and I. Procaccia, Physica D **13**, 34 (1984).
[14] O.E. Rössler, Phys. Lett. **57A**, 397 (1976).
[15] E. Olbrich, R. Hegger, and H. Kantz, Phys. Rev. Lett. **84**, 2132 (2000).
[16] J.-P. Eckmann and D. Ruelle, Physica D **56**, 185 (1992).