Route to noise-induced synchronization in an ensemble of uncoupled chaotic systems

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(Received 6 July 2009; revised manuscript received 4 December 2009; published 3 March 2010)

We investigate the route to synchronization in an ensemble of uncoupled chaotic oscillators under common noise. Previous works have demonstrated that, as the common-noise amplitude is increased, both chaotic phase synchronization and complete synchronization can occur. Our study reveals an intermediate state of synchronization in between these two types of synchronization. A statistical measure is introduced to characterize this noise-induced synchronization state and the dynamical origin of the transition to it is elucidated based on the Lyapunov dimension of the set formed by all oscillator states.

DOI: 10.1103/PhysRevE.81.036201 PACS number(s): 05.45.Xt, 05.40.Ca, 05.45.Pq

I. INTRODUCTION

The study of synchronization in nonlinear dynamical systems has been an active field of research [1,2]. For a system of coupled chaotic oscillators, the route to synchronization has been understood reasonably well. In particular, as the coupling strength is increased from zero, a weak type of synchronization, namely phase synchronization where the phases of the oscillators are confined with respect to each other but their amplitudes remain uncorrelated, can occur. As the coupling is strengthened further, partial synchronization in the form of synchronous clusters of oscillator can arise [3–5]. Amplitude or complete synchronization occurs when the coupling is sufficiently strong.

It has been known that chaotic synchronization in the absence of any coupling can also arise when the oscillators are under common noise, the phenomenon of noise-induced synchronization [6–9]. For example, phase synchronization can arise when the common-noise amplitude ε exceeds a small value ε_P. As ε is increased through a relatively large value ε_C, complete synchronization can occur [7]. The critical values of ε_P and ε_C can be related to the spectrum of the Lyapunov exponents of an individual oscillator under noise. In particular, ε_P is the critical noise amplitude at which the originally null Lyapunov exponent begins to become negative, and for $\varepsilon=\varepsilon_C$, the largest Lyapunov exponent crosses zero toward negative values. We note that, for limit-cycle oscillators, physical theories have been developed for common-noise-induced synchronization [10,11].

In this paper, we report a state of synchronization, which is referred to as intermediate synchronization, in an ensemble of uncoupled chaotic oscillators under common noise. The critical value of the noise amplitude, denoted by ε_I, lies in between ε_P and ε_C. To explain this new synchronization state, we consider an infinite number of identical chaotic oscillators (the phenomenon persists for nonidentical oscillators, as we will demonstrate in this paper). To be concrete but without loss of generality, we first assume that each oscillator exhibits a chaotic attractor with one positive Lyapunov exponent (hyperchaotic systems will also be treated in this work). The information (or Lyapunov) dimension d_L^0 of the attractor thus assumes a value between 2 and 3. Suppose the ensemble of chaotic oscillators starts with different but random initial conditions. Without common noise, at any given time the trajectory points of these oscillators will be distributed on the chaotic attractor according to its natural measure. Visually we expect the trajectory points to spread over the entire attractor. The details of this "covering" change with time, but we expect the statistical properties to remain invariant under time evolution. For example, the dimension d_L of the set of trajectory points is equal to d_L^0, which is approximately the case even when ε is increased through ε_P so that phase synchronization occurs, as the amplitudes of the oscillators are still uncorrelated. For $\varepsilon>\varepsilon_C$ so that complete synchronization occurs, all trajectories will collapse into a single point that moves randomly on the chaotic attractor in time.

The intermediate-synchronization state is characterized by a significant deviation in the distribution of the oscillator trajectory points from the natural distribution of the attractor. In particular, as ε is increased through ε_I, the distribution of the trajectory points becomes localized on the attractor and the dimension d_L becomes less than 2. As ε tends to ε_C, d_L approaches unity. For $\varepsilon=\varepsilon_C$, d_L changes discontinuously from one to zero. The synchronization state that occurs for $\varepsilon_I<\varepsilon<\varepsilon_C$ is thus "stronger" than phase synchronization because there exists some degree of correlation among the amplitudes of the oscillators, but it is weaker than complete synchronization. A probable reason that this synchronization state was not uncovered previously may be that most existing works on common-noise-induced synchronization focused on two or a few chaotic oscillators, while the revelation of the state requires a large number of chaotic oscillators under common noise.

In Sec. II, we demonstrate and characterize in detail the intermediate-synchronization state in dynamical systems with three examples, including an ensemble of high-dimensional chaotic oscillators. The issue of robustness with respect to system mismatch and heterogeneity in noise is also discussed. Conclusions are presented in Sec. III.

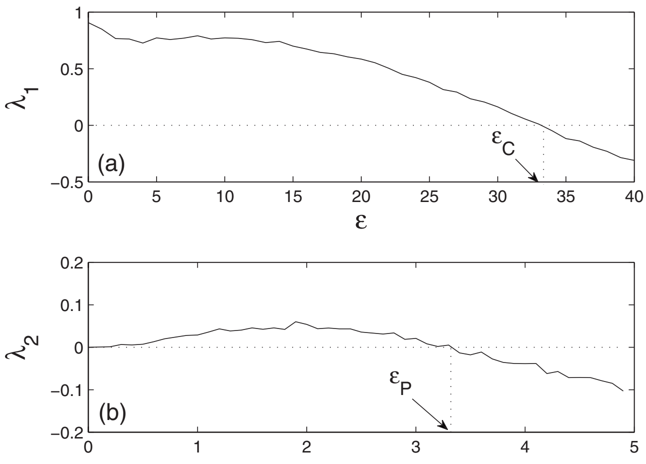

FIG. 1. For the classical chaotic Lorenz oscillator under noise, (a) largest Lyapunov exponent λ_1 and (b) originally null exponent λ_2 as a function of the noise amplitude ε. For an ensemble of such oscillators under common noise, chaotic phase synchronization occurs for $\varepsilon=\varepsilon_P$ at which λ_2 becomes negative, and complete synchronization occurs for $\varepsilon=\varepsilon_C$ when λ_1 crosses zero from the positive side.

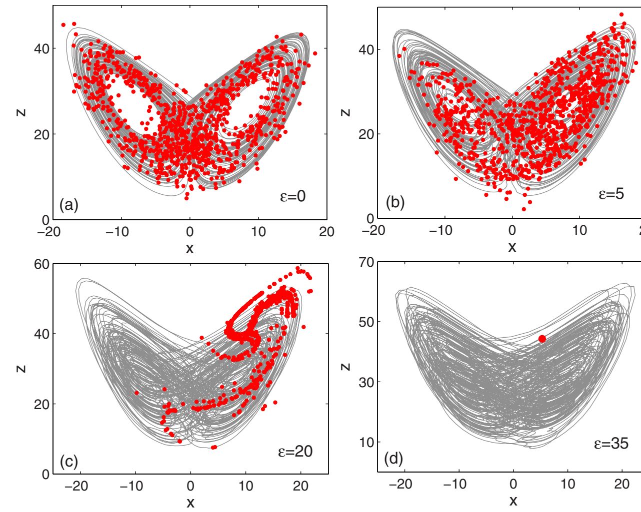

FIG. 2. (Color online) For an ensemble of $N=1000$ chaotic Lorenz oscillators under common noise, spread of trajectory points from all oscillators at an instant of time for four different values of the noise amplitude: (a) $\varepsilon=0$, (b) $\varepsilon=5.0>\varepsilon_P$, (c) $\varepsilon=20.0<\varepsilon_C$, and (d) $\varepsilon=35>\varepsilon_C$. The background represents a typical (noisy) Lorenz attractor at the corresponding noise amplitude.

II. NOISE-INDUCED INTERMEDIATE SYNCHRONIZATION

A. Ensemble of chaotic Lorenz oscillators

To illustrate our finding, we consider the following ensemble of N uncoupled chaotic Lorenz oscillators [12] under both common and heterogeneous noise:

$$\dot{x}_i = 10(y_i - x_i),$$

$$\dot{y}_i = \rho_i x_i - y_i - x_i z_i + \xi(t) + \eta_i(t),$$

$$\dot{z}_i = x_i y_i - 8/3, \quad i = 1, \ldots, N, \tag{1}$$

where ρ_i is a parameter of the Lorenz oscillator, the white noise term $\xi(t)$ is common to all oscillators, which is characterized by $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t')\rangle = \varepsilon^2\delta(t-t')$, and $\eta_i(t)$ are independent Gaussian processes that satisfy $\langle \eta_i(t)\rangle=0$ and $\langle \eta_i(t)\eta_j(t')\rangle = \sigma^2\delta_{ij}\delta(t-t')$. We write the system parameter as $\rho_i=\rho_0+\delta\rho_i$, where $\rho_0=28.0$ and $\delta\rho_i$ represent the parameter mismatches among the oscillators. We shall consider both situations of identical oscillators ($\delta\rho_i=0$) and nonidentical oscillators ($\delta\rho_i \neq 0$). We numerically solve the set of stochastic differential equations Eq. (1) by using a standard second-order stochastic Runge-Kutta algorithm [13].

We first consider the case of identical oscillators under common noise by setting $\delta\rho_i=0$ for all i and $\sigma=0$. Figures 1(a) and 1(b) show, for any oscillator in the ensemble, the largest Lyapunov exponent λ_1 and the originally null Lyapunov exponent λ_2 versus the common-noise amplitude ε, respectively. We see that λ_1 becomes negative as ε is increased through the critical value $\varepsilon_C \approx 33.3$, indicating that complete synchronization among all oscillators in the ensemble can be achieved for $\varepsilon>\varepsilon_C$. The second largest Lyapunov exponent, which is zero for $\varepsilon=0$, first becomes positive [14] as ε is increased from zero and then becomes negative as ε is increased through $\varepsilon_P \approx 3.3$, indicating the occurrence of phase synchronization for $\varepsilon>\varepsilon_P$ [7]. Intermediate synchronization occurs for $\varepsilon_P<\varepsilon_I<\varepsilon<\varepsilon_C$, where $\varepsilon_I \approx 13.0$.

Figures 2(a)–2(d) show, for four different values of the common-noise amplitude, trajectory points of an ensemble of $N=1000$ oscillators at a fixed instant of time (snapshot attractor [15,16]), where the background represents a typical (noisy) Lorenz attractor at the corresponding noise amplitude. We see that, for $\varepsilon=0$ [Fig. 2(a)], the trajectory points spread over the deterministic attractor. The distribution of the points follows the natural measure of the attractor. This situation holds approximately for relatively small value of ε even when phase synchronization has already set in, as shown in Fig. 2(b) for $\varepsilon=5.0>\varepsilon_P$. However, as ε is increased further, at any time the set of points tends to occupy only part of the natural measure of the noisy attractor, as shown in Fig. 2(c) for $\varepsilon=20.0<\varepsilon_C$. The part of the attractor that the set of points cover changes from time to time, but its statistical properties are invariant in time (to be demonstrated below). This is an intermediate-synchronization state. When complete synchronization is realized, all trajectories collapse into a single point, as shown in Fig. 2(d).

We introduce a statistical quantity to characterize the intermediate-synchronization state. The basic observation is that, as the degree of synchronization among the oscillators is increased, the "region" occupied by the oscillators in a closure that contains the attractor decreases. In particular, in a desynchronized state, at any given time the trajectory points from all oscillators will "fill" the entire attractor, but for complete synchronization all trajectories will occupy the same point. Our idea is thus to divide a phase-space region that encloses the attractor into a grid of small cells of size δ. For any given amplitude ε of the common noise, we evolve

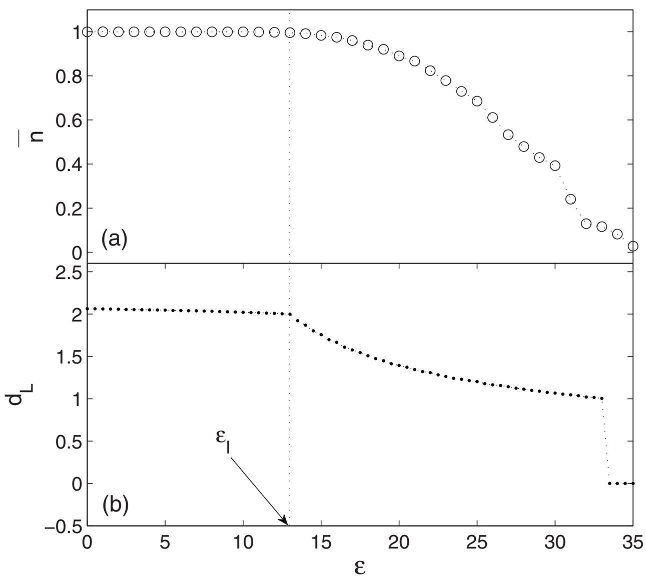

FIG. 3. For an ensemble of $N=1000$ chaotic Lorenz oscillators, (a) time-averaged fraction of attractor volume occupied by all oscillators and (b) the Lyapunov dimension versus the common-noise amplitude ε. The onset of intermediate synchronization occurs at $\varepsilon_I \approx 13.0$ at which the Lyapunov dimension begins to decrease from 2.

an ensemble of N oscillators using random initial conditions. Discarding transients, we count the number of nonempty cells occupied by the oscillators at large time t, which is denoted by $n_\varepsilon(\delta,t)$. We expect $n_\varepsilon(\delta,t)$ to exhibit fluctuations with both δ and t, but if δ is small, the time average of $n_\varepsilon(\delta,t)$ is well defined and depends only on ε. We first define

$$\tilde{n}_\varepsilon \equiv \lim_{\delta\to 0} \lim_{T\to\infty} \frac{\int_0^T n_\varepsilon(\delta,t)dt}{T} \tag{2}$$

as the time-averaged attractor volume occupied by all oscillators in the ensemble, i.e., the average number of occupied cells. We next define

$$\bar{n}(\varepsilon) \equiv \frac{\tilde{n}_\varepsilon}{\tilde{n}_0} \tag{3}$$

as the time-averaged fraction of attractor volume, where $\tilde{n}_0$ denotes the time-averaged number of occupied cells in the absence of noise, so $\bar{n}(\varepsilon)$ lies between 0 and 1 due to the fact that $0<\tilde{n}_\varepsilon \leq \tilde{n}_0$. An example of the behavior of $\bar{n}(\varepsilon)$ is shown in Fig. 3(a), where the simulation parameters are δ=0.01, N=1000, T=10000, and the three-dimensional phase-space region that covers the chaotic Lorenz attractor is chosen to be $[-30,30]\times[-55,55]\times[0,90]$ [in (x,y,z)]. We observe that, for small values of ε, $\bar{n}(\varepsilon)$ is essentially unity, indicating that the attractor is fully covered by oscillator trajectories, signifying lack of amplitude correlation among the oscillators (although phase synchronization can still occur). As ε is increased, $\bar{n}(\varepsilon)$ begins to decrease from unity and tends to zero as complete synchronization is achieved. The value ε_I at which $\bar{n}(\varepsilon)$ begins to turn downward marks the onset of the intermediate-synchronization state.

From the physical meaning and behavior of $\bar{n}(\varepsilon)$, we can see that, associated with intermediate synchronization, not only are the phases of the oscillators strongly correlated, there is also amplitude correlation [17]. In particular, when there is phase but not intermediate synchronization, the amplitudes of oscillators under common-noise driving are statistically uncorrelated. As a result, at any instant of time the trajectory points from different oscillators tend to fall in different cells, as in the case where there is no noise and the dynamics of oscillators are independent of each other. This suggests that, before intermediate synchronization sets in, for any cell in the phase space, $\bar{n}(\varepsilon)$ is about unity. After intermediate synchronization arises, $\bar{n}(\varepsilon)$ begins to deviate significantly from unity, which means that, statistically, there is a *nonvanishing* fraction of oscillators with similar amplitudes for a long time, in sharp contrast to the case of phase synchronization only. The fraction of both phase and amplitude correlated oscillators increases with the strength of the common noise, and the fraction becomes unity when complete synchronization has been achieved. The behavior of $\bar{n}(\varepsilon)$ as a function of the common-noise amplitude thus suggests that intermediate synchronization is characteristically different from and in fact stronger than phase synchronization in the usual sense. Our results thus reveal that intermediate synchronization is a distinct dynamical state beyond phase synchronization in the evolution of the ensemble of oscillators under common noise toward complete synchronization, and the quantity $\bar{n}(\varepsilon)$ may be regarded as an order parameter to characterize how far (or how close) the system is from complete synchronization.

The onset of the intermediate-synchronization state can in fact be predicted by examining the fractal dimension of the subset of the attractor occupied by the oscillator trajectories. In this regard the Lyapunov dimension [18] is convenient, which is given by

$$d_L \equiv 2 + \frac{\lambda_1+\lambda_2}{|\lambda_3|}, \tag{4}$$

where λ_i ($i=1,2,3$) are the Lyapunov exponents of any oscillator in the ensemble under noise. In the deterministic case, we have $\lambda_1>0$, $\lambda_2=0$, and $\lambda_3<0$ but $|\lambda_3|>\lambda_1$ so that $2<d_L^0<3$. Under noise, if d_L is larger than 2, most part of the attractor will be covered by oscillator trajectories. Essentially partial covering occurs when d_L is decreased from the value of 2. This occurs when

$$\lambda_1+\lambda_2=0. \tag{5}$$

Figure 3(b) shows d_L versus ε. We observe a sharp decrease in d_L from the value of 2 as ε exceeds the critical value ε_I. This value coincides with that determined by the time-averaged fraction of attractor volume occupied by all oscillators, as in Fig. 3(a).

Following an arbitrary oscillator in the ensemble, we will find that the trajectory point of this reference oscillator continuously evolves in the phase space from one cell to another. When there is no common noise, at any given instant of time,

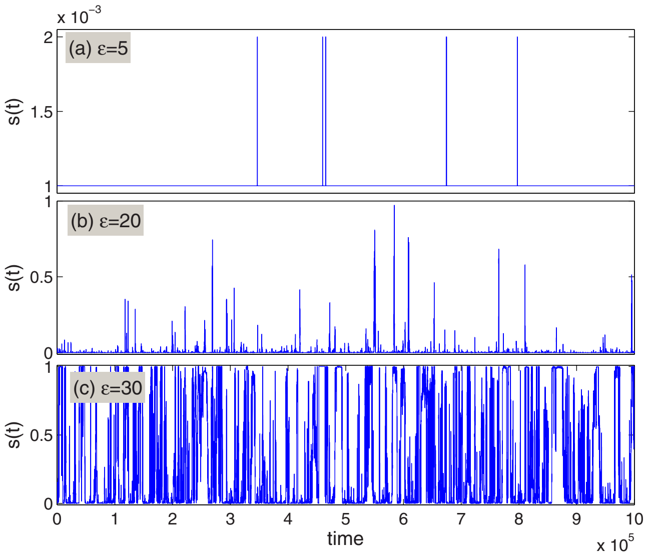

FIG. 4. (Color online) Time evolution of the normalized number $s(t)$ of oscillators clustered into a single phase-space cell for (a) $\varepsilon=5.0<\varepsilon_I$, (b) $\varepsilon_I<\varepsilon=20.0<\varepsilon_C$, and (c) $\varepsilon=30.0\lesssim\varepsilon_C$. We observe a strong intermittency near the transition to complete synchronization.

the number of oscillators whose phase points cluster into the same cell as that for the reference oscillator is expected to be unity for small enough cell size so that the normalized number s (by N) is of the order of $1/N$. This also holds approximately before the onset of intermediate synchronization, i.e., for $\varepsilon<\varepsilon_I$, as shown by the time evolution of s in Fig. 4(a). For $\varepsilon>\varepsilon_I$, we expect the probability for more trajectory points to cluster into a single cell to increase, as shown in Fig. 4(b) for $\varepsilon=20.0$. We see that, occasionally, almost all oscillators fall into a single cell and $s(t)$ exhibits an intermittent behavior. As complete synchronization is approached, the frequency with which most oscillators cluster into a single cell becomes higher and, in fact, $s(t)$ exhibits a strong intermittency between the values of zero and one, as shown in Fig. 4(c). The intermittency can be explained by the snapshot-attractor theory developed for transition to chaos in random dynamical systems [16] and its connection to chaotic synchronization under common noise [6].

B. Ensemble of Hindmarsh-Rose neurons

The Hindmarsh-Rose neural oscillator is a model often used in computational neuroscience. We consider an ensemble of $N=1000$ Hindmarsh-Rose neurons under noise [19],

$$\dot{x}_i = y_i + a_i x_i^2 - b x_i^3 - z_i + I + \xi(t) + \eta_i(t),$$

$$\dot{y}_i = c - d x_i^2 - y_i,$$

$$\dot{z}_i = r[S(x_i-\chi) - z_i], \quad i=1,\ldots,N, \tag{6}$$

where the parameters are $b=1.0$, $c=1.0$, $d=5.0$, $S=4.0$, $r=0.006$, $\chi=-1.56$, and $I=3.0$. The parameter $a_i=a_0+\delta a_i$ can be slightly different for each system, where $a_0=3.0$ and

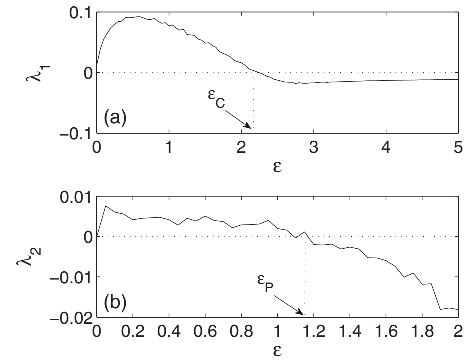

FIG. 5. For the chaotic Hindmarsh-Rose neuron under noise, (a) largest Lyapunov exponent λ_1 and (b) originally null exponent λ_2 as a function of the noise amplitude ε.

δa_i denotes parameter mismatches. In the absence of parameter mismatches and heterogeneous noise, i.e., $\delta a_i=0$ and $\sigma=0$, the ensemble of Hindmarsh-Rose neurons can be phase and completely synchronized at the respective transition points $\varepsilon_P\approx 1.2$ and $\varepsilon_C\approx 2.3$, as shown in Fig. 5.

Figure 6 demonstrates different snapshot attractors for increasing noise amplitudes from $\varepsilon=0$ to $\varepsilon=3.0$. A gradual clustering of the snapshot attractor as ε is increased is evident. To calculate $\bar{n}(\varepsilon)$, we choose the phase-space region that encloses the attractor to be $[-4.5,4.5]\times[-22,0]\times[1.5,3.5]$ [in (x,y,z)]. This region is divided into small cells with size $\delta=0.001$, and the simulation time is $T=10\,000$. As shown in Fig. 7(a), $\bar{n}(\varepsilon)$ begins to decrease toward zero for $\varepsilon>\varepsilon_I$, which coincides with the transition of the Lyapunov dimension d_L to below the value of 2. We obtain $\varepsilon_I\approx 1.96$.

To better distinguish intermediate from phase synchronization, we plot the profile of the x components of this ensemble of neurons at different noise amplitudes, as shown in Fig. 8, where different colors denote different values of x. In the noise free case, the states of each oscillator are totally uncorrelated. When phase synchronization is established,

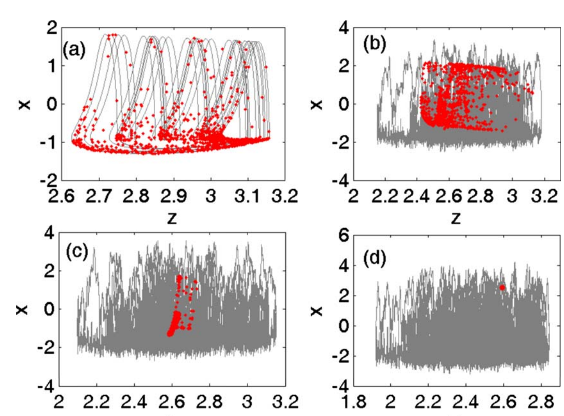

FIG. 6. (Color online) For an ensemble of $N=1000$ chaotic Hindmarsh-Rose neurons under common noise, spread of trajectory points from all oscillators at an instant of time for four different values of the noise amplitude: (a) $\varepsilon=0$, (b) $\varepsilon=1.5>\varepsilon_P$, (c) $\varepsilon=2.0<\varepsilon_C$, and (d) $\varepsilon=3.0>\varepsilon_C$. The background represents a typical (noisy) attractor at the corresponding noise amplitude.

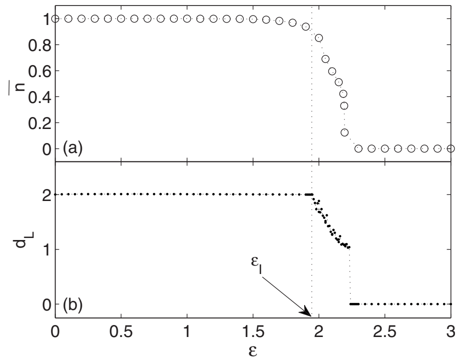

FIG. 7. For an ensemble of $N=1000$ chaotic Hindmarsh-Rose oscillators, (a) time-averaged fraction of attractor volume occupied by all oscillators and (b) the Lyapunov dimension versus the common-noise amplitude ε. The onset of the intermediate-synchronization state occurs at $\varepsilon_I \approx 1.96$ where the Lyapunov dimension begins to decrease from 2.

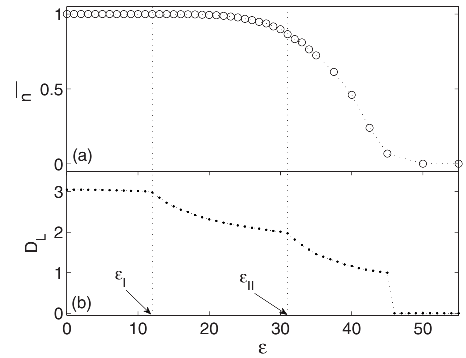

FIG. 9. For an ensemble of $N=1000$ generalized Lorenz oscillators, (a) time-averaged fraction of attractor volume occupied by all oscillators and (b) Lyapunov dimension d_L versus the common-noise amplitude ε. Onsets of multiple stages of intermediate-synchronization state occur at $\varepsilon_I \approx 12.0$ and $\varepsilon_{II} \approx 31.5$, where d_L begins to decrease from the values of 3 and 2, respectively.

they begin to oscillate between local extremes in a roughly coherent way but with rather blurred boundaries that define the coherent motion. We stress that $\bar{n} \approx 1$ means that, although some degree of coherence is achieved, statistically the states of each oscillator cannot coincide for any long stretch of time. Only when the intermediate-synchronization state is established is it statistically possible for the coincidence of oscillator states to appear persistently. At this stage, the previously blurred boundaries become sharp.

These results indicate that in a neuronal system, synchronization stronger than phase synchronization but weaker than complete synchronization can be expected when an ensemble of neural oscillators are subject to common noise. It would certainly be interesting to look for the "biological usage" of intermediate synchronization.

C. Ensemble of high-dimensional chaotic oscillators

Intermediate synchronization can also occur in high-dimensional systems. To demonstrate this, we consider an ensemble of $N=1000$ hyperchaotic generalized Lorenz oscillators [20] under common noise,

$$\dot{w}_i = -(25\beta + 10)(w_i - x_i),$$

$$\dot{x}_i = (\rho_i - 35\beta)w_i + (29\beta - 1)x_i - w_i y_i + z_i + \xi(t) + \eta_i,$$

$$\dot{y}_i = -(8+\beta)y_i/3 + w_i x_i,$$

$$\dot{z}_i = -6w_i, \tag{7}$$

where $\beta=0.01$ so that each individual oscillator possesses two positive Lyapunov exponents, ρ_i, ξ, and η_i are the same as in Eq. (1). The region of the phase space that covers the attractor is chosen to be $[-35,35]\times[-60,60]\times[0,90]\times[-100,100]$ [in (w,x,y,z)], and the size of cell δ is 0.01. Simulation time is $T=10\,000$. In Fig. 9(a), the behavior of $\bar{n}(\varepsilon)$ versus ε is displayed. We observe that $\bar{n}(\varepsilon)$ starts to decrease when ε is increased through the value of about 10, signaling the occurrence of intermediate synchronization. To quantify the transition, we examine the Lyapunov dimension as a function of ε. In a high-dimensional dynamical system, the Lyapunov dimension of an attractor is given by [21]

$$d_L = K + \frac{1}{|\lambda_{K+1}|}\sum_{i=1}^{K}\lambda_i, \tag{8}$$

where K is the largest integer that satisfies $\Sigma_{i=1}^{K}\lambda_i \geq 0$. Figure 9(b) shows d_L versus ε. We observe that for $\varepsilon < \varepsilon_I \approx 12.0$, d_L assumes the constant value of slightly above 3, which is the dimension of the hyperchaotic attractor of an individual generalized Lorenz oscillator. As ε is increased through ε_I, d_L starts to decrease from the value of 3 in a smooth way. This behavior continues until d_L reaches the value of 2 when ε arrives at another critical point $\varepsilon_{II} \approx 31.5$. For $\varepsilon > \varepsilon_{II}$, d_L decreases from the value of 2 smoothly but in a manner that is different from that for $3 > d_L > 2$. Finally, complete synchronization sets in when d_L reaches the value of unity and then drops abruptly to zero. These behaviors of d_L suggest the

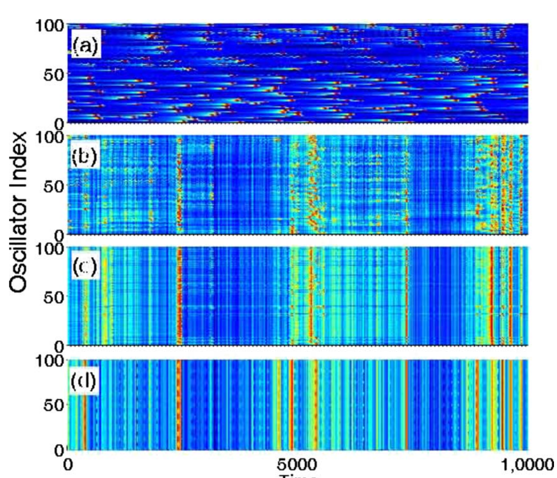

FIG. 8. (Color online) Profile of the x components of the ensemble of Hindmarsh-Rose neurons at different noise amplitudes: (a) $\varepsilon=0.0$; (b) $\varepsilon=1.5$; (c) $\varepsilon=2.0$; and (d) $\varepsilon=3.0$. Different colors denote different values of x.

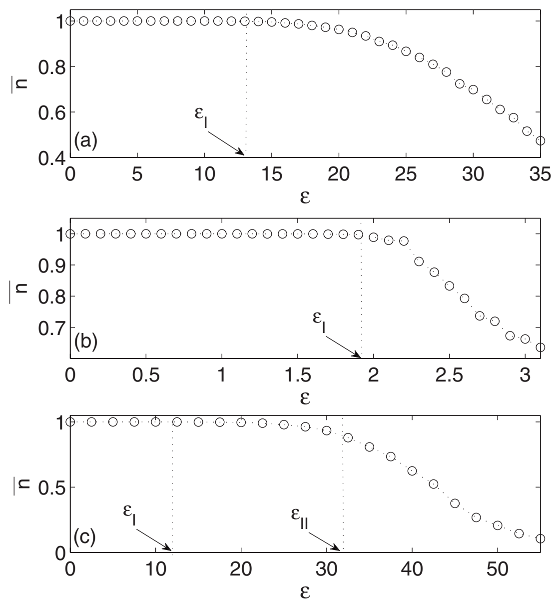

FIG. 10. Intermediate synchronization in the presence of system parameter mismatches and inhomogeneous noise. Shown are time-averaged fraction of attractor volume occupied by all oscillators versus the common-noise amplitude for: (a) an ensemble of 1000 Lorenz oscillators; (b) an ensemble of 1000 Hindmarsh-Rose oscillators; and (c) an ensemble of 1000 generalized Lorenz systems.

occurrence of different degrees of intermediate synchronization in multiple stages.

D. Robustness of intermediate synchronization

The phenomenon of intermediate synchronization is not restricted to systems of identical chaotic oscillators. When there are parameter mismatches among the oscillators and/or there is a heterogeneous component in the noise, we have also found this synchronization state in all three systems. For example, Fig. 10 shows the time-averaged fraction of attractor volume for (a) the Lorenz system with $\delta\rho_i$ uniformly distributed in [0,0.1], (b) the Hindmarsh-Rose neural system with δa_i uniformly distributed in [−0.005,0.005], and (c) the generalized Lorenz system with $\delta\rho_i$ uniformly distributed in [0,0.1]. In all cases, the amplitude of the heterogeneous noise is chosen as σ=0.001. We observe a strong similarity between Figs. 10(a) and 3(a), Figs. 10(b) and 7(a), and Figs. 10(c) and 9(a). These results suggest that intermediate synchronization among chaotic oscillators under common noise is a robust phenomenon.

III. CONCLUSIONS

In summary, we have investigated the transition to various synchronization states among an ensemble of uncoupled chaotic oscillators under common noise and uncovered an intermediate-synchronization state in between chaotic phase synchronization and complete synchronization. Statistically, this synchronization state is characterized by a decrease in the time-averaged fraction of attractor volume occupied by all oscillators from unity. Dynamically, intermediate synchronization sets in when the Lyapunov dimension of the set of trajectory points from all oscillators decreases from some integer value. Despite extensive works on chaotic synchronization in the past, to our knowledge this intermediate-synchronization state has not been reported before. Our finding indicates that, similar to systems of coupled chaotic oscillators, noise-induced synchronization in systems of uncoupled chaotic oscillators can also exhibit a rich variety of manifestations. In particular, a typical route of transition to synchronization in an ensemble of uncoupled chaotic systems can be a successive process where, with the increase of the noise amplitude, phase synchronization, intermediate synchronization, and complete synchronization occur in turn.

ACKNOWLEDGMENT

Y. C. L. was supported by ONR under Grant No. N00014-08-1-0627 and by AFOSR under Grant No. F9550-09-1-0260. The visit of Y.W. to Arizona State University was sponsored by the State Scholarship Fund of China. Y.W. and Z.G.Z. were partially supported by NNSF of China under Grant No. 10875011, the 973 Program under Grant No. 2007CB814805, and the Foundation of Doctoral Training under Grant No. 20060027009.

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