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Communicating with chaos using two-dimensional symbolic dynamics

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Abstract

Symbolic representations of controlled chaotic orbits produced by signal generators can be used for communicating. In this Letter, communicating with chaos is investigated by using more realistic dynamical systems described by two-dimensional invertible maps. The major difficulty is how to specify a generating partition so that a good symbolic dynamics can be defined. A solution is proposed whereby hyperbolic chaotic saddles embedded in the attractor are exploited for digital encoding. Issues addressed include the channel capacity and noise immunity when these saddles are utilized for communication. © 1999 Published by Elsevier Science B.V. All rights reserved.

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Communicating with chaos has become a field of recent interest [1–5]. There have been two different approaches to the problem. The first is to use the principle of synchronous chaos [1,2,6] to embed and transmit digital information. In the second approach, the principle of controlling chaos [7] is extended to dynamical systems with well-defined symbolic dynamics [3,4]. This second approach makes explicit use of the fundamental principle that chaotic systems are natural information sources. By manipulating the symbolic dynamics of a chaotic system in an intelligent way, the system produces trajectories in which digital information is embedded in the corresponding symbolic dynamics.

This Letter addresses communicating with chaos by controlling symbolic dynamics in two-dimensional maps (equivalently three-dimensional flows). Our motivation comes from the fact that, although the principle of utilizing chaotic symbolic dynamics for communication is quite general, so far examples illustrating this idea exclusively utilize chaotic systems whose dynamics can be approximated by one-dimensional maps [3,4]. Many chaotic systems encountered in practice, however, cannot be described by one-dimensional dynamics. It is thus of interest to study whether communicating using controlled symbolic dynamics can be realized in higher dimensions.

The major difficulty when two-dimensional maps are utilized for communication is to locate a generalized partition so that a good symbolic dynamics can be defined. This difficulty arises due to nonhyperbolicity. In

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smooth, noninvertible two-dimensional maps such as those arising on the Poincaré surface of section of three-dimensional autonomous flows, nonhyperbolicity is typically characterized by the existence of an infinite number of points embedded in a chaotic attractor at which the stable and unstable directions coincide - the set of tangency points. Analogous to the critical point in one-dimensional chaotic maps [e.g., $x_c = 1/2$ in the logistic map $f(x) = rx(1-x)$], which is naturally the generating partition point for defining symbolic dynamics, in two dimensions the generating partition is a zig-zag curve, the curve connecting all primary tangency points in the chaotic attractor [9]. It is generally quite difficult to locate precisely the partition curve even for well studied two-dimensional systems such as the Hénon map [10]. To overcome this difficulty, in this Letter we propose a general solution: we exploit various *hyperbolic chaotic invariant sets* embedded in the nonhyperbolic chaotic attractors. Due to hyperbolicity, it is straightforward to locate a generating partition for trajectories restricted to these saddles [1]. We can choose the chaotic saddles so that the symbolic dynamics are robust against small random noise. Thus, utilizing hyperbolic chaotic saddles for communication also provides a solution to overcome the influence of noise [5].

To begin, we briefly describe symbolic partitions in two dimensions. The fundamental requirement that qualifies a chaotic system for communication is whether a good symbolic dynamics can be defined which faithfully represents the dynamics in the phase space. That is, there should be a one-to-one correspondence between points in the phase space and those in the symbolic space. To generate a symbolic dynamics, one first partitions the phase space into cells C_i $(i = 1,2,\dots,m)$ covering the entire attractor and then assigns symbols s_i to cells C_i, respectively. Consider a point $\mathbf{x}$ in the cell C_i, together with a segment of its unstable manifold. Let $\mathbf{a}$ and $\mathbf{b}$ be the two intersecting points of the unstable manifold segment with the cell boundaries, as shown in Fig. 1. A primary condition for a good symbolic dynamics is that the images of $\mathbf{a}$ and $\mathbf{b}$ under the map $\mathbf{F}(\mathbf{x})$, denoted by $\mathbf{F}(\mathbf{a})$ and $\mathbf{F}(\mathbf{b})$, respectively, should still be at the cell boundaries within which the curve $\mathbf{F}(\mathbf{a})\mathbf{F}(\mathbf{b})$ lies, as shown in Fig. 1(a). However, for an arbitrary partition, situation may arise where one of the end points, say $\mathbf{b}$, is no longer on a cell boundary, thus creating a ‘‘dangling end,’’ as shown in Fig. 1(b). Dangling ends may also occur for the stable manifold of $\mathbf{x}$ under the inverse map $\mathbf{F}^{-1}$. In both cases, there is no one-to-one correspondence between points in the phase space and those in the symbolic space. Such an ill-defined symbolic representation of phase-space points is not desirable for communication application, as ambiguities will arise when one attempts to assign symbols to different cells. Nonetheless, if the chaotic attractor is hyperbolic, the partition into cells can be chosen in such a way that the situation of dangling ends depicted in Fig. 1(b) does not occur [8]. Such partitions are called Markov partitions [11], the dynamics of which is schematically illustrated in Fig. 1(a). The partition is *generating* if every infinitely long symbol sequence created by the partition corresponds to a single point in the phase space [9].

Since chaotic attractors arising in most two-dimensional maps are nonhyperbolic, the key issue becomes how to find hyperbolic subsets embedded in the attractor. To illustrate the principle, we use the Hénon map [10]: $(x,y) \to (1.4 - x^2 + 0.3y, x)$, which is widely believed to admit a chaotic attractor. The partition is a zig-zag curve connecting all primary tangency points in the phase space, which lies near $y = 0$ [9]. Trajectory points above the curve bear symbol **1** and those below bear symbol **0**. This curve is a generating partition but it is difficult to compute. To overcome this difficulty while preserving the generating partition, we look for chaotic saddles embedded in the attractor with a gap region, or a forbidden region, defined by $y = \pm s/2$, which covers the partition curve. Due to the gap, a trajectory restricted to the chaotic saddle will never visit the vicinity of the zig-zag partition curve. The partition for the chaotic saddles thus becomes easy to locate: it is the gap itself. In particular, trajectory points above the gap bear symbol **1** and those below bear symbol **0**, and this partition is *generating*. Fig. 2(a) shows such a chaotic saddle with gap size $s = 0.2$. The chaotic saddles are numerically computed by the Proper-Interior-Maximum triple (PIM-triple) procedure [12]. Since the forbidden region

[1] Strictly speaking, the hyperbolic subsets do not cover the entire attractor and, hence, the corresponding ‘‘generating partition’’ is not the generating partition for the original map but for a ‘‘truncated’’ map.

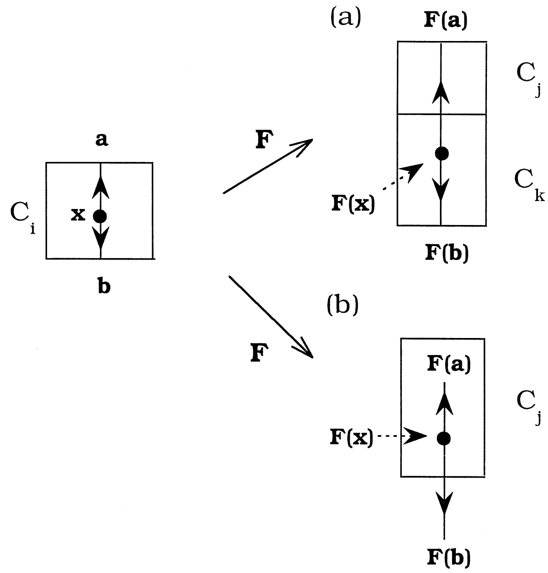

Fig. 1. (a) The forward dynamics of a Markov partition. (b) For an arbitrary partition, a ‘‘dangling end’’ of the unstable manifold. This dangling end destroys the one-to-one correspondence between the phase space and the symbolic space.

contains *all* the primary tangency points, the chaotic saddle in Fig. 2(a) is apparently *hyperbolic*: it does not contain any tangency points between the stable and unstable manifolds. As such, a Markov partition can be defined for such a hyperbolic saddle, which naturally admits a good symbolic dynamics. In fact, there are infinitely many gap sizes s which correspond to different hyperbolic chaotic saddles embedded in the attractor.

The hyperbolic chaotic saddle shown in Fig. 2(a) is a subset embedded in the chaotic attractor and, hence, its topological entropy cannot be larger than that of the attractor. A question is then, how severe is the reduction in the topological entropy. This question is important for communication because the topological entropy of a chaotic set characterizes, quantitatively, how much information can be encoded into the trajectories on the set (the channel capacity) [13,3–5]. To address this, we compute the topological entropy $h_T(s)$ of the chaotic saddle

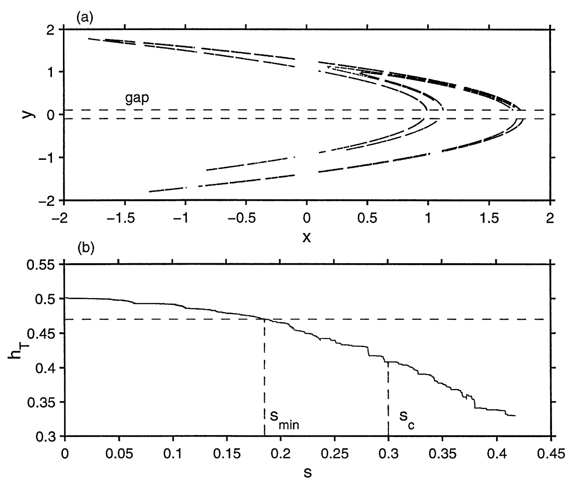

Fig. 2. (a) A hyperbolic chaotic saddle embedded in the Hénon attractor with gap size $s = 0.2$. (b) The topological entropy $h_T(s)$ of the chaotic saddle as a function of the gap size s.

as the gap size s is increased from zero [5], as shown in Fig. 2(b) for $0 \leq s < s_{\max} \approx 0.42$, where the dashed horizontal line at $h_T = 0.466$ defines the minimum gap size $s_{\min}$ above which the generating partition for the chaotic saddle is simply [2] $y = 0$. We see that as s increases from 0, h_T decreases slowly at first, and then faster. The slowly decreasing behavior warrants a relatively large regime $s < s_c \approx 0.3$ in which h_T decreases only slightly (less than 10% reduction in h_T). The key implication is that utilizing chaotic saddles with gap size smaller than s_c but larger than $s_{\min}$ seems to be practically beneficial in communication applications: the specification of the symbolic dynamics is straightforward, yet the channel capacity is close to that obtained when one utilizes the original chaotic attractor. We conjecture that the function of h_T versus s is a devil's staircase, a statement that can be made rigorous for certain one-dimensional maps [5].

In general, it is advantageous to use chaotic saddles, such as the ones depicted in Fig. 2(a), for communication, because the symbolic dynamics on the chaotic saddle are immune to small noise. If the system is in a noisy environment, and the original chaotic attractor is used to encode messages, then a bit error (i.e., **0** becomes **1** or vice versa) may occur whenever the trajectory comes close to the partition curve, because noise can kick the trajectory over the curve in both directions. However, trajectories on the chaotic saddles do not come close to the partition point because of the forbidden region. Thus, the possibility for bit error due to noise can be substantially reduced when a chaotic saddle is utilized to encode messages if the noise amplitude is smaller than $s_{\min}$. Generally, there is a trade-off between the channel capacity and noise resistance.

We now give an example of coding a specific message. Suppose we wish to encode the message ''BEAT ARMY!'' into a trajectory in the chaotic saddle in Fig. 2(a). The message ''BEAT ARMY!'' has the following ASCII representation:

$$\overbrace{1000010}^{B}\overbrace{1100101}^{E}\overbrace{1100001}^{A}\overbrace{1110100}^{T}\overbrace{0100000}^{\text{space}}\overbrace{1000001}^{A}\overbrace{1110010}^{R}\overbrace{1101101}^{M}\overbrace{1111001}^{Y}\overbrace{0100001}^{!}.$$

If the chaotic saddle were equivalent to the fullshift grammar symbolic dynamics, i.e., no grammatical restrictions, then we could simply find a trajectory in the x,y plane such that its y itinerary exactly follows the above digital message. However, the symbolic dynamics of the chaotic saddle are subshift-type because its topological entropy is less than ln2.

Dynamics on the saddle is representable by the Bernoulli shift map, on a bi-infinite symbol space Σ of two symbols [9,15]. A bi-infinite symbol sequence is a point in the symbolic space: $\sigma = \ldots \sigma_{-2}\sigma_{-1}\sigma_0 . \sigma_1\sigma_2\sigma_3 \ldots \in \Sigma$, where $\sigma_i = \mathbf{0}$ or $\mathbf{1}$, and σ_i is the position of $(x_i, y_i) \in \mathbf{R}^2$, relative to a partition curve, on the ith (pre)iterate for $(i<0) i \geq 0$. Shifting the decimal to the right represents a forward iteration, and shifting the decimal to the left represents an inverse iteration. To quantify the correspondence between a point $\mathbf{x}$ in the phase space and a point (points) in the symbolic space, it is necessary to use a vector function (the so-called coding function [3], corresponding to the ''symbolic plane'' discussed in Ref. [15]): $\mathbf{G} = (\delta, \gamma)$, where δ and γ are determined by

$$\delta = 1 - 0.d_1 d_2 \ldots d_\infty \equiv 1 - \sum_{k=1}^{\infty} d_k 2^{-k}, \quad \gamma = 0.c_1 c_2 \ldots c_\infty \equiv \sum_{k=1}^{\infty} c_k 2^{-k}, \tag{1}$$

where $d_k = \sum_{i=1}^{k} (1 - a_{-i}) \bmod(2)$ and $c_k = \sum_{i=1}^{k} a_i \bmod(2)$. The phase-space dynamics can then be represented by the following map in the coding space: $(\delta_{n+1}, \gamma_{n+1}) = D(\delta_n, \gamma_n)$, where $D(\delta,\gamma) = \{(1-\delta)/2, 2\gamma\}$ if

[2] When the noise-gap size is small so that the zigzag generating partition curve cannot be covered entirely, utilizing $y = 0$ as the partition line for the symbolic dynamics leads to an error ΔN in $N(n)$, the number of possible symbol sequences of length n. We have compared the values of h_T for $s = 0$ (the chaotic attractor) (i) by counting $N(n)$, with x-axis as the partition line, and (ii) by using a procedure developed by Newhouse and Pignataro [14]. From the counting method (i), we obtain $h_T(s=0) \approx 0.500$, while the Newhouse–Pignataro algorithm (ii) gives $h_T(s=0) \approx 0.466$. This suggests that when a chaotic saddle has entropy less than about 0.466, its noise-resisting gap has already covered the zigzag generating partition of the attractor. This, in turn, gives an estimation for the value $s_{\min}$ in Fig. 2(b), the minimum gap size for which a good symbolic dynamics can be defined by simply using $y = 0$ as the partition.

$\gamma < 1/2$ and $D(\delta,\gamma) = \{(1+\delta)/2, 2-2\gamma\}$ if $\gamma \geq 1/2$. A trajectory of 50000 points in the symbolic plane corresponding to the chaotic saddle in Fig. 2(a) is shown in Fig. 3. The forbidden points (blank regions) in the symbolic plane is generated by the *pruning front* [15]. Fig. 3 thus determines, completely, the grammar on the chaotic saddle, from which a controlling scheme can be derived to encode messages into the trajectories in the chaotic saddle. We note that the pruning front of an embedded chaotic saddle must be ordered less than or equal to the pruning front of the full chaotic attractor, following the fact that the subshift grammar of the chaotic saddle must be a subset of the subshift grammar of the attractor. Furthermore, the pruning front must be a monotone nonincreasing curve (i.e., receding), as a function of the increasing gap. This corresponds to the fact that we observe a monotone nonincreasing topological entropy.

In what follows we present a practical method to learn the grammar and then to encode digital messages. In physical or numerical experiments, only finite precision can be achieved and, hence, it is reasonable to choose an n-bit precision approximation (subshift of finite type). A way to represent the transitions between the allowed n-bit words is to use the directed-graph method in Ref. [4] which was originally discussed for one-dimensional noninvertible chaotic maps (with an infinite shift space). The directed-graph representation is, however, more general: two-dimensional invertible maps (with a bi-infinite shift space) requires little modification, for n-bit words, or truncations of the bi-infinite symbol sequences, which represent $n/2$ pre-iterates and $n/2$ future iterates. The main point of a symbol dynamics representation is that each (x,y) state in phase space occupies a neighborhood which corresponds to an n-bit code, labeled as a node on the graph. There are two possible situations: (1) either a **0** or a **1** may be shifted into the n-bit register, and this choice means that one of the message bits may be controlled; or alternatively, (2) only a **0** or a **1** exclusively may be shifted into the bit register, and this must be a non-message bearing ''buffer-bit'' *even if the bit happens to coincide with the next message bit* because according to Shannon's information theory [13], an event only carries information if that event is not pre-determined. It is exactly this time spent transmitting the buffer-bits which causes decreased channel capacity, as measured by the topological entropy. The more of the n-bit words which have the two possible outcomes, **0** or **1**, the higher the channel capacity. In our numerical experiments, we approximate the

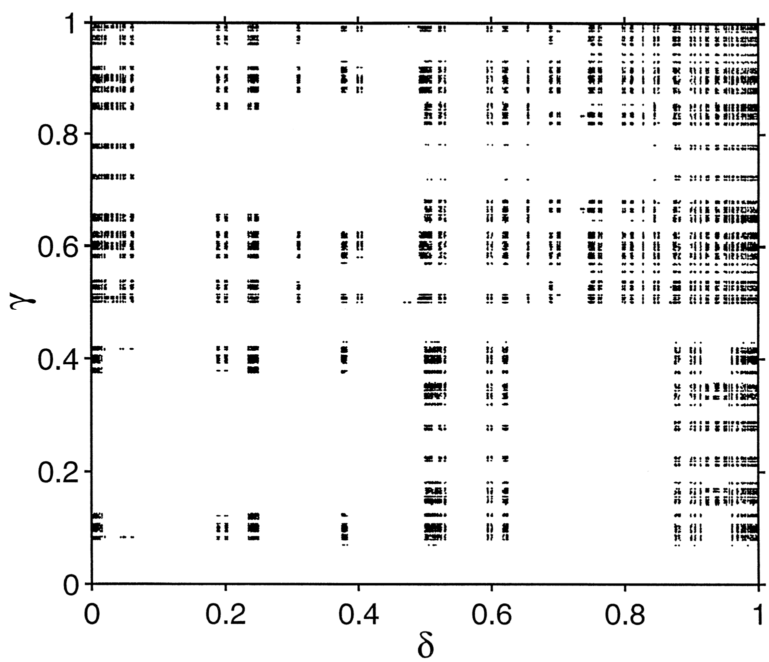

Fig. 3. The symbolic plane for the hyperbolic chaotic saddle in Fig. 2(a).

symbolic dynamics of the chaotic saddle by using 12-bit words. By using the method outlined in Ref. [4], we encode the message into a trajectory on the chaotic saddle, where the actual phase-space trajectory is shown in Fig. 4(a) and the corresponding time series y_n is shown in Fig. 4(b). The receiver can completely recover the original message, given the time-series, the location of the symbol partition $y = 0$, and the grammar in the form of the 2^n list of transitions.

We remark that an alternative method to record the grammar of a two-dimensional map is to use the pruning fronts [15] which was originally developed for the Hénon map as the analogy in the symbol plane to the one-dimensional kneading theory of Milnor and Thurston [16]. Both theories give a partial order for the symbol representation of a given point, relative to a ''maximum'' grammatically allowed word. Given a particular *n*-bit sequence, it is only necessary to check whether both possibilities are grammatically permitted. In the case of the kneading theory, one checks, in the Gray-code ordering, whether both shifting in a **0** and a **1** give new *n*-bit words which are also below the kneading sequence, which is the maximum sequence corresponding to the symbolic code of the critical point. In the case of the two-dimensional pruning-front theory, one must check that both **0** and **1** lead to symbolic codes ordered ''below'' the pruning front; if either shift, say a **0** (or **1**), is greater than the pruning front, then that word is grammatically forbidden on the chaotic saddle, and therefore the alternative shift, say the **1** (or **0**), is determined. In either case, just as with the directed-graph method of book-keeping the grammar, information theory demands that when the two possible outcomes are permitted, the

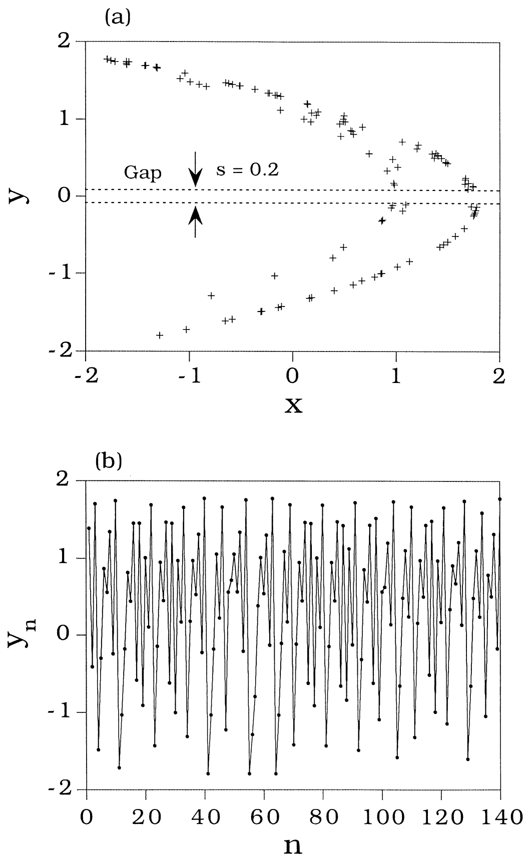

Fig. 4. Example of encoding a message into a chaotic saddle: (a) controlled trajectory in the phase space; and (b) the corresponding time series.

message bit can be transmitted, but when one of the alternatives (**0** or **1**) leads to a symbolic code ordered larger than the pruning front, the transmitted bit must be a non-message bearing buffer bit.

In conclusion, we have studied the feasibility of utilizing two-dimensional symbolic dynamics for communicating with chaos. The main difficulty for chaotic attractors in two-dimensional invertible maps, arising from three-dimensional flows, is that due to nonhyperbolicity, the generating partition for defining a good symbolic dynamics is extremely difficult to compute. Our idea is that there typically exists an infinite number of hyperbolic chaotic saddles embedded in the chaotic attractor for which the generating partition can be easily specified. The hyperbolic chaotic saddles have the additional property that their symbolic dynamics are immune to small environmental noise. When chosen properly, the topological entropies of the chaotic saddles can be close to that of the original attractor. These advantages make dynamical systems described by two-dimensional invertible maps potential candidates for nonlinear digital communication.

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