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Metamorphosis of chaotic saddle

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Abstract

Chaotic saddles are nonattracting dynamical invariant sets that can lead to a variety of physical phenomena. We report our finding and analysis of a type of discontinuous global bifurcation (metamorphosis) of chaotic saddle that occurs in high-dimensional chaotic systems with an invariant manifold. A metamorphosis occurs when a chaotic saddle, lying in the manifold, loses stability with respect to perturbations transverse to the invariant manifold. The fractal dimension of the chaotic saddle increases *abruptly* through the bifurcation. We illustrate our finding by using a system of coupled maps.

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Bifurcations of dynamical chaotic invariant sets, i.e., how qualitative changes of the invariant sets occur as a system parameter changes, have been a central and important issue in the study of chaotic systems. There are two types of chaotic invariant sets that occur commonly in situations of physical interest: chaotic attractors and nonattracting chaotic saddles. The former are responsible for the sustained random behavior observed in a large variety of deterministic processes, and the latter are responsible for phenomena related to transient chaos such as associated with fractal basin boundaries [1] and chaotic scattering [2]. It is known that almost all discontinuous changes of a chaotic attractor are due to crisis [3]: a dynamical event that can cause a sudden destruction or a sudden enlargement of the chaotic attractor. There has been a large literature on crisis (see Ref. [4], references therein). There has also been work on bifurcation of chaotic saddles. For instance, two chaotic saddles can collide with each other as a system parameter changes through a critical value, resulting in physically observable phenomena such as basin boundary metamorphosis [5] and enhancement of chaotic scattering [6]. More recently, a homoclinic bifurcation has been identified after which a chaotic saddle acquires new pieces that were lo-

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cated at a finite distance from the saddle and were not part of the chaotic saddle before the bifurcation [7].

In this Letter, we present a type of discontinuous bifurcation of chaotic saddles that occur in dynamical systems with an invariant manifold. This bifurcation is fundamentally different from those previously reported [5–7]. Before the bifurcation, there is a low-dimensional chaotic saddle in the invariant manifold and, trajectories on the chaotic saddle are stable with respect to perturbations that are transverse to the invariant manifold. As a system parameter changes, the transverse stability of the chaotic saddle changes. At the bifurcation, the chaotic saddle is neutrally stable in the transverse direction. After the bifurcation, the chaotic saddle becomes transversely unstable and it acquires infinitely more new pieces outside the invariant manifold, which were not existent before the bifurcation. As a consequence, the fractal dimension of the chaotic saddle increases *abruptly* at the bifurcation. Because of this sudden dimension increase, we call the bifurcation a *metamorphosis* of the chaotic saddle [1]. We note that systems with an invariant manifold occur in a large variety of high-dimensional processes [2]: it can arise in systems with a natural symmetry or in systems of coupled oscillators. Thus, metamorphoses of chaotic saddles reported here are in principle a high-dimensional phenomenon, whereas to our knowledge, all previously reported bifurcations of chaotic saddles are for low-dimensional chaotic systems.

To illustrate our findings, we present our results using the following class of high-dimensional chaotic systems that naturally admit an invariant manifold – the coupled map lattices [8] [3]:

$$\boldsymbol{x}_{n+1}^{i} = \boldsymbol{f}\left(\boldsymbol{x}_{n}^{i}\right) + \epsilon \sum_{j=1}^{N} g_{ij} \boldsymbol{h}\left(\boldsymbol{x}_{n}^{j}\right), \quad i = 1, \dots, N, \tag{1}$$

where $\boldsymbol{x}_i$ is a D-dimensional vector, $\boldsymbol{f}$ and $\boldsymbol{h}$ are D-dimensional vector functions, ϵ is a parameter characterizing the coupling strength, and g_{ij} denotes the elements of the coupling matrix. For such a system, of great importance is the synchronization state (manifold) $\mathscr{M}$ defined by $\boldsymbol{x}^1 = \boldsymbol{x}^2 = \dots = \boldsymbol{x}^N$. If the elements of the coupling matrix satisfies $\sum_j g_{ij} = 0$, then the synchronization state is a solution of Eq. (1). In this case, if the system starts from an initial condition in $\mathscr{M}$, the state of the system remains synchronized in the absence of random noise. The synchronization manifold $\mathscr{M}$ is thus an *invariant* manifold for the system. It has dimension D, while the full dynamics lies in a manifold of dimension $N \times D$.

We consider a fairly general situation where the map $\boldsymbol{f}(\boldsymbol{x})$ describing the dynamics in the invariant manifold can exhibit typical nonlinear behaviors, e.g., a period-doubling cascade to chaos and the existence of periodic windows in the chaotic regime. We focus on the situation where there is a chaotic saddle in the invariant manifold, which occurs when $\boldsymbol{f}(\boldsymbol{x})$ falls in one of the infinite number of periodic windows where there is also an attracting periodic orbit that coexists with the chaotic saddle. There is thus transient chaos in the invariant manifold: a trajectory starting from a random initial condition in the invariant manifold wanders in the vicinity of the chaotic saddle for a finite amount of time before settling into the stable periodic orbit. For initial conditions off the invariant manifold, there are four possibilities: (1) being attracted to the periodic attractor directly; (2) being repelled from the periodic attractor directly (possibly going to another attractor in the phase space); (3) being attracted to the chaotic

[1] The terminology metamorphosis is inspired from the phenomenon of basin-boundary metamorphosis [5] in which a chaotic saddle collides with another one, leading to a sudden enlargement of the fractal basin boundary and consequently a sudden increase in the fractal dimension of the basin boundary. However, the mechanism for a metamorphosis of a chaotic saddle described in this paper is different from that of the basin boundary metamorphosis. In that case, both saddles were present before the bifurcation. Our metamorphosis is more like a blowout bifurcation of chaotic saddles, a bifurcation documented previously only for chaotic attractors [13].

[2] While there is at present no formal definition of high-dimensional chaos, we use the following notion that seems to gain wide acceptance by researchers in the chaos community: high-dimensional chaotic systems are systems with more than one positive Lyapunov exponent, and low-dimensional chaotic systems are those with only one positive Lyapunov exponent.

[3] We stress that our result holds for any high-dimensional chaotic systems with an invariant manifold.

saddle, exhibiting transient chaos, and then asymptoting to the periodic attractor; and (4) being repelled away from the chaotic saddle (possibly going to another attractor). Whether situations (1) and (3) or (2) and (4) occur depends on the transverse stabilities of the periodic attractor and the chaotic saddle.

To quantify the transverse stability of a trajectory (either chaotic or periodic) in $\mathscr{M}$, we follow Pecora and Carroll [9] and examine the following variational equation of Eq. (1):

$$\delta \boldsymbol{x}^i_{n+1} = \boldsymbol{Df}\left(\boldsymbol{x}^i_n\right)\delta \boldsymbol{x}^i_n + \epsilon \sum_{j=1}^{N} g_{ij} \boldsymbol{Dh}\left(\boldsymbol{x}^i_n\right)\delta \boldsymbol{x}^j_n, \quad (2)$$

where $\boldsymbol{Df}$ and $\boldsymbol{Dh}$ denote the Jacobian matrices of the vector functions $\boldsymbol{f}$ and $\boldsymbol{h}$, respectively. On $\mathscr{M}$, where $\boldsymbol{x}^1 = \ldots = \boldsymbol{x}^N = \boldsymbol{x}$, Eq. (2) can be written concisely as:

$$\boldsymbol{\delta} X_{n+1} = \left[\boldsymbol{I}_N \otimes \boldsymbol{Df}(\boldsymbol{x}) + \epsilon \boldsymbol{g} \otimes \boldsymbol{Dh}(\boldsymbol{x})\right] \cdot \boldsymbol{\delta} X_n, \quad (3)$$

where $\boldsymbol{\delta} X = (\delta \boldsymbol{x}^1, \ldots, \delta \boldsymbol{x}^N)^T$, and $\boldsymbol{I}_N$ denotes the $N \times N$ identity matrix. Assuming that the coupling matrix $\boldsymbol{G}$ is diagonalizable, we write: $\boldsymbol{g} = \boldsymbol{T}^{-1}\boldsymbol{\Gamma}\boldsymbol{T}$ with $\boldsymbol{\Gamma} = \mathrm{Diag}(\gamma_0, \ldots, \gamma_{N-1})$, where $\boldsymbol{T}$ denotes the similarity transform. In the kth eigenspace of $\boldsymbol{g}$, we have:

$$\boldsymbol{\delta} y^k_{n+1} = \left[\boldsymbol{DF}(\boldsymbol{x}) + \epsilon\gamma_k \boldsymbol{DH}(\boldsymbol{x})\right] \cdot \boldsymbol{\delta} y^k_n, \qquad k = 0,1, \ldots, N-1, \quad (4)$$

where $\delta \boldsymbol{y}^i = \sum_j T^i_j \delta \boldsymbol{x}^j$. Since $\sum_{ij} g_{ij} = 0$, the coupling matrix $\boldsymbol{g}$ has at least one zero eigenvalue which we take to be $\gamma_0 = 0$, the corresponding equation determines the stability of a trajectory orbit in $\mathscr{M}$. The remaining $N-1$ equations determine the stability of the trajectory in the $D(N-1)$ directions *transverse* to $\mathscr{M}$. Let $\{\boldsymbol{x}_n\}^{\infty}_{n=1}$ be a trajectory on the chaotic saddle and let $\{\boldsymbol{x}(i)\}^{p}_{i=1}$ denote the stable periodic attractor, where $\boldsymbol{x}(i+1) = \boldsymbol{f}[\boldsymbol{x}(i)]$ $(i = 1, \ldots, p-1)$ and $\boldsymbol{x}(1) = \boldsymbol{f}[\boldsymbol{x}(p)]$. The transverse stabilities of the chaotic saddle and the periodic attractor are then determined by the singular values of the following matrix products, respectively:

$$\prod_{n=1}^{\infty}\left[\boldsymbol{Df}(\boldsymbol{x}_n) + \epsilon\gamma_k \boldsymbol{Dh}(\boldsymbol{x}_n)\right],$$

and

$$\prod_{i=1}^{p}\left\{\boldsymbol{DF}[\boldsymbol{x}(i)] + \epsilon\gamma_k \boldsymbol{Dh}[\boldsymbol{x}(i)]\right\}, \qquad k = 1, \ldots, N-1. \quad (5)$$

In the transverse subspaces, typically we have $\gamma_k \neq 0$ so that the chaotic saddle and the periodic attractor can be transversely stable or transversely unstable, depending on the coupling ϵ. Overall, the transverse stabilities are determined by Λ^s_{T} and Λ^p_{T}, the *largest* transverse Lyapunov exponent of the chaotic saddle and the periodic attractor computed from Eq. (5).

We now argue that a metamorphosis of the chaotic saddle can occur when Λ^s_{T} passes through zero from the negative side as the coupling ϵ changes. We recall that a chaotic saddle is globally nonattracting and it has a basin of attraction of vanishing volume in the phase space. Nonetheless, a *conditionally invariant measure* can still be defined on the saddle [4]. Consider a trajectory on the chaotic saddle in $\mathscr{M}$ with respect to the conditionally invariant measure: call it a *conditional* trajectory. By continuity, a trajectory in the vicinity of $\mathscr{M}$ in the transverse subspaces is also a conditional trajectory. For $\Lambda^s_{\mathrm{T}} < 0$, the conditionally invariant measure on the chaotic saddle is transverse stable. That is, the chaotic saddle tends to attract nearby conditional trajectories in the transverse directions. In this case, the chaotic saddle

[4] A conditionally invariant measure on a chaotic saddle can be defined as follows [10]. Imagine that we enclose the saddle by a cube C in the phase space and we sprinkle a very large number $N(0)$ of initial conditions uniformly in the cube. The number of trajectories that still remain in C is: $N(t) \approx N(0)\mathrm{e}^{-t/\tau}$, where τ is the average lifetime of the chaotic saddle. Let $N_0(\eta,t,C)$ be the number of trajectories that are in C at time ηt, where $0 < \eta < 1$. The conditionally invariant measure is defined to be:

$$\mu_0(C) = \lim_{t \to +\infty} \lim_{n(0) \to \infty} \frac{N_0(\eta,t,C)}{N(t)}.$$

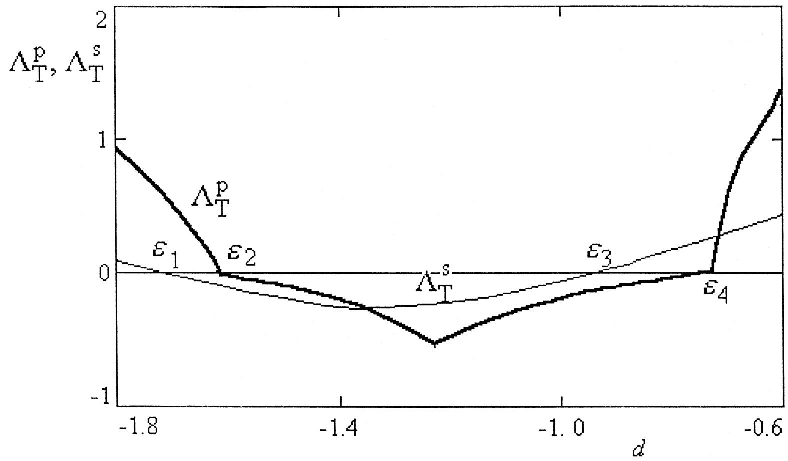

Fig. 1. For $r = 3.63$ in the logistic map (a period-6 wondow), the transverse Lyapunov exponent of the chaotic saddle Λ_T^s (solid line) and the period-6 attractor Λ_T^p (dashed line) versus the coupling strength ϵ.

is confined within the invariant manifold and it is *isolated* from the remaining of the phase space. For $\Lambda_T^s > 0$, the conditionally invariant measure on the chaotic saddle becomes unstable. In this case, a conditional trajectory in the neighborhood of the chaotic saddle is typically repelled away from it asymptotically. Since the trajectory is conditional, it remains conditional under the dynamics even though it is no longer confined within $\mathscr{M}$. The corresponding measure that supports the trajectory must be conditional and, hence, it defines a chaotic saddle that *extends* beyond the invariant manifold $\mathscr{M}$ in the transverse subspaces. As a consequence, a sudden enlargement, or a metamorphosis, of the chaotic saddle occurs as Λ_T^s becomes positive. The newly acquired infinite pieces of the chaotic saddle do not exist before the metamorphosis: they are created at the metamorphosis when the chaotic saddle in $\mathscr{M}$ becomes transversely unstable.

What is the change in the fractal dimension d_s of the chaotic saddle through a metamorphosis? Since $\mathscr{M}$ has a dimension D, before the metamorphosis, the chaotic saddle has dimension $d_s^- < D$. After the metamorphosis, the saddle gains infinite new pieces and it extends in all subspaces which are transversely unstable. Thus, typically we expect the increase in the fractal dimension of the chaotic saddle to be $\Delta d_s \sim m d_s^-$, where m is the number of transversely unstable directions.

We now present numerical evidence for metamorphoses of chaotic saddles. In order to be able to numerically compute and visualize chaotic saddles [5], we consider the following system of two coupled logistic maps, which is a two-dimensional version of Eq. (1):

$$x_{n+1} = r x_n (1 - x_n) + \epsilon (y_n - x_n),$$
$$y_{n+1} = r y_n (1 - y_n) + \epsilon (x_n - y_n), \tag{6}$$

where r is the parameter in the logistic map. We choose r so that the logistic map falls in one of the infinite number of periodic windows [6]. The synchronization manifold $\mathscr{M}$ is one-dimensional, so is the transverse subspace. The transverse Lyapunov

[5] The primary reason that we choose to illustrate our results using two dimensional maps is that for such systems, there exists a procedure, the Proper-Interior-Maximum triple (PIM-triple) procedure [11], for computing an arbitrarily long trajectory on a chaotic saddle with high precision. We are not aware of any procedure that can be utilized to compute trajectories on chaotic saddles in higher dimensions.

[6] Dynamics of Eq. (6), where r is chosen to yield a chaotic attractor in the synchronization manifold, was studied recently [12].

exponent for a conditional trajectory $\{x_n\}_{n=1}^{\infty}$ on the chaotic saddle in $\mathscr{M}$ is given by:

$$\Lambda_T^s = \lim_{M\to\infty} \sum_{n=1}^{M} \ln|r(2x_n - 1) - 2\epsilon|$$

(note that Λ_T^p can be computed similarly). We fix $r = 3.63$ for which the logistic map exhibits a period-6 window in which there is a chaotic saddle and a period-6 attractor. Fig. 1 shows Λ_T^s (solid line) and Λ_T^p (dashed line) versus the coupling strength ϵ for ϵ in the interval $-1.8 < \epsilon < -0.6$, where we see that the chaotic saddle is transversely stable for $-1.65 \approx \epsilon_1 < \epsilon < \epsilon_3 \approx -0.96$ and transversely unstable in the remaining of the interval. Metamorphoses of the chaotic saddle thus occur when ϵ decreases through ϵ_1 and ϵ increases through ϵ_3. The period-6 attractor, however, is transversely stable for $-1.61 \approx \epsilon_2 < \epsilon < \epsilon_4 \approx -0.76$. The fact that the chaotic saddle and the period-6 attractor are transversely stable in different parameter intervals has intricate implications for the basin structure of the system [12]. Our focus, however, is on the matamorphosis of the chaotic saddle. Fig. 2(a) and 2(b) show, for $\epsilon = -1.05 \lesssim \epsilon_3$ and $\epsilon = -0.9 \gtrsim \epsilon_3$, respectively, a long trajectory on the chaotic saddle. Apparently, before the metamorphosis [Fig. 2(a)] the chaotic saddle is confined to $\mathscr{M}$ (the diagonal in the $x-y$ plane). It therefore has a fractal dimension which is less than 1. After the metamorphosis [Fig. 2(b)], the chaotic saddle appears to spread in the entire plane off $\mathscr{M}$ with infinitely many new pieces. At the metamorphosis, there is a sudden enlargement of the chaotic saddle, which can be best seen in the plot of the fractal (information) dimension versus ϵ as ϵ varies through the transition point ϵ_3, as shown in Fig. 3. The fractal dimension jumps, in a rather abrupt fashion, from $d_s \approx 0.8$ before the metamorphosis to $d_s \approx 1.72$ after the metamorphosis, which is consistent with the plots of the chaotic saddles in Fig. 2(a) and 2(b).

In summary, we outline a mechanism for metamorphoses of nonattracting chaotic sets in chaotic systems. The metamorphosis discovered and analyzed in this Letter is a high-dimensional phenomenon that can be expected in systems such as coupled map lattices or coupled ordinary differential equations, which arise naturally when one discretizes a nonlinear partial differential equation.

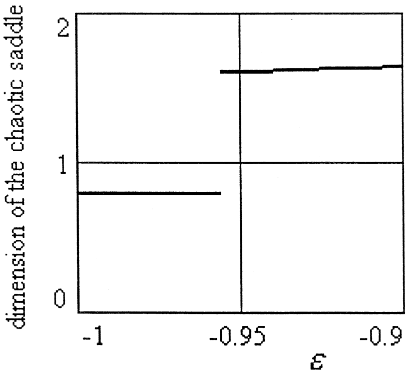

Fig. 3. Information dimension of the chaotic saddle versus ϵ as ϵ increases through the metamorphosis. Apparently, there is an abrupt jump in the dimension at the metamorphosis.

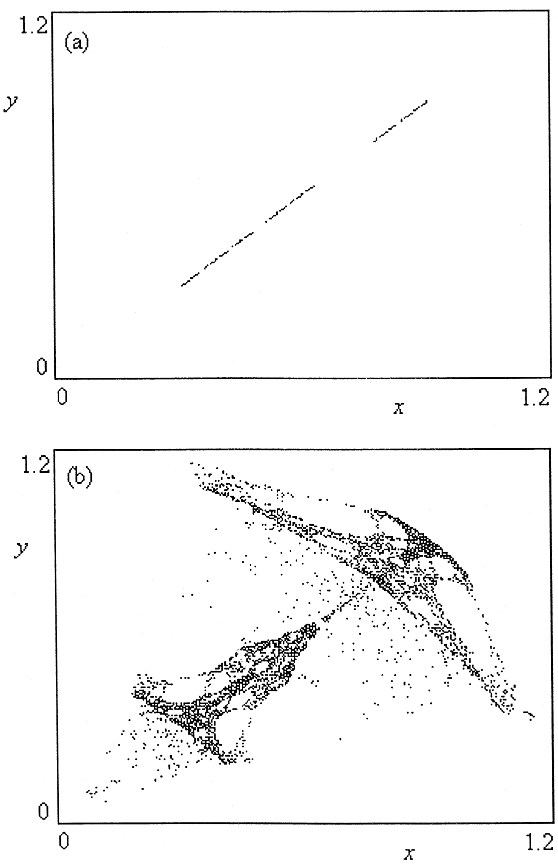

Fig. 2. For $\epsilon = -1.05$ (a) before the metamorphosis and $\epsilon = -0.9$ (b) after the metamorphosis, a long trajectory on the chaotic saddle.

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