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Extreme final state sensitivity in inhomogeneous spatiotemporal chaotic systems

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Abstract

Recently it has been found that spatiotemporal chaotic systems modeled by coupled map lattices with translational symmetry exhibit an extreme type of final state sensitivity characterized by a near-zero uncertainty exponent in both phase space and parameter space. A perturbation in initial condition and parameter, no matter how small from the point of view of computation, has a significant probability of altering the system's asymptotic attractor completely. In this paper we demonstrate that such a final state sensitivity persists for spatiotemporal systems without symmetry. This suggests that extreme final state sensitivity is a robust dynamical phenomenon in spatiotemporal chaotic systems.

The asymptotic attractor, or the final state, of a dynamical system depends both on the choice of the initial condition and the parameter specifying the system. Dynamical systems with multiple attractors can exhibit *final state sensitivity* in phase space, i.e., the attractor to which the system asymptotes depends sensitively on the initial condition [1,2]. In such a case, boundaries separating basins of attraction of distinct attractors are fractal sets. Final state sensitivity can also arise in the parameter space of chaotic systems, such as the logistic map [2,3], where, for certain parameter regimes, small changes in the parameter can alter the final attractor from chaotic to nonchaotic. However, this parameter dependence is rather "weak" [2,3], insofar as errors in the numerical approximation of the parameter value on a digital computer are unlikely to affect predictions of the asymptotic attractor. Recently, it has been found that low-dimensional dynamical systems with certain symmetries exhibit an extreme type of final state sensitivity in phase space due to the occurrence of "riddled basins" [4]. In such a case, any perturbation in the initial condition, no matter how small, has a positive probability to completely alter the asymptotic attractor. It is therefore impossible to compute correctly the final state for such low-dimensional systems.

Spatiotemporal chaotic systems are high-dimensional dynamical systems. Recently, extreme final state sensitivity in both phase space and parameter space has been established in spatiotemporal systems modeled by globally coupled map lattices with periodic boundary conditions [5]. In certain parameter regimes, there

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are multiple coexisting attractors of different types (e.g., chaotic, quasiperiodic and periodic), and the attractor to which the system asymptotes depends extremely sensitively on the choice of both the parameters and the initial conditions. A scaling exponent which quantitatively characterizes the final state sensitivity, called the uncertainty exponent [1,2], has been found to be near zero [5], indicating a significant probability of error in the computation of the final state for particular parameter values and initial conditions. A similar behavior has also been observed qualitatively on a system of diffusively coupled ordinary differential equations (the Duffing oscillator) [5]. A common feature of these spatiotemporal systems is a high degree of symmetry with respect to the spatial site of individual elements. In particular, in Ref. [5] it was assumed that each coupled element is identical and the coupling is uniform and, therefore, the spatiotemporal systems are homogeneous.

The type of perfect spatial symmetry assumed in previous studies [5] describes some highly idealized situation. In practice, the existence of external random noise and system imperfect identification renders inevitable non-uniformity among spatial elements. In physical situations it is often the case that certain properties which exist in systems with a good symmetry would no longer be present when the symmetry is broken. For example, the occurrence of "riddled basins" in certain low-dimensional dynamical systems relies on the system's possessing a perfectly symmetric invariant manifold [4]. Most well studied low-dimensional chaotic systems do not exhibit riddled basins because these systems do not have such a symmetric invariant manifold. *It would be surprising if certain dynamical properties which exist in symmetric systems continue to exist even if the symmetry is broken.* It is, therefore, interesting to investigate whether the extreme type of final state sensitivity which exists in homogeneous spatiotemporal chaotic systems would persist when the homogeneity no longer exists. This would, in turn, address the question of whether such an extreme final state sensitivity is common in more realistic spatiotemporal dynamical systems. The main point of the paper is to demonstrate that extreme final state sensitivity also exists in spatiotemporal chaotic systems when the translational symmetry is broken. Both the phase space and parameter space uncertainty exponents have been found to be near zero. This strongly suggests that extreme final state sensitivity may be a generic feature of spatiotemporal chaotic systems.

We consider the following globally coupled Hénon map lattice [6],

$$\begin{aligned} x_{n+1}(i) &= a(i) \\ &\quad - \left((1-\delta)x_n(i) + \frac{\delta}{N-1}\sum_{j,j\neq i} x_n(j)\right)^2 + by_n(i), \\ &\quad i = 1,\dots,N \\ y_{n+1}(i) &= x_n(i), \end{aligned} \tag{1}$$

where i and n denote discrete spatial sites and time, respectively, $a(i)$ and b are the parameters of the Hénon map [7], and δ is a parameter specifying the coupling strength between maps at different sites. For simplicity we assume that the coupling is uniform. To break the translational symmetry, we assume that the parameter $a(i)$ is different at each site [1]. The Jacobian of the map is $(-b)^N$. In the numerical computations to be described, b is fixed at a value $b = 0.3$, and $a(i)$ is chosen randomly with a uniform probability density in the interval $1.2 \leqslant a(i) \leqslant 1.4$.

Our general diagnostic tool to determine the nature of asymptotic attractors is the maximum Lyapunov exponent [8]. For a system of N coupled Hénon maps, there are $2N$ Lyapunov exponents. Let λ_1 be the maximum exponent. For given parameter and initial condition, if $\lambda_1 > 0$ (< 0), then the asymptotic attractor is chaotic (periodic), and $\lambda_1 = 0$ indicates quasiperiodic attractor. In numerical experiments, we compute $\lambda_1(n)$ using up to 500000 iterations and 10000 preiterations. Such a computational setting is in general sufficient to detect different asymptotic attractors accurately, i.e., to distinguish between chaotic, quasiperiodic, and periodic attractors [5]. To explore parameter space and phase space sensitivity, it is necessary to compute λ_1 for many parameters and initial conditions. Since Eq. (1) is high dimensional, the compu-

[1] In Ref. [5], the two-dimensional parameter space defined by a and δ are investigated for Eq. (1) in the case where both a and δ are constants. It is found that there are parameter regions satisfying the property that for a parameter pair that leads to chaotic attractors, there are parameter pairs arbitrarily nearby that lead to nonchaotic attractors. Therefore, breaking the translational symmetry either in a (as we do for this paper), or in the coupling δ, or even in an arbitrary line in the a–δ plane, would in principle produce qualitatively similar results.

Table 1
Values of $a(i)$ in Eq. (1) used for numerical experiments

map site index i	$a(i)$
1	1.278008502024068
2	1.240084527170116
3	1.211922865324661
4	1.299184331464945
5	1.230301155273752
6	1.336431156209852
7	1.301978877177085
8	1.205272215658278
9	1.239986797398543
10	1.329232468081715

tation is thus very intensive. We have used the massively parallel Connection Machines CM5. The strategy is to map the whole system Eq. (1) onto each processor of the CM5, and evolve the system forward in time in parallel, assigning different initial conditions or parameters to different processors. In this fashion, λ_1 can be computed in parallel. This strategy provides a significant improvement over conventional workstations in the computation time required.

We study the case where the values of $a(i)$ are listed in Table 1. The first step is to compute λ_1 as δ increases from 0. Fig. 1a plots, for $N = 10$, λ_1 (computed using 500000 iterations) versus δ, where 1024 values of δ were chosen uniformly in a range $0 \leqslant \delta \leqslant 1.4$. The set of initial conditions was fixed as δ is varied. In general, when δ is small (< 0.1 in Fig. 1a), the dynamics of maps at different spatial sites are almost independent of each other [5]. The interesting dynamics occur when $0.1 < \delta < 0.3$. In this coupling regime, λ_1 fluctuates wildly. The fluctuations of λ_1 persist as smaller scales of δ ranges are examined. Fig. 1b shows a blowup of part of Fig. 1a for $0.15 \leqslant \delta \leqslant 0.25$ (1024 values of δ). In this small parameter range, the values of λ_1 can be positive, zero or negative, indicating the existence of chaotic, quasiperiodic and periodic attractors, respectively. For an arbitrarily small change in the coupling δ, if λ_1 is positive, then it may change to zero or negative. This suggests that in this coupling regime, the asymptotic attractor of the system depends extremely sensitively on the parameter δ, a phenomenon similar to that observed when each coupled map is identical, i.e., $a(i) = a_0$ (constant) [5].

The extreme sensitive parameter dependence seen

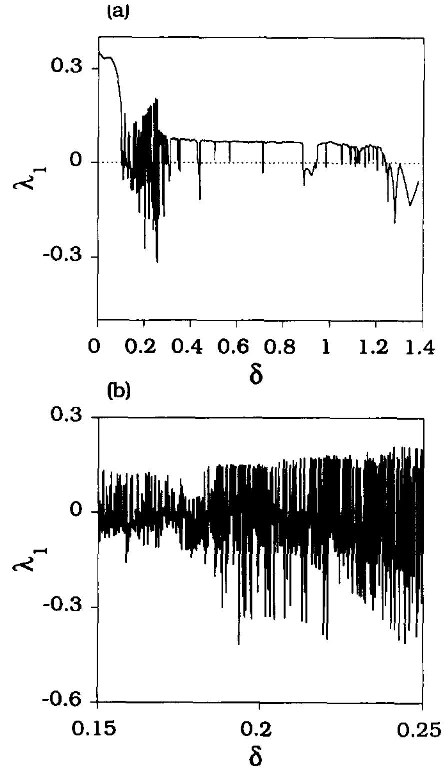

Fig. 1. (a) The maximum Lyapunov exponent λ_1 versus the coupling δ for Eq. (1), where $N = 10$, $b = 0.3$, and values of $a(i)$ ($i = 1, \ldots, 10$) were randomly chosen in $1.2 \leqslant a(i) \leqslant 1.4$ (listed in Table 1). There are wild oscillations of λ_1 when $0.1 < \delta < 0.3$. (b) An expanded view of (a) for $0.15 < \delta < 0.25$.

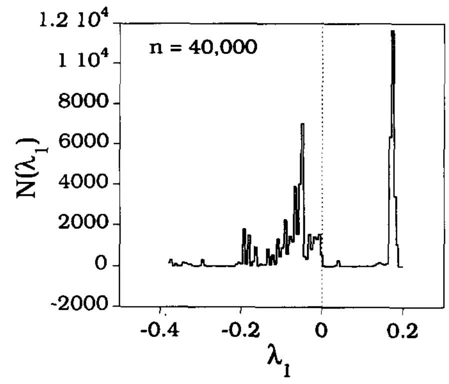

Fig. 2. A histogram of 65536 values of λ_1 computed from initial conditions on a uniform 256 × 256 grid in the two-dimensional phase space region $-1 \leqslant x(8), y(8) \leqslant 1$, where $\delta = 0.22$. There is one peak at $\lambda_1 \approx 0.18$ (chaotic attractor) and numerous peaks in $\lambda_1 \leqslant 0$.

Table 2
Initial values of $x(i)$ and $y(i)$ ($i = 1, \ldots, 7, 9, 10$) chosen for computing basins of attraction and the phase space uncertainty exponent. Values of $x(8)$ and $y(8)$ are chosen systematically on a uniform grid (see text for details)

map site index i	$x(i)$	$y(i)$
1	−0.8838959932327271	−0.4387297630310059
2	−0.9126765727996826	0.4580087661743164
3	−0.8723410367965698	−0.2822475433349609
4	0.4781551361083984	0.8546810150146484
5	−0.1393966674804688	0.5457124710083008
6	−0.9611161053180695	0.7243089675903320
7	−0.4340405464172363	−0.1071848869323730
9	−0.1845588684082031	0.7604818344116211
10	−0.5194680690765381	−0.4350042343139648

above can be related to a similar type of dependence in phase space [5]. This can be understood heuristically by noting that small perturbations in phase space at fixed parameter values can be regarded as equivalent to small perturbations in the parameter space at fixed initial conditions, provided that the map has a smooth dependence on both state variables and parameters [2]. In order to explore the phase space, we choose a two-dimensional plane among the $2N$ state variables and systematically examine the type of attractors resulting from many initial conditions chosen on this plane. Fig. 2 shows, for $\delta = 0.22$, a histogram of λ_1 values (computed using 40000 iterations) resulting from 65536 initial conditions chosen from a uniform 256×256 grid in the region $-1 \leqslant x(8), y(8) \leqslant 1$. Values of state variables at other sites are chosen randomly and then fixed (listed in Table 2). There is one peak at $\lambda_1 \approx 0.18$ and numerous peaks in $\lambda_1 \leqslant 0$. This indicates the existence of one chaotic attractor and many nonchaotic attractors. To assure that the peak at $\lambda_1 \approx 0.18$ indeed corresponds to an attractor rather than a chaotic transient, we have computed snapshots of the histogram using a smaller 32×32 grid of initial conditions at every 20000 iterations up to 500000 iterations. Figs. 3a and 3b show the snapshot histograms at 300000 and 500000 iterations, respectively. It can be seen that the histogram does not change as the number of iterations is increased. This suggests that the peak at $\lambda_1 \approx 0.18$ is due to the existence of a chaotic attractor [3].

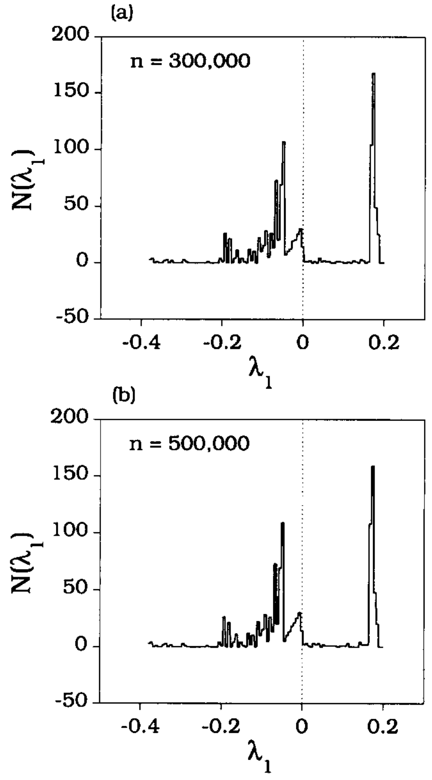

Fig. 3. Snapshots of histogram of 1024 values of λ_1 from a 32×32 uniform grid of initial conditions at 300000 (a), and 500000 (b) iterations. The histogram appears to be unchanged when the number of iterations is increased, indicating that the peak at $\lambda_1 \approx 0.18$ corresponds to a chaotic attractor.

[2] This observation was first used by Moon to detect fractal basin boundaries in experiments [9].

[3] We have not ruled out the possibility that this corresponds to a chaotic transient. However, such a transient would have an extremely long lifetime, which we estimated to be about 3.3×10^7.

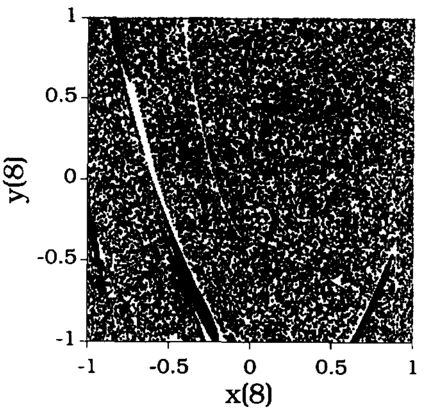

Fig. 4. The basin of the chaotic attractor (black dots). Blank regions are the combined basin of all nonchaotic attractors.

Fig. 4 shows the basin of the chaotic attractor (black dots) on the $x(8)$–$y(8)$ plane, where the blank regions correspond to the basins of the nonchaotic attractors. For most regions, the basins of the chaotic and nonchaotic attractors are extremely intermingled. For initial conditions that lead to the chaotic attractor, there appear to be initial conditions nearby that lead to nonchaotic attractors. There are also open regions in which all initial conditions asymptote to the chaotic attractor. This indicates that the basin of the chaotic attractor is not riddled [4].

Final state sensitivity in both phase space and parameter space can be quantified by the uncertainty exponent α, which was first introduced by Grebogi et al. to characterize fractal basin boundaries [1,2]. The exponent α is defined as follows. Randomly choose an initial condition or a parameter value r_0. Define $r_0' = r_0 + \epsilon$, where ϵ is a small perturbation. Determine whether the asymptotic dynamics of the system using these two parameters or initial conditions are qualitatively different (e.g., chaotic versus nonchaotic). Parameters or initial conditions leading to distinct asymptotic attractors upon small perturbation are called uncertain parameter values or uncertain initial conditions, respectively. For a given perturbation ϵ, the fraction of uncertain parameter values or uncertain initial conditions, $f(\epsilon)$, can be computed by randomly choosing many parameter values or initial conditions. For fractal sets, $f(\epsilon)$ decreases with decreasing ϵ, typically scaling with ϵ as $f(\epsilon) \sim \epsilon^{\alpha}$, where α is the uncertainty exponent [1,2]. In numerical simulation of orbits, ϵ can be viewed as the precision with which a parameter or an initial condition is specified. Then α determines the probability that the computed asymptotic behavior inaccurately reflects the true dynamics of the system. If $\alpha > 1$, reducing ϵ can improve the probability of correct computation of the final state. If $\alpha = 1$, a reduction in ϵ results in an equal reduction in the probability of incorrect computation of the final state. If $\alpha < 1$, then a reduction of ϵ will result in only a small reduction of $f(\epsilon)$. In particular, in the extreme case where $\alpha \approx 0$, improvement in the precision ϵ with which δ is specified, even over many orders of magnitude, may result in only an incremental improvement in ability to predict the asymptotic state correctly. The uncertainty exponent is equivalent to the exponent introduced in Ref. [3] to characterize sensitive parameter dependence in dynamical systems. For the spatiotemporal systems studied in Ref. [5], it was found that $\alpha \approx 0$. Hence, there is an extreme final state sensitivity.

Fig. 5a plots $f(\epsilon)$ versus ϵ in the base-10 logarithmic scale, where ϵ is the perturbation in initial conditions chosen randomly on the line defined by $y(8) = 0$ at site 8, and $f(\epsilon)$ was computed by accumulating the number of uncertain initial conditions to 200. The threshold λ_1 value for distinguishing chaotic and nonchaotic attractors is set to be 0.1, i.e., an initial condition is uncertain of it yields $\lambda_1 > 0.1$ and its ϵ-perturbation yields $\lambda_1 \leqslant 0.1$, or vice versa. Choosing a different (but similar) threshold value does not affect the computation of α. The slope of the plot in Fig. 5a, which is approximately the uncertain exponent α in phase space, is estimated to be 0.012 ± 0.003, a value very close to zero. Fig. 5b shows $\log_{10} f(\epsilon)$ versus $\log_{10} \epsilon$, where ϵ is the perturbation in the coupling δ (chosen randomly in the range $0.1 \leqslant \delta \leqslant 0.3$). The uncertainty exponent in parameter space is therefore 0.0089 ± 0.0063, also a near zero value. Hence, for Eq. (1), although the basins of the chaotic attractor is not riddled, there is nevertheless an extreme final state sensitivity.

To appreciate the significance of a near zero uncertainty exponent, assume that α takes its upper bound value of about 0.014 in Figs. 5a and 5b. Assume the value of initial conditions or parameters can be specified to within 10^{-16} (computer double precision arithmetic), then there is a probability of $f(\epsilon) \sim 10^{0.014\times(-16)} \approx 0.6$ that the final asymptotic state

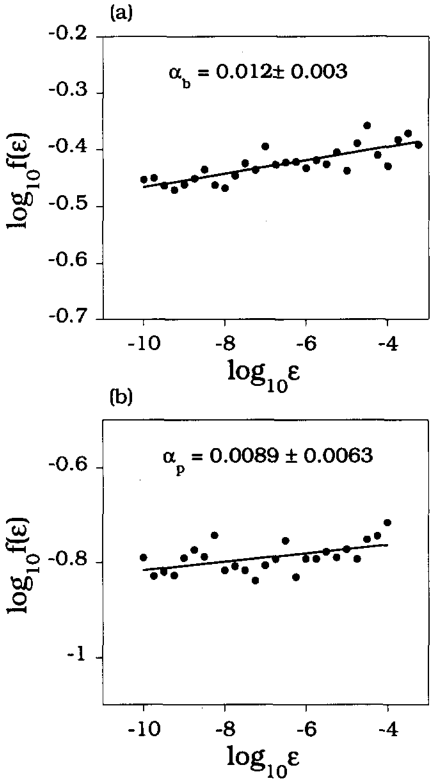

Fig. 5. (a) Plot of $\log_{10} f(\epsilon)$ versus $\log_{10} \epsilon$, where ϵ is the perturbation in initial conditions. The phase space uncertainty exponent is $\alpha_b = 0.012 \pm 0.003$. (b) A similar plot where ϵ is the perturbation in the parameter δ. The parameter space uncertainty exponent is $\alpha_p = 0.0089 \pm 0.0063$. Both uncertainty exponents are close to zero.

computed is incorrect. Improving the precision with which initial conditions or parameters are specified offers little improvement in the probability of computing the final state of the system correctly. For example, suppose computer precision is improved by 16 decades to 10^{-32}. Then the probability of incorrectly computing the asymptotic state is still about $10^{0.014\times(-32)} \approx 0.36$. This means that vast improvement in the computer precision only yields a very small improvement in uncertainty to determine the asymptotic attractor. These near zero uncertainty exponents thus indicate that the globally coupled inhomogeneous Hénon map lattice also exhibits an extreme sensitive dependence of asymptotic attractors on both initial conditions and parameters [4].

Remarks. Uniformly coupled map lattices consisting of identical elements have been discovered to exhibit an extreme type of final state sensitivity in both phase space and parameter space [5]. Such a dependence can be characterized by near zero uncertainty exponents in both phase space and parameter space. The work described in this paper demonstrates that when the perfect translational symmetry of the system is broken, there is still a similar type of extreme final state sensitivity. This strongly suggests that extreme final state sensitivity is a robust dynamical phenomenon in spatiotemporal chaotic systems.

The most profound consequence of such an extreme final state sensitivity is that asymptotic attractors of spatiotemporal systems cannot be computed reliably on machines with finite precision arithmetic. Statistical properties of the asymptotic attractor, such as the Lyapunov exponents and fractal dimensions, are consequently unpredictable in parameter regimes where such a sensitivity occurs.

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