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Temporal crossover from classical to quantum behavior: a Markov-chain approach

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Previous renormalization group analyses that for chaotic systems near their critical points, the crossover time from classical to quantum behaviors scales with the Planck constant $\hbar$ as $t_{\max} \sim \hbar^{-\mu}$. We argue that the same scaling relation also holds for *typical* two-degrees-of-freedom and time-independent chaotic Hamiltonian systems. Our analysis makes use of a self-similar Markov-chain model which was previously used to qualitatively explain the algebraic decay law in Hamiltonian systems.

The quantum mechanical properties of dynamical systems whose classical behaviors are chaotic have attracted studies for many years ([1], reference therein. References on quantum chaos can also be found in, e.g., ref. [2]). The major characteristic that distinguishes a quantum mechanical system from its corresponding classical system is that in quantum mechanics, the system has a finite wavelength or equivalently, finite $\hbar$. Here we use the symbol $\hbar$ to denote the Planck constant nondimensionalized by normalizing to characteristic length and momentum values, so that $\hbar \to 0$ corresponds to the classical limit, and $1 \gg \hbar > 0$ corresponds to the semiclassical regime. When the wavelength is set to zero, the system is classical. In a classically chaotic system, a particle can spend an arbitrarily long time resolving finer and finer phase-space structures. For the same system considered quantum mechanically in the semiclassical regime, however, the phase-space structures with volume less than $O(\hbar^N)$ (where N is the number of degrees of freedom) cannot be resolved. Thus, if we initialize an ensemble of particles in the phase space and evolve them under the dynamics, there comes a time $t_{\max}$ after which the phase-space structures resolved by classical orbits are quantum mechanically not resolved. Therefore, one might expect that the classical mechanics might be relevant for $t \lesssim t_{\max}$ but that the evolution for $t \gtrsim t_{\max}$ is entirely governed by quantum mechanics. We then say that at $t_{\max}$, there is a crossover from classical behaviour to quantum behavior. As the wavelength becomes longer (or $\hbar$ becomes larger), one expects that the quantum effects will take over earlier, i.e. $t_{\max}$ will become smaller. (We should note that quantum effects (e.g. weak localization) are also possible for $t < t_{\max}$. Thus our $t_{\max}$ is an upper bound past which quantum effects are dominant.)

It was shown by Fishman, Grempel and Prange [3] that for a dynamical system near its critical parameter value (e.g., Feigenbaum point in a Hamiltonian period-doubling cascade, or the parameter value at which the last KAM surface is destroyed), $t_{\max}$ scales with $\hbar$ like

$$t_{\max} \sim \hbar^{-\mu}, \tag{1}$$

where μ is a scaling exponent. By using a renormal-

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ization group theory, they were able to calculate the scaling exponent μ for the system near the critical points. In particular, they found that for the system near the Feigenbaum point $\mu \approx 6.04$, while for a Hamiltonian system near a critical point in which the last bounding KAM surface is destroyed $\mu \approx 3.05$. However, it is difficult to apply the renormalization group theory to chaotic Hamiltonian systems not near their critical points. The objective of this Letter is to show that the same scaling relation (1) can be established using a self-similar Markov-chain model (as we will describe below). This model was used to qualitatively explain [4] the algebraic decay law [5] in classical two-degrees-of-freedom, time-independent chaotic Hamiltonian systems. Because this model is only heuristic, we cannot expect it to predict the value of the exponent μ. Nonetheless, the scaling relation (1) derived from it applies to typical two-degrees-of-freedom, time-independent chaotic Hamiltonian systems. Thus, our goal is only to lend credance to the proposition that the scaling relation (1) holds *not only* for dynamical systems near their critical points, *but also* for typical Hamiltonian systems that exhibit algebraic decay.

The phase space of a typical Hamiltonian system consists of KAM islands and chaotic regions. The KAM islands exist in all scales and typically they form island-around-island phase-space structure. Hence, a particle initialized in the chaotic region may spend an arbitrarily long time penetrating the island-around-island structure. Orbits contributing to longer times penetrate more deeply into this structure. A physical consequence is that particles decay algebraically [5] from some pre-defined (large) phase-space region containing the island-around-island structure. To be specific, imagine that there is some large region surrounded by an outermost KAM island curve and that outside this island there is a connected chaotic region. We then draw a large circle which contains the large island at its center and which also encloses a substantial part of the outer chaotic region. Now initialize a large number N_0 of randomly chosen initial conditions in the connected chaotic region outside the large islands and evolve them under the dynamics. If a particle leaves the circle, we regard it as being lost from the system. Let $N(t)$ denote the number of particles inside the circle at time t; then one typically observes [5] that $N(t) \sim t^{-z}$, where z is the decay exponent.

A self-similar Markov-chain model was introduced in ref. [4] to heuristically explain the algebraic decay in two-dimensional Hamiltonian map systems. (We henceforth restrict consideration to such systems.) The Markov-chain model is characterized by states and transition probabilities between adjacent states. The states in the model correspond to phase-space regions of a typical Hamiltonian system (with KAM surfaces) with different scales. The states are denoted by integers: 0, 1, ..., where "0" specifies a reference state in which the particles are considered to have escaped, and "1", "2", ... correspond to the largest chaotic region in between KAM islands, the second largest chaotic region in between KAM islands, etc. A Markov chain is defined to be self-similar if the transition probability p_{ij} from the state i to state j has the scaling property

$$p_{i+k,j+k} = \epsilon^k p_{ij} , \tag{2}$$

where ϵ is a scaling constant. Another property of the Markov chain is that the probability of a transition from j to $j+1$ differs from the probability of the reverse transform by a factor β (constant):

$$p_{j,j+1}/p_{j+1,j} = \beta . \tag{3}$$

In this paper, following the treatment of Hanson, Cary and Meiss [4], we consider the general nearest-neighbor interacting self-similar Markov chain. For such a model, the transition probability matrix has the form

$$p_{ij} = a_i \delta_{i-1,j} + b_i \delta_{ij} + c_i \delta_{i+1,j} , \tag{4}$$

where the self-similarity property eq. (2) implies

$$a_i = a\epsilon^i, \quad b_i = b\epsilon^i, \quad c_i = c\epsilon^i . \tag{5}$$

We set an appropriate time scale so that $b_1 = -1$. Equation (3) implies $c/a = \beta\epsilon$. Use of particle conservation $\sum_j p_{ij} = 0$ gives

$$a = \frac{1}{\epsilon(1+\beta\epsilon)}, \quad b = -\frac{1}{\epsilon}, \quad = \frac{\beta}{1+\beta\epsilon} . \tag{6}$$

When the transition probabilities are small, the Markov-chain model can be modified to have continuous time:

$$\frac{dN_j}{dt} = \sum_i N_i p_{ij}, \tag{7}$$

where N_j is the mean number of particles in state j. Substituting eqs. (4), (5) and (6) into eq. (7), we obtain the following equation for N_j,

$$\frac{dN_j}{dt} = \epsilon^{j-1}\left(\frac{\beta}{1+\beta\epsilon} N_{j-1} - N_j + \frac{\epsilon}{1+\beta\epsilon} N_{j+1}\right). \tag{8}$$

It can be seen from eq. (8) that self-similarity implies that time is effectively rescaled by ϵ when j is increased by unity.

To proceed, we consider the first passage time distribution $R_{ij}(t)$, which is the probability that a particle in state i at time zero *first reaches* the state j at time t. An important quantity is $R_{10}(t)$, which is the probability of a particle entering the chain in state 1 at time zero and leaving the chain in state 0 at time t. The physical significance of $R_{10}(t)$ is that the states $i=1,2,...,\infty$, can be imagined as being deep in the Cantori [6] sequence, so that a particle enters the chain by first being in state 1 and leaves the chain when it arrives at state 0. Thus, R_{10} gives the statistics of leaving the chain given that the particle enters the chain at $t=0$. Clearly, $R_{10}(t)$ consists of the following two components: (1) the direct first passage time distribution from state 1 to state 0 $R^{d}_{10}(t)$ and, (2) the sum of all the probabilities that the particle reaches some high state i ($i=2,...,\infty$) at some earlier time t_i ($t_i \leqslant t$) and then arrives at state 0 at time t. The direct first passage time distribution $R^{d}_{10}(t)$ is the probability that a particle in state 1 first reaches state 0 at time t *without* having been in another state between times zero and t. This probability is given by [4] $R^{d}_{10}(t) = p_{10}\exp(p_{11}t)$. We denote $R_{1i,i0}(t)$ to be the probability that the particle enters the chain at time zero, reaches some high state i at some earlier time t_i and then reaches the state 0 at time t ($t_i < t$). $R_{1i,i0}(t)$ can be written as $R_{1i,i0}(t) = R_{1i}(t) \otimes R_{i0}(t)$, where $\otimes$ denotes the convolution operation. Using these notations, $R_{10}(t)$ can then be expressed as

$$R_{10}(t) = R^{d}_{10}(t) + \sum_{i=2}^{\infty} R_{1i,i0}(t). \tag{9}$$

It was shown [4,7] analytically that $R_{10}(t)$ decays algebraically for large t, as expected for typical chaotic Hamiltonian systems.

Since $R^{d}_{10}(t)$ decays exponentially, for large time its contribution to $R_{10}(t)$ is negligible. Hence, for large time, the major contribution to $R_{10}(t)$ comes from different $R_{1i,i0}(t)$. In particular, particles stay in the system for a long time by crossing Cantori with a smaller scale. The longer the time is, the more deeply particles penetrate into the chaotic region adjacent to KAM tori. To see this, we have numerically solved eq. (8) using a Markov chain with 30 states. We then solve for $R_{10}(t)$ and for the different $R_{1i,i0}(t)$'s. Figure 1 shows, on a logarithmic scale, plots of $R_{10}(t)$ and $R_{1i,i0}(t)$ ($i=5$, 10, 15, 20) for $\epsilon=0.4$ and $\beta=0.95$. From fig. 1, we see that $R_{10}(t)$ decays algebraically with time, as expected. Moreover, in different times scales, different $R_{1i,i0}(t)$ makes the major contribution to $R_{10}(t)$. In particular, $R_{1i,i0}(t)$ with larger i contributes to $R_{10}(t)$ in larger time. Physically, this means that for a given state i, there comes a time after which the particle can resolve this state or higher.

In the Markov-chain model, the phase-space area in state i scales like $\sim A^i$, where $A<1$ is some scaling constant. At finite wavelength, the minimum phase-space area that the particle can resolve $\sim\hbar$. Hence, this finite $\hbar$ determines the highest accessible state N in the chain. In particular, we have $\hbar \sim A^N$ and hence

$$N \sim |\log(\hbar)|. \tag{10}$$

Our goal is to find the scaling relation between the crossover time t_{max} and $\hbar$ (or N). Given a highest accessible state N, there comes a time t_{max} after which

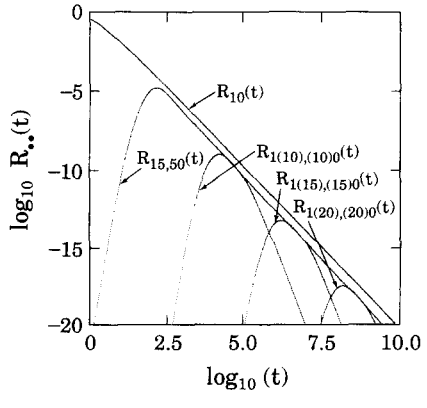

Fig. 1. Plots of $R_{10}(t)$ and $R_{1i,i0}(t)$ ($i=5$, 10, 15, 20) on a logarithmic scale for $\epsilon=0.4$ and $\beta=0.95$ in a Markov chain of total 30 states.

the contribution of $R_{1i,i0}(t)$ $(i \geqslant N)$ to $R_{10}(t)$ becomes significant. Physically, this means that for $t < t_{max}$, the particle has not reached phase-space regions with scale finer than $\hbar$. Hence, for $t < t_{max}$, the particle motion is governed by classical mechanics, while for $t \geqslant t_{max}$, the particle can resolve phase-space regions finer than $\hbar$, which is prohibited by the uncertainty principle. Hence, for $t \geqslant t_{max}$, the particle motion obeys quantum mechanics. Therefore, t_{max} is the crossover time from classical to quantum behavior. To determine t_{max}, it is required that the contribution of $R_{1i,i0}(t)$ to $R_{10}(t)$ at $t = t_{max}$ is negligible, i.e. the probability that the particle reaches state N or higher and then arrives at state zero is negligible. Therefore, we have adopted the following rough criterion for determining the crossover time t_{max},

$$\left.\frac{R_{1i,i0}(t)}{R_{10}(t)}\right|_{t=t_{max}} = 0.01 . \tag{11}$$

The precise value of the right-hand side is not important for our conclusion.

Figure 2a shows the scaling relation of $\log t_{max}$ with N numerically determined from eqs. (2)–(9) and the criterion (11) for the same parameter values of ϵ and β as in fig. 1. From figs. 1 and 2a we have

$$\log t_{max} \sim N \sim |\log(\hbar)| = -\log(\hbar) . \tag{12}$$

We have also checked several different combinations of parameters ϵ and β in the Markov-chain model. The resulting scaling relations are shown in figs. 2b–2d. In particular, the scaling relation associated with particle transport near a critical noble torus is shown in fig. 2b. For all the cases we have considered, the

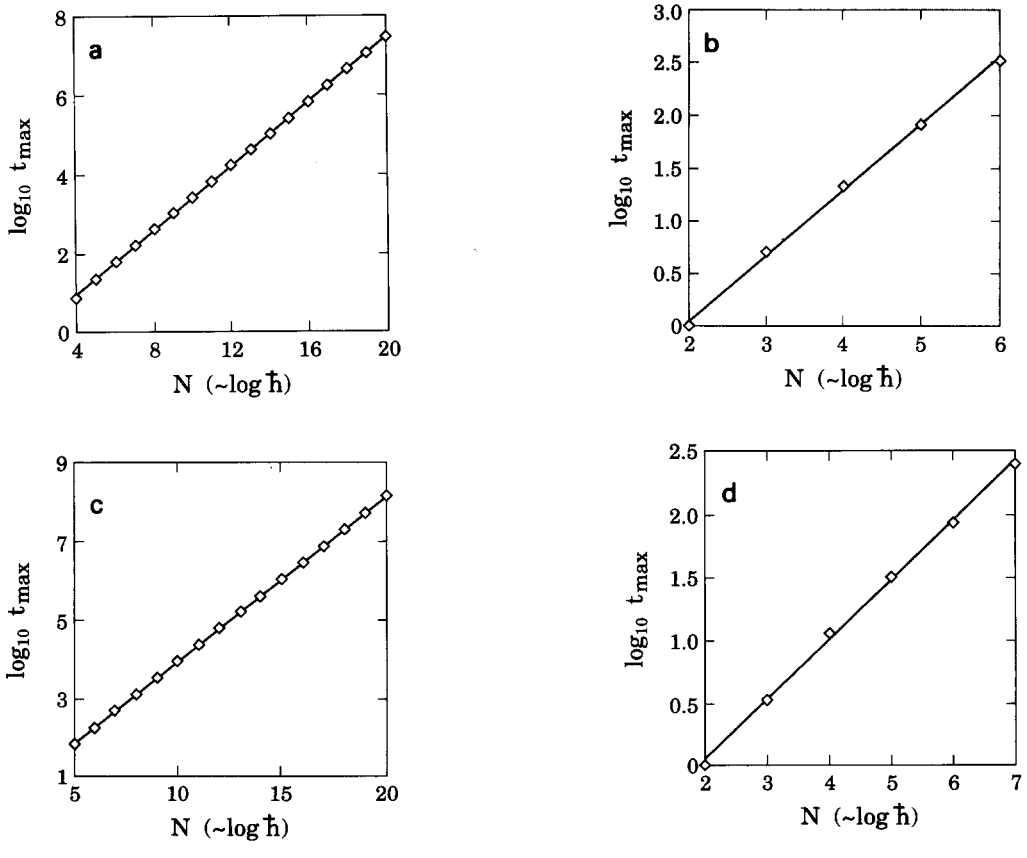

Fig. 2. In a Markov chain of total 30 states, the scaling of $\log t_{max}$ with the highest accessible state N ($\sim |\log \hbar|$) for (a) $\epsilon = 0.4, \beta = 0.95$, (b) $\epsilon = 0.381966, \beta = 0.1390449$ (particle motion near a critical noble torus), (c) $\epsilon = 0.381966, \beta = 1$ and (d) $\epsilon = 0.5, \beta = 0.2$.

scaling relation (12) holds. Equation (12) is equivalent to the scaling relation of eq. (1).

In conclusion, we have established the scaling relation between the quantum crossover time and the Planck constant $\hbar$ by using the Markov-chain model to describe the particle transport in two-degrees-of-freedom, time-independent chaotic Hamiltonian systems. Although our approach does not yield the specific scaling exponent, our analysis indicates that the scaling relation (1) holds, not only for systems near their critical points, but also for typical Hamiltonian systems [#1].

[#1] In a recent paper [8] the authors found that for hyperbolic phase-space regions, the scaling relation between the time for which the semiclassical theory holds and the Planck constant appears to be a power law, in contradiction to the logarithmic law which can be obtained using an argument similar to that of our paper. However, we emphasize that their argument is for the case where hyperbolicity and strong stretching produce highly linear stretched-out phase-space structures. Since we are dealing with orbits approaching to KAM surfaces, linear stretching is weak and the dynamics is strongly nonhyperbolic. We believe that in such cases our argument, based on the Planck constant action area, is qualitatively correct. This is reinforced by the fact that the result (1) is also obtained in the renormalization analysis of Fishman et al. [3].

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