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NOISE-INDUCED CHAOS: A CONSEQUENCE OF LONG DETERMINISTIC TRANSIENTS

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We argue that transient chaos in deterministic dynamical systems is a major source of noise-induced chaos. The line of arguments is based on the fractal properties of the dynamical invariant sets responsible for transient chaos, which were not taken into account in previous works. We point out that noise-induced chaos is a weak noise phenomenon since intermediate noise strengths destroy fractality. The existence of a deterministic nonattracting chaotic set, and of chaotic transients, underlying noise-induced chaos is illustrated by examples, among others by a population dynamical model.

Keywords: Noise-induced chaos; population dynamics; transient chaos; nonattracting chaotic sets; fractal dimension.

1. Introduction

When a system is under noise, several interesting phenomena can happen. A particularly important one is the phenomenon of noise-induced chaos. In a general sense, noise-induced chaos is referred to as a noisy attractor with sensitive dependence on initial conditions, which disappears upon switching off the noise. The largest Lyapunov exponent calculated from a typical trajectory on the noisy attractor is positive.

The discovery and first description of the phenomenon of noise-induced chaos goes back to Iansiti *et al.* [1985], Herzel *et al.* [1987], Bulsara *et al.* [1990], and Hamm *et al.* [1994]. The first biological examples were reported by Rand and Wilson [1991], Drepper *et al.* [1994], and Engbert and

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Drepper [1994], in the context of epidemics. The recent interest in the ecological relevance of noise-induced chaos (see [Dennis *et al.*, 2003; Ellner & Turchin, 2005]) makes it worth reconsidering its relation to underlying chaotic sets.

We emphasize that a clear definition of noise-induced chaos can be obtained by taking into account fractality, which is an undetachable property of chaos (see e.g. [Ott, 1993; Tél & Gruiz, 2006]). Guided by the requirement that noise-induced chaotic attractors of invertible systems should be fractal, one concludes that this can happen in the *weak noise limit* only, since stronger noise washes out any kind of fractality. Chaos in this limit can only be of deterministic origin. The chaotic set of the deterministic problem must be nonattracting, otherwise long term chaos would also be present without noise. Nonattracting chaotic sets can only generate chaos of finite lifetime, therefore we conclude that transient chaos is a prerequisite of noise-induced chaos. It is worth mentioning that the relevance of deterministic chaotic transients has already been pointed out in different contexts of ecology (see e.g. [Huisman & Weissing, 2001; Hastings, 2004]). The observation above can be considered as a further evidence for the importance of deterministic transients.

Thus, the following picture emerges. The underlying deterministic dynamics is such that a typical initial condition leads to a trajectory that behaves chaotically only for a finite amount of time (transient chaos), which occurs when the trajectory is near the nonattracting chaotic set. All such trajectories must end at the periodic attractor, due to the nonattracting nature of the chaotic set. Noise makes it possible that a trajectory on the periodic attractor occasionally leaves it and visits the chaotic set again transiently, comes back to the periodic attractor, and so on. Noise of sufficient (but small) amplitude can thus dynamically link the periodic attractor with the nonattracting chaotic set, leading to a noise-induced attractor that contains the original chaotic set as subset. Such an attractor typically possesses at least one positive Lyapunov exponent.

Note that noise-induced chaos can always occur in settings where, in the absence of noise, there is one or more periodic attractor and a coexisting nonattracting chaotic set, as in any periodic window of a nonlinear dynamical system. Noise-induced chaos is thus a very generic dynamical phenomenon.

We start the argument by calling the attention to the fact that the fractality of chaotic sets remains *unchanged* in the presence of weak noise, and briefly review the properties of transient chaos in deterministic systems. Via simple examples of noise-induced chaos we demonstrate that by adding stronger noise, the fractality of the noise-induced attractor quickly becomes lost. The underlying nonattracting chaotic set is constructed in several cases, including a population dynamical model of fennoscandian voles. Finally, further noise effects are mentioned, and our conclusions are formulated.

2. Chaotic Sets Under Noise

The inclusion of fast external or internal perturbations into the dynamics happens via a noise term, which converts the purely deterministic equations of motion into *stochastic* equations. For illustrative purposes, we consider the case of additive noise only. In our considerations the system variables are assumed to be continuous. As for the role of noise in discrete state systems, we refer the reader to a recent paper by Scheuring and Domokos [2007]. We are thus interested in the noisy version of both continuous and discrete time dynamics, which are described (in dimensionless forms) by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, p) + \sigma\xi(t), \tag{1}$$

and

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p) + \sigma\xi_n, \tag{2}$$

respectively. Here p is the set of system parameters, $\sigma > 0$ represents the noise strength, and the ξ-s are independent, identically distributed random variables of zero mean and unit variance. The distribution $P(\xi)$ is assumed to be known and be *independent of time*, which expresses that the noise dynamics is in a kind of stationary state. The simplest example of P is a Gaussian distribution,

$$P(\xi) \sim \exp\left(-\frac{\xi^2}{2}\right). \tag{3}$$

This form shows that (even for small noise strength σ) the random perturbation can be arbitrarily large, but the probability for large perturbations decreases exponentially. For discrete-time dynamics, we shall also consider homogeneous noise with a uniform distribution, e.g. $|\xi| \in (-1, 1)$.

A very strong noise makes the dynamics fuzzy, and suppresses all the deterministic features. If the dynamics bears some chaotic character, this can only happen in the presence of *weak* noise, which is often the case in real situations, when

$$\sigma \ll 1. \tag{4}$$

It is a general observation due to Ben-Mizrachi *et al.* [1984] that such a weak noise does *not* modify the fractal dimensions characteristic of the chaotic dynamics. What it does is to make the dynamics fuzzy below a certain threshold scale ε_c in phase space, which grows with the noise strength. Below the threshold scale, i.e. in an ε_c-neighborhood of the deterministic fractal sets, trajectories fill the phase space. Beyond that scale, however, the dimension of the deterministic system is found when measuring the noisy system. It is the breakdown of the fractal scaling which depends on the noise strength. The effect of weak noise is thus merely a shrinking of the scaling region where a nontrivial fractal dimension can be observed, but this does not influence the value of fractal dimension itself.

Since the average amount of noisy perturbations for Gaussian noise is σ, just like for homogeneous noise, these types of noise lead to *qualitatively similar* weak noise behavior.

3. Transient Chaos

3.1. *Basic features*

Transiently chaotic time series have the following characteristic properties [Kantz & Grassberger, 1985]:

1. For a fixed initial condition, the time series will be chaotic up to a certain time and then switch over, often quite abruptly, into a different, often nonchaotic behavior which governs all the rest of the time series. The actual length depends very sensitively on the initial condition: nearby trajectories might have drastically different lifetimes.
2. Nevertheless, the *distribution* of lifetimes is a smooth function, which tends to zero for large arguments, with an average value τ.
3. There exist *arbitrarily* long transients. They are, however, exceptional, and the corresponding initial conditions form a set of *measure zero*. In the phase space, these infinitely long-living orbits form a *fractal object*, a *nonattracting chaotic set*.

It is thus an essentially new feature of transiently chaotic signals that by observing them in the phase space one finds, besides the actual attractor(s), *a novel* phase-space object responsible for the irregular transients. These objects do not attract global trajectories from their surroundings.

Trajectories starting *exactly* from points of a nonattracting set *never* leave the set and exhibit chaotic motion forever. It is, however, completely unlikely to hit such a point by random choice since the nonattracting set is a fractal set of zero measure. What is *observable* is not the nonattracting set but rather a *small neighborhood* of it. Trajectories starting close to the set can stay for a long time in its neighborhood and show chaotic properties, but sooner or later they *escape* the neighborhood. These are exactly the trajectories producing transiently chaotic signals.

3.2. *Nonattracting chaotic sets*

Any saddle-like unstable point possesses two different invariant manifolds. A *stable manifold*, a surface along which the point can be reached, and an *unstable manifold*, which is the stable manifold of the time-reversed dynamics (see e.g. [Ott, 1993; Tél & Gruiz, 2006]). These concepts play an important role in the following arguments.

In systems described by differential equations (flows in brief) or invertible maps (due to the invertibility of differential equations, maps following from differential equations are always invertible themselves), the nonattracting chaotic sets are not fully repelling (and should thus not be called repellors). They repell everywhere in the phase space with the exception of a surface, the stable manifold. They are, therefore, called *chaotic saddles*. Qualitatively speaking, a chaotic saddle is an infinite union of saddle points, and has a fractal pattern. An example of a chaotic saddle and its manifolds is shown in Fig. 1, where the "double fractal" pattern so characteristic of chaotic saddles can be clearly seen [Tél, 1990]. This pattern is due to the fact that a chaotic saddle is the intersection of its stable and unstable manifolds, which are both fractal curves. The pattern is, therefore, markedly different from that of chaotic attractors in which only the unstable manifolds are fractals (the stable manifolds are space filling, otherwise a finite basin of attractions cannot exist).

The construction of chaotic saddles (and their manifolds) plays a central role in the field of transient chaos. Several methods have been developed to this end, and we present here a particularly simple one [Lai *et al.*, 1993] which were applied in all the examples presented in this paper.

On a two-dimensional map or on a Poincaré section of a flow, start $N_0 \gg 1$ trajectories distributed

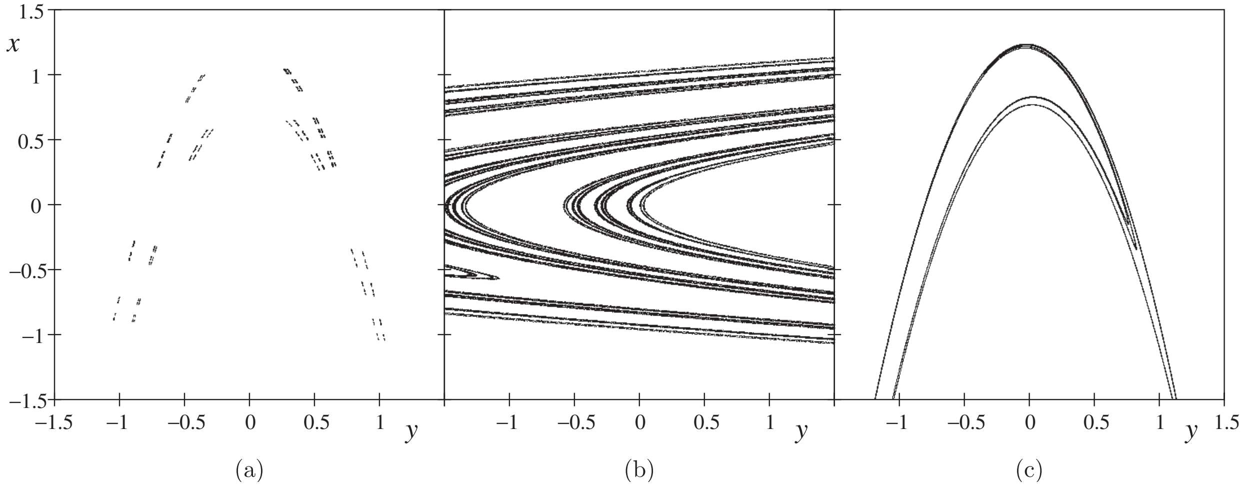


Fig. 1. The (a) chaotic saddle and (b) stable and (c) unstable manifolds of the Hénon map $x_{n+1} = 1 - ax_n^2 + by_n$, $y_{n+1} = x_n$ at parameters $a = 2.0, b = 0.3$, which coexist with a single attractor at infinity. (The well-known Hénon attractor belongs to $a = 1.4, b = 0.3$.) To generate the invariant sets, region Γ was chosen as a square of size 1.5×1.5 centred at the origin, $N_0 = 10^7$, $\tau = 2.9$. Panels (b), (a) and (c) show points of trajectories with a minimum lifetime $n_0 = 16$ at iteration numbers $n = 8$, $n = 0$ and $n = n_0 = 16$, respectively.

uniformly over a region Γ containing the saddle. Next, choose an iteration number n_0 corresponding to a multiple of the average lifetime τ, and follow the time evolution of each initial point up to exactly time n_0. Keep only trajectories that do not escape Γ in n_0 steps. If n_0/τ is sufficiently large (but not so large that only a few points remain inside) then we can be sure that trajectories with this long lifetime get close to the saddle in the course of the motion. This necessarily implies that their initial conditions were in the immediate vicinity of the stable manifold of the saddle (or of the saddle itself). Simultaneously, the endpoints must be close to the unstable manifold of the saddle since most points still inside after n_0 steps are already in the process of leaving the region. Points around the middle of these trajectories (e.g. with $n = n_0/2$) are then certainly in the vicinity of the saddle. It applies in general that the initial, intermediate and endpoints of trajectories with lifetimes of at least n_0 trace out the stable manifold, the saddle and the unstable manifold, respectively, within region Γ to a good approximation which is also illustrated in Fig. 1.

3.3. *The escape rate*

A quantity measuring how quickly particles lead any neighborhood of the nonattracting chaotic set is the so-called *escape rate*. Imagine that a large number N_0 of initial points is distributed (e.g. uniformly) in a region Γ which is supposed to be a simple region with a smooth boundary, e.g. a rectangle. By iterating trajectories starting from the initial points, many will leave the region Γ after a certain number of steps. Let N_n denote the number of trajectories staying still inside Γ after n steps, and take N_0 so large that $N_n \gg 1$. As n gets large one observes, in general, an exponential decay in the number of survivors (see e.g. [Ott, 1993; Tél & Gruiz, 2006]), that is, one finds

$$\frac{N_n}{N_0} \sim e^{-\kappa n} \quad \text{for } n \gg 1, \tag{5}$$

where κ is the escape rate.

The definition of the escape rate tells us that the number of survivors decreases by a factor of $1/e$ after about $1/\kappa$ steps. This implies that the majority of trajectories do not live longer than $1/\kappa$ in a region containing the nonattracting set. Therefore, it is natural to identify this number with the average lifetime of transients, i.e. to write:

$$\tau = \frac{1}{\kappa}. \tag{6}$$

Besides the escape rate, which characterizes the globally unstable dynamics, there is another important characteristic number, the average Lyapunov exponent $\overline{\lambda}$, which is a measure of the local dynamical instablity. The average is to be taken with the distribution on the nonattracting chaotic set, generated by trajectories of long lifetime.

3.4. *Fractal properties*

As Fig. 1 illustrates, the chaotic saddle has a direct product structure: its fractal dimension is the sum of the so-called *partial fractal dimensions* [Kantz & Grassberger, 1985]. In two-dimensional invertible maps, we have $D_0 = D_0^{(1)} + D_0^{(2)}$, where $D_0^{(1)}$ and $D_0^{(2)}$ are the partial dimensions along the unstable and stable directions, respectively, and $0 < D_0^{(j)} < 1$, $j = 1, 2$. By taking into account the distribution on the saddle generated by long-living trajectories, partial *information* dimensions $D_1^{(1)}$ and $D_1^{(2)}$ can also be defined [Ott, 1993; Tél & Gruiz, 2006]. They cannot exceed the value of the corresponding fractal dimension, but they are often rather close to them. The partial information dimensions are thus often good approximants to their fractal counterparts.

It is a central result of transient chaos theory that these information dimensions can be expressed via the Lyapunov exponents and the escape rate. The so-called dimension formulae [Kantz & Grassberger, 1985] state that

$$D_1^{(1)} = 1 - \frac{\kappa}{\bar{\lambda}} \quad \text{and} \quad D_1^{(2)} = \frac{\bar{\lambda} - \kappa}{|\bar{\lambda}'|}. \tag{7}$$

Here $\bar{\lambda}'$ denotes the negative average Lyapunov exponent on the saddle. The dimension of the chaotic saddle is the sum of the partial dimensions: $D_1 = D_1^{(1)} + D_1^{(2)}$. These relations express a link between the fractal geometry and the chaotic dynamics on the chaotic saddle. The respective information dimensions of the unstable and stable manifolds are

$$D_1^{(u)} = 1 + D_1^{(2)} = 1 + \frac{\bar{\lambda} - \kappa}{|\bar{\lambda}'|} \quad \text{and} \quad D_1^{(s)} = 1 + D_1^{(1)} = 2 - \frac{\kappa}{\bar{\lambda}}, \tag{8}$$

since these manifolds are locally smooth one-dimensional curves.

The case of one-dimensional maps is the limit of infinitely strong contraction, i.e. $|\bar{\lambda}'| \to \infty$. Since contraction is immediate, the concept of stable manifolds becomes ill-defined. The nonattracting sets of one-dimensional maps are therefore chaotic *repellors*.

It is worth mentioning that analogous (but more complicated) formulae exist for maps of any dimension [Hunt *et al.*, 1996]).

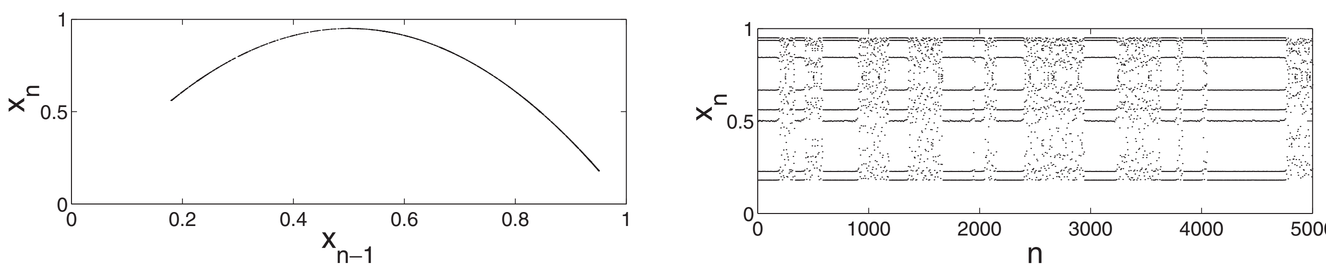
4. Noise-Induced Chaos

The parameter p in the governing dynamics (1), (2) is chosen such that the attractor of the deterministic system $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p)$ is not chaotic, but nonetheless the system may possess nonattracting chaotic sets. Now place the system in a noisy environment. As the noise strengths are increased through a critical value, the asymptotic attractor of the system becomes chaotic, as characterized, e.g. by a sudden enlargement into a fractal shape, or by the appearance of a positive Lyapunov exponent.

4.1. *Examples*

Since one-dimensional models are widely used, our first example is from such maps (see Fig. 2). We consider the logistic map $x_{n+1} = ax_n(1-x_n)$ at $a = 3.8008$, which lies in a periodic window of period-8. Figures 2(a) and 2(b) show, for a Gaussian noise of strength $\sigma = 10^{-4.8}$, the noisy chaotic attractor in the (x_{n-1}, x_n) plane and the time series x_n versus n, respectively. An intermittent behavior can be seen, where the trajectory visits the period-8 attractor and the interval containing a chaotic repellor.

The effect of noise can then be interpreted as kicking the point out from the attractor. It might fall close to a point of the repellor, and



(a) (b)

Fig. 2. For the logistic map at parameters $a = 3.8008$ with Gaussian noise of strength $\sigma = 10^{-4.8} \approx 1.6 \times 10^{-5}$: (a) noise-induced chaotic attractor and (b) intermittent time series $\{x_n\}$. The critical noise strength below which no noise-induced attractor exists is $\sigma_c = 10^{-5.1}$. From [Lai *et al.*, 2003].

start jumping chaotically on the infinitely many repellor points (more precisely, on their small neighborhoods). Since, however, the repellor was not hit exactly, and noise is active later on, as well, the point will move away from the repellor, along its instable manifold, the x axis. The permanent wandering between the attractors and the repellor traces out the noise induced attractor. Due to the restricted dimensionality of the phase space, this attractor is an interval, and hence atypical. Nevertheless, the repellor existing between the attractor points (not shown) is a fractal, a Cantor set.

Our next example is a generic two-dimensional map which possesses two fixpoint attractors with a fractal basin boundary (the stable manifold of a chaotic saddle) in between [Fig. 3(a)]. In the presence of weak uniform noise, above a certain threshold value σ_c, the attractors merge with some surroundings of the chaotic saddle to form a noisy chaotic attractor of fractal shape [Fig. 3(b)]. Figure 4 presents the chaotic saddle of the deterministic dynamics, along with its manifolds. A comparison with the noise-induced attractor of Fig. 3(b) clearly shows that it is the *unstable* manifold of

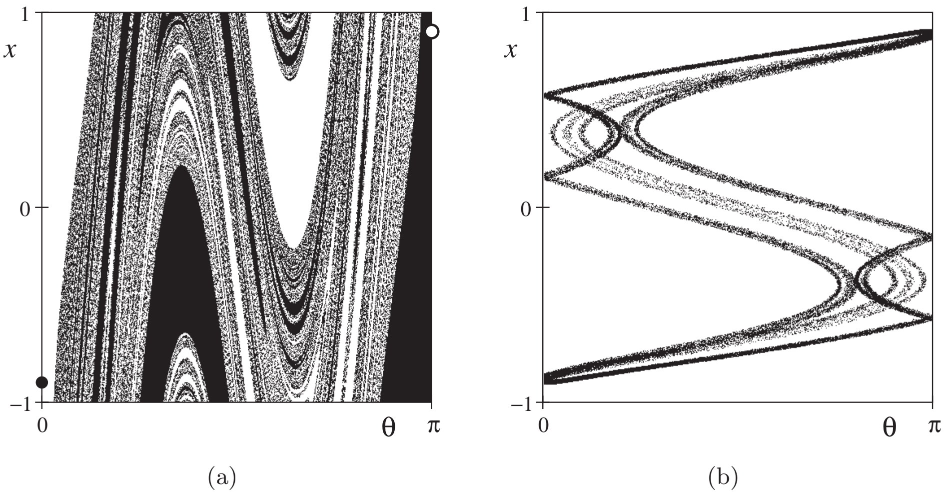


Fig. 3. Noise-induced chaos in the map: $\theta_{n+1} = \theta_n + 1.32 \sin 2\theta_n - 0.9 \sin 4\theta_n - x_n \sin \theta_n + \sigma\xi_n^{(1)}$, $x_{n+1} = -0.9 \cos \theta_n + \sigma\xi_n^{(2)}$, where $\xi_n^{(1,2)}$ represent independent uniform noises. (a) Deterministic case, two fixed point attractors (white and black dots) and their basins of attraction (in black and white). (b) Noise-induced chaotic attractor at noise strength $\sigma = 0.01$. The critical noise strength is $\sigma_c = 0.009$.

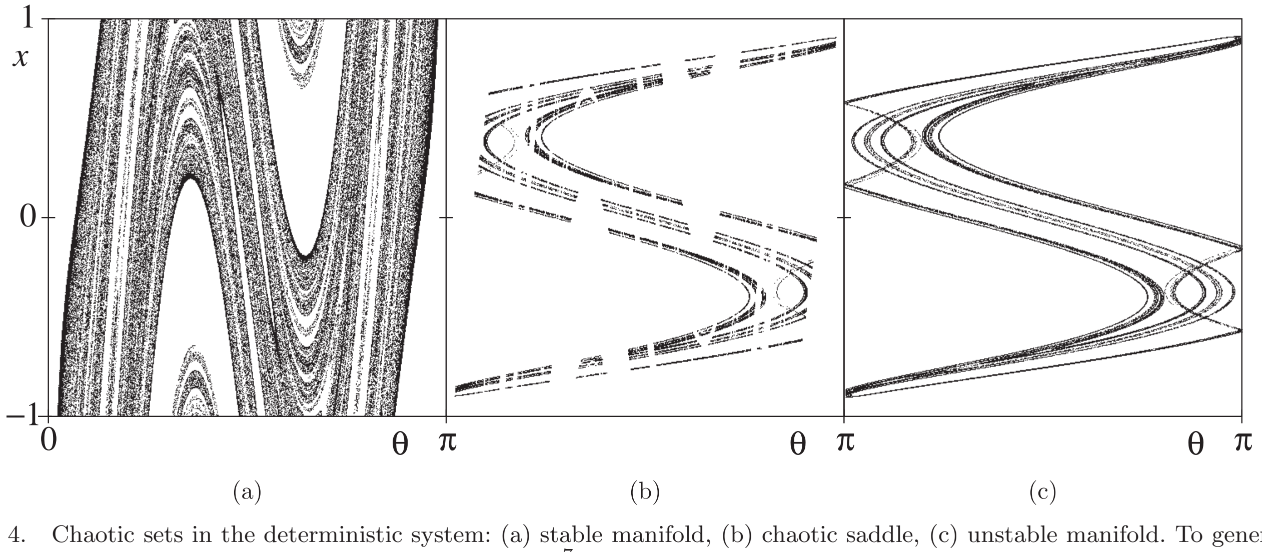


Fig. 4. Chaotic sets in the deterministic system: (a) stable manifold, (b) chaotic saddle, (c) unstable manifold. To generate these sets Γ was chosen as the rectangle shown, $N_0 = 10^7$, $\tau = 20$. Panels (a)–(c) show points of trajectories with a minimum lifetime $n_0 = 40$ at iteration numbers $n = 0$, $n = 5$ and $n = n_0 = 40$, respectively.

the saddle which becomes part, in the presence of noise, of the chaotic attractor. The reason why the stable manifold does not appear as part of the new attractor is that the approach of the saddle is exponentially fast along the stable manifold (with a rate set by $|\lambda'|$). The time spent around the saddle and its unstable manifold is thus much longer than the time spent along the stable manifold.

When applying a somewhat stronger noise, the fractal pattern is washed out. For $\sigma = 0.03$ [Fig. 5(a)], for example, only three bands can be resolved, which are washed together into a single band by $\sigma = 0.1$ [Fig. 5(b)]. In these cases noise dominates the dynamics, illustrated by the lack of a clear fractality.

It is worth mentioning that noise-induced chaos may also occur when there are only periodic saddles besides the regular attractors. If the stable and unstable manifolds of these do not yet cross, but are about to form intersections in the deterministic system, the presence of noise can materialize at the intersections, creating a chaotic saddle in the noisy problem, the so-called *stochastic chaotic saddle* [Billings & Schwartz, 2002]. It is then the unstable manifold of this saddle which merges with the regular attractor(s) to form an extended noisy chaotic attractor. The positivity of the Lyapunov exponent in the noisy logistic map before reaching the accumulation point [Crutchfield *et al.*, 1982] is also a similar phenomenon.

Finally, we consider an example of ordinary differential equations. This is the ecological model treated in [Ellner *et al.*, 2005] to describe the population dynamics of fennoscandian voles. The equations of motions for the scaled pray (vole) density, n, and predator (weasel) density, p, are

$$\frac{dn}{dt} = 4.5n(1 - \sin(2\pi t) - n) - \frac{gn^2}{n^2 + 0.01} - \frac{8np}{n + 0.04}, \tag{9}$$

$$\frac{dp}{dt} = 1.25p\left(1 - \sin(2\pi t) - \frac{p}{n}\right), \tag{10}$$

with the parameters taken from Turchin and Ellner [2000]. The seasonal variation is of the period of $t = 1$ year. A stroboscopic section is taken with a sampling of once per year (at $t = 1, 2, \ldots$), and corresponds to an invertible two-dimensional map. The attractor of the deterministic problem for $g = 0.12$, a 13-cycle, was plotted in [Ellner & Turchin, 2005] (their Fig. 4), along with the attractor induced by noise. We demonstrate in Fig. 6 the presence of a chaotic saddle coexisting with the 13-cycle, obtained by the method described in Sec. 3.2. To complete the picture, Fig. 7 displays the unstable manifold of this chaotic saddle. A comparison with Turchin and Ellner's plot clearly suggests that the noise-induced attractor is the union of the old attractor and the saddle's unstable manifold.

4.2. *Fractal features of noise-induced chaotic attractors*

In the absence of noise, since the attractor is regular, the largest Lyapunov exponent of the asymptotic attractor is zero in flows, and a negative number in maps. As noise is turned on and

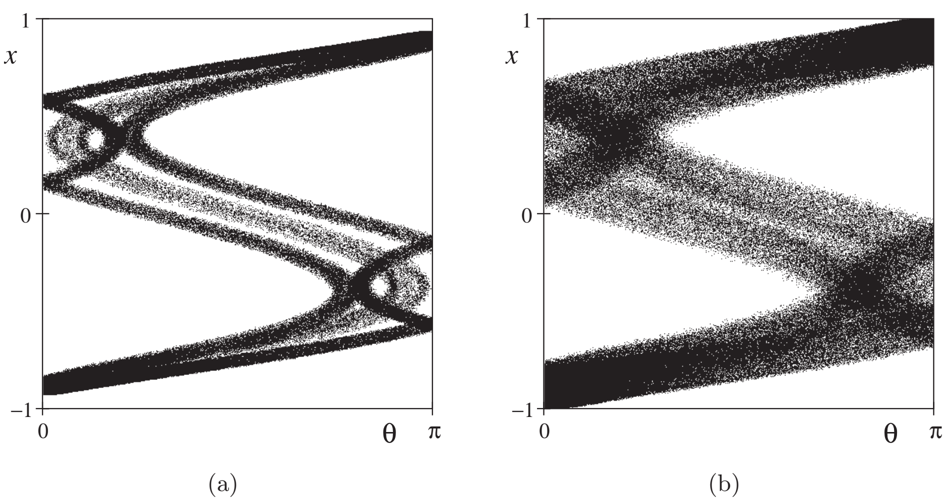


Fig. 5. Attractors in the presence of stronger noise: (a) $\sigma = 0.03$, (b) $\sigma = 0.1$. These attractors are no longer fractal, and therefore we propose to not call them chaotic.

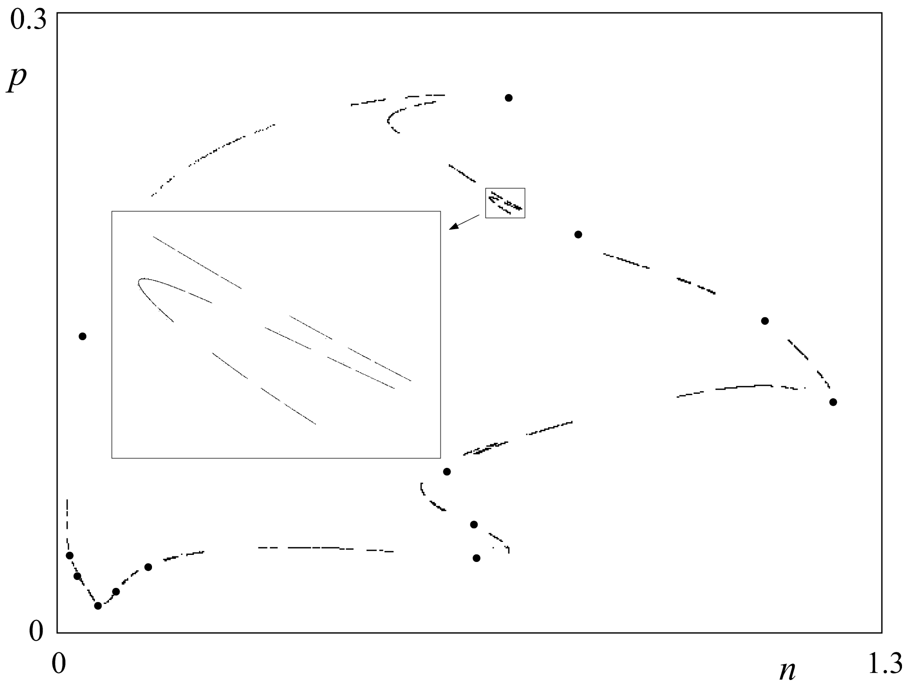


Fig. 6. The chaotic saddle of the ecological model (9), (10) for $g = 0.12$. It is obtained by starting at $N_0 = 5 \times 10^5$ points uniformly distributed on the rectangle Γ: $0.001 < n < 1.3$, $0 < p < 0.3$. The band $0 \leq n \leq 0.001$ is not used since numerical instability would show up there due to the term p/n in (10). The lifetime around the saddle is quite long: $\tau = 56$ years! Trajectories not entering a circle of size 0.0005 around any of the attractor points (shown by black dots) up to $n_0 = 100$ years are kept and their points taken at year $n = 25$ provide a good approximant to the saddle. The saddle is quite compact, nearly a Cantor set, like the repellor of one-dimensional maps. The inset of magnification 22 indicates, however, the double fractal character. A fourth order Runge–Kutta method of fixed time step of 0.001 year was used.

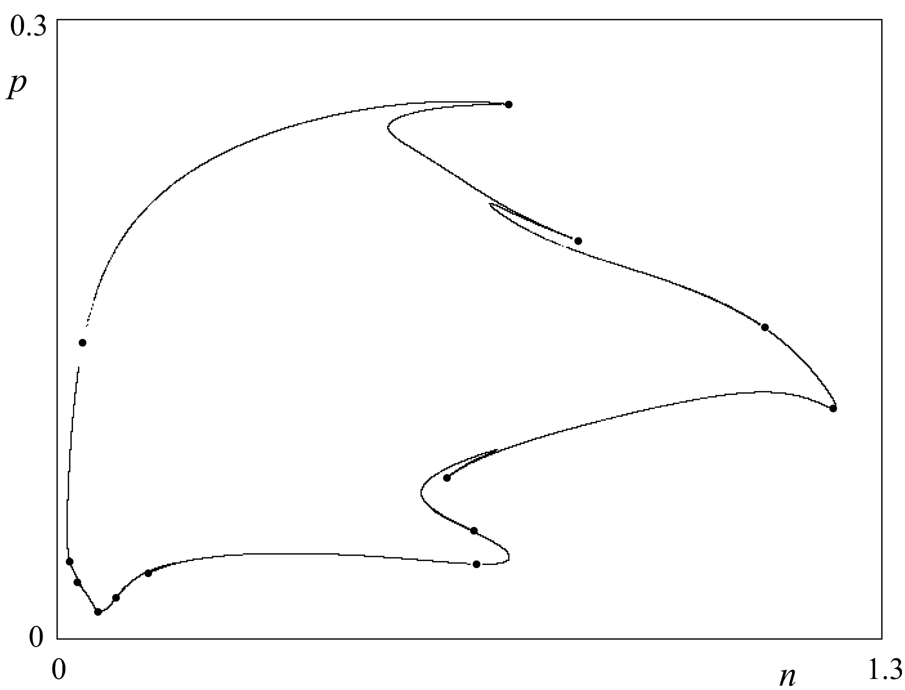
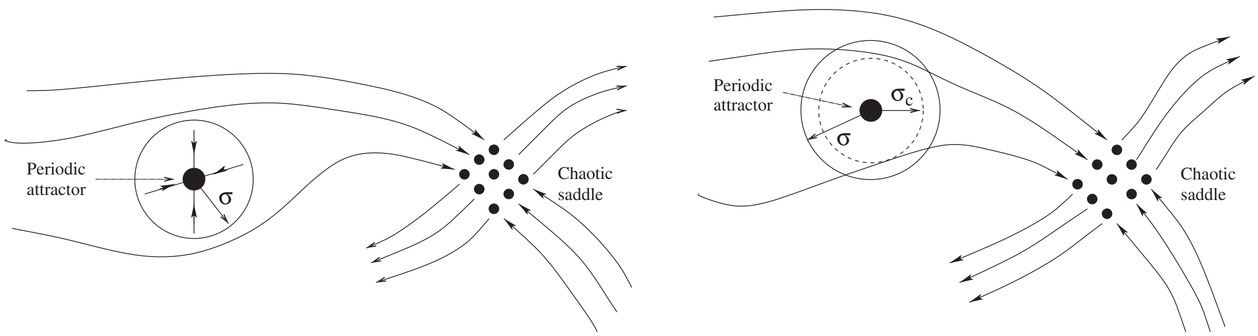


Fig. 7. The unstable manifold of the chaotic saddle of Fig. 6 obtained as described in the previous caption, just the endpoints ($n = n_0 = 100$) are plotted. This is practically the same as the noise-induced attractor displayed in Fig. 4(b) of [Ellner & Turchin, 2005].



(a) $\sigma \leq \sigma_c$ (b) $\sigma \stackrel{\sim}{>} \sigma_c$

Fig. 8. Schematic illustration of the meaning of the critical noise strength σ_c on a two-dimensional Poincaré surface of section: (a) trajectories are confined near the periodic attractor for σ below σ_c, and (b) a typical trajectory can move intermittently between the periodic attractor and the chaotic saddle for σ slightly above σ_c.

its amplitude becomes sufficiently large (but yet small), there is a nonzero probability that a trajectory originally on the attractor escapes it and wanders near the coexisting nonattracting chaotic set. In this case, the largest Lyapunov exponent λ_1 of the noisy system becomes positive, indicating that the asymptotic attractor has become chaotic for trajectories starting from random initial conditions.

Consider a two-dimensional Poincaré map, as shown schematically in Fig. 8, where there are a periodic attractor and a coexisting chaotic saddle. The circular region surrounding the periodic attractor denotes the effective range of the influence of noise of strength σ.

There exists a *critical noise strength* σ_c so that for $\sigma < \sigma_c$, there is no overlap between the stable manifold of the chaotic saddle and the noisy periodic attractor, as shown in Fig. 8(a). A rigorous interpretation of the critical noise strength is given in Appendix. For $\sigma > \sigma_c$, a subset of the stable manifold of the chaotic saddle is located in the range of the noisy periodic attractor, as shown in Fig. 8(b). As a result, there is a nonzero probability that a trajectory near the periodic attractor is kicked out of its range and moves toward the chaotic saddle along its stable manifold. Because the chaotic saddle is nonattracting, the trajectory can stay in its vicinity for only a finite amount of time before leaving it along its unstable manifold and then, enter the noisy periodic attractor again, and so on. The noise-induced chaotic attractor is thus *the union of the simple deterministic attractors and the saddle's unstable manifold*. We have seen in Sec. 2 that the unstable manifold's dimension is unchanged under weak noise. Since the attractors are zero dimensional objects on a Poincaré plane, the overall dimension D_0 or D_1 of the noise-induced attractor is the same as that of the saddle's unstable manifold in the noise-free system $D_0 = D_0^{(u)}$, $D_1 = D_1^{(u)}$. In particular, using (8), we find that

$$D_1 = 1 + \frac{\overline{\lambda} - \kappa}{|\overline{\lambda}'|} \tag{11}$$

is an approximant to the information dimension of the noisy attractor. Note that the dimension is independent of the noise strength, σ, in accord with the claim that the weak noise limit is relevant. The noise-induced chaotic attractor's dimension is uniquely determined by the parameters of the underlying chaotic saddle of the noise-free problem!

For σ slightly above σ_c, the probability for the trajectory to leave the noisy periodic attractor is, however, small. Thus, as our examples also illustrate (see e.g. Fig. 2), an intermittent behavior can be expected where the trajectory spends long stretches of time near the periodic attractor, with occasional bursts out of it wandering near the nonattracting chaotic set.

We note that another determining feature of noise-induced chaos is that the positive Lyapunov exponent of the attractor exceeds zero according to a power law (see [Lai *et al.*, 2003]).

5. Conclusions

Phenomena similar to noise-induced chaos also exist. In the presence of a deterministic small size (or multi-piece) chaotic attractor, the attractor may suddenly widen by adding weak noise. This is the so-called noise-induced crisis [Sommerer *et al.*, 1991]. In such cases there is always a nonattracting chaotic set coexisting with the chaotic attractor. It is again the *unstable manifold* of this set which merges with the old attractor into the new one. The term comes from the observation that the motion on the new attractor is intermittent, with long stays on the old attractor and short excursions to the unstable manifold, just like at a slightly shifted parameter, where the deterministic system undergoes a crisis. The properties of this intermittent behavior are uniquely related to the properties of the noise-free system around crisis [Sommerer *et al.*, 1991]. In particular, (11) provides the dimension of this new attractor, as well, where the characteristic numbers belong to the deterministic nonattracting chaotic set lying around the chaotic attractor. This indicates that the assumption of weak noise is useful in all these noise-induced phenomena.

Recently, a debate has been developed on noise-induced chaos in the journal *Oikos*. In their paper Dennis *et al.* [2003] claim that "...chaotic dynamics can be revealed in stochastic systems through the strong influence of underlying deterministic chaotic invariant sets." By invariant set they mean, however, chaotic attractors, and just mention by passing the possibility of the existence of nonattracting invariant sets. In a critical response, Ellner and Turchin [2005] state that "Even when an estimated skeleton predicts a system's short time dynamics with extremely high accuracy [i.e. if the attractor is simple], the skeleton's long term dynamics and attractor may be very different from those of the actual noisy system." This formulation contains the possibility of the coexistence of nonattracting chaotic sets with the attractor, but as one of their examples shows, the authors would also consider noise-induced chaos as a chaos-like noisy behavior in a system with a deterministic point attractor. Based on fractality, we have argued for a version in-between these points of view: *Chaotic dynamics is typically revealed in stochastic systems through the strong influence of underlying nonattracting chaotic invariant sets.*

It is exactly the fractal property of the attractor which can be used as a condition to decide whether the noise is weak. As can be seen from Fig. 9, exhibiting the results of the box-counting algorithm carried out for the example in Figs. 3–5, noise makes the dynamics space filling at short scales, smaller than $\varepsilon_c \approx e^{-4} = 0.018$ for $\sigma = 0.01$. For weak noise, there is always a scaling region with the slope of the noise-free fractal dimension, which is 1.5 in this case. This property is in full harmony with the statement of Ben Mizrachi *et al.* [1984] mentioned earlier. As soon as this scaling region becomes too short, fractality cannot be identified, not even on gross scales, as is the case for $\sigma = 0.03$. Then noise smearers out the dynamics into large, finite bands of the phase space (cf. Fig. 5). This is a sign of noise being more influential than the original deterministic dynamics, and noise can be considered from here on to be strong. This is the case of the first examples, noisy logistic flow and noisy Ricker map, of both Dennis *et al.* [2003] and Ellner and Turchin [2005]. Here a fixed point is turned into an extended attractor of the dimension of the phase space. Due to the lack of fractality, we suggest not to call this a chaotic attractor, despite the positivity of the Lyapunov exponent (which is only one feature of a chaotic attractor, the other one, fractality, is missing here). The term "stability masked by noise", suggested by Ellner and Turchin [2005] perfectly characterizes the situation.

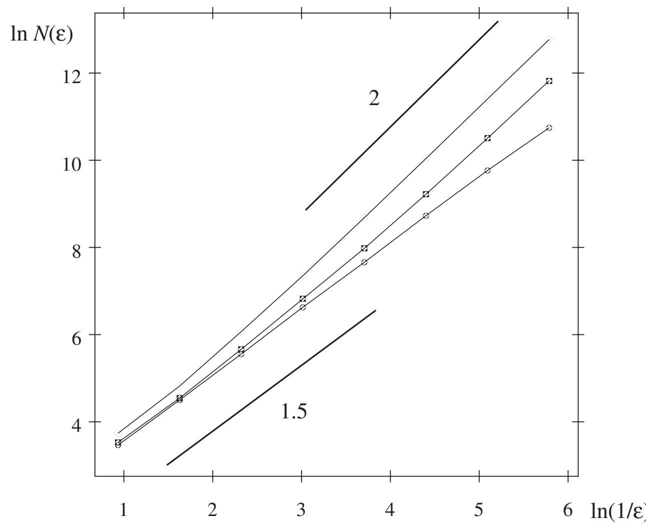


Fig. 9. Results of the box counting algorithm carried out for the deterministic unstable manifold of Fig. 4(c) (black dots), and for the noisy attractors of Fig. 3(b) (black squares, $\sigma = 0.01$) and 5(a) (black diamonds, $\sigma = 0.03$). The slope of the bold lines corresponds to the fractal dimension $D_0^{(u)} = 1.5$ of the deterministic saddle's unstable manifold and to the phase space's dimension $d = 2$ (cf. Fig. 1). The threshold scale beyond which fractality holds is $\varepsilon_c \approx 0.018$ for $\sigma = 0.01$. Such a value does not exist at all for $\sigma = 0.03$: this case is thus noise-dominated. Similar plot can be obtained for the scaling of the information contents with the box size. The noninteger slope is then provided by the information dimension of the deterministic unstable manifold: $D_1^{(u)} = 1.4$.

We are aware of the problem that the selection of noisy and deterministic dynamics is often an important task. Our arguments indicate that the determinism dominated behavior, i.e. noise-induced chaos or intermittency, can clearly be understood in terms of the noise-free dynamics. Much less is known, however, about cases where the noisy and deterministic effects become comparable. This belongs to the realm of strong noise when fractality is washed out, and the dynamics do not rely any longer on any fingerprint (the skeleton, in terms of Ellner and Turchin [2005]) of the deterministic problem. In such cases only methods borrowed from the praxis of time series analysis [Drepper *et al.*, 2003; Ellner & Turchin, 1995; Kantz & Schreiber, 2003] can be used to judge how relevant is the deterministic influence.

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Appendix: Physical Meaning of the Critical Noise Strength

We have argued that noise-induced chaos sets in when a periodic attractor and a chaotic saddle are dynamically linked by noise. In the case of Gaussian noise (3), if one is allowed an infinite amount of computational or experimental time, the two sets will connect for arbitrarily weak noise. Thus the meaning of a finite critical noise strength for the onset of noise-induced chaos needs to be clarified.

To define a critical noise strength for a finite physical time, we note that under Gaussian noise, the steady-state probability distribution for point $\mathbf{x}$ on the attractor can be written as [Freidlin & Wentzell, 1984; Hamm *et al.*, 1994]: $W(\mathbf{x}) \sim Z(\mathbf{x})e^{-\Phi(\mathbf{x})/\sigma^2}$, a form similar to that describing fluctuations in thermal equilibrium and $\Phi(\mathbf{x})$ is analogous to the free energy. While the explicit form of $Z(\mathbf{x})$ and $\Phi(\mathbf{x})$ cannot be obtained from the mere knowledge of the equations of motion, the interesting feature is that they are *both independent of the noise strength*. This allows for a proper threshold to be defined for dynamical events such as noise-induced chaotic attractors [Hamm *et al.*, 1994]. In particular, for chaotic attractor induced by noise in a periodic window, before the transition, the periodic attractor appears to be fuzzy in the presence of noise. The noisy attractor can be defined as the region on which the probability distribution is close to its maximum. The potential value of the attractor can be conveniently set to zero and as such the probability density $W(\mathbf{x})$ around the attractor is large. By choosing a small threshold value χ according to the resolution of the probability variation, we can define the noisy attractor as the set of points where $W \sim \exp(-\Phi/\sigma^2) \geq \chi$. Collision of this noisy attractor with the nonattracting set occurs at *a critical noise strength* σ_c. We thus have $\exp(-\Delta\Phi/\sigma_c^2) = \chi$, which gives $\sigma_c = \sqrt{\Delta\Phi/\ln\chi^{-1}}$, where $\Delta\Phi$ is the potential difference between the attractor and the non-attracting chaotic set. Thus, despite the unbounded nature of the Gaussian noise, the finite observational time allowed in any physical application renders meaningful a proper noise strength for noise-induced chaos.