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Limits to chaotic phase synchronization

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Abstract. – Phase synchronization in coupled chaotic oscillators, a situation where the phase differences of the oscillators are bounded while their amplitudes remain uncorrelated, has been shown to occur for chaotic attractors having a proper structure of rotation in phase space. As applications of phase synchronization become popular, it is important to understand its limit. Here we show that phase synchronization in the above sense cannot occur for the general class of coupled Lorenz type of chaotic oscillators. For such a system, intermittent synchronization between the dynamical variables sets in as soon as an originally null Lyapunov exponent becomes negative.

Phase synchronization in coupled chaotic oscillators, since its discovery in 1996 [1], has received a great deal of attention [2], both theoretically [3–8] and experimentally [9–12]. As described in ref. [1], phase synchronization means that, due to coupling (typically weak), the phase differences of the oscillators are bounded but their amplitudes, or the dynamical variables of the oscillators themselves, remain uncorrelated. Phase synchronization in this sense is *nontrivial*, *vs.* the situation where the dynamical variables of the oscillators are synchronized so that the phases are trivially synchronized (here we do not deal with such trivial phase synchronization, so in what follows we refer phase synchronization only to the nontrivial sense). So far, phase synchronization has been investigated mostly for chaotic attractors of the Rössler type [13], where they possess a well-defined structure of rotation in phase space and therefore are phase coherent [8]. As applications of phase synchronization become increasingly popular [3,4], it is important to understand its limit [14,15]. The aim of this letter is to show that phase synchronization cannot occur in dynamical systems having attractors with multiple scrolls in phase space. A typical example is the chaotic Lorenz oscillator [16].

We have conducted extensive numerical experiments on coupled chaotic Lorenz oscillators to search for phase synchronization. Because of the double-scroll structure of the attractor, the definition of a proper phase variable is not as straightforward as that for Rössler type of chaotic attractors. We utilize three approaches: 1) the filtering procedure as introduced in ref. [5], 2) the Poincaré return-time method as detailed in ref. [6], and 3) the frequency method as described in ref. [7]. In a typical numerical experiment on two coupled Lorenz oscillators, for each definition, the phase difference is computed for a range of the coupling parameter K. The well-accepted scenario of transition to phase synchronization [1], established based on coupled phase-coherent Rössler type of chaotic oscillators, suggests that we would have observed phase synchronization for $K > K_0$, where K_0 is the parameter value for which one of the originally null Lyapunov exponents becomes negative. Exhaustive numerical experiments give, however, a consistent lack of phase synchronization for $K > K_0$. Instead, phase variables tend to synchronize with each other only when $K > K_+ \gg K_0$, where K_+ is the coupling parameter value for which one of the *positive* Lyapunov exponents becomes negative. Note that, for $K > K_+$, the dynamical variables of the two oscillators, which contain both the amplitude and phase variables, are actually synchronized, so phase synchronization observed in this regime is trivial.

The failure to observe phase synchronization motivates us to analyze the phase dynamics of the chaotic Lorenz attractor([1]). The basis of our analysis is that the phase evolution is governed by the dynamics along the neutral direction of the underlying chaotic flow. We find that for Lorenz type of attractors, this dynamics is generally altered by coupling in a way that is fundamentally different from that for Rössler type of attractors. In particular, a neutral direction can be destroyed immediately and becomes unstable as the coupling is increased from zero. This is due to the interaction between the amplitude and the phase dynamics, which is such that *phase coherence is impossible insofar as the amplitude dynamics remains uncorrelated*. Alternatively, because of this interaction between the amplitude and phase dynamics, we expect to observe a degree of amplitude synchronization for $K > K_0$ where an originally neutral direction becomes contracting. Indeed, we observe a transition to intermittent synchronization in the dynamical variables as K is increased through K_0. In particular, let $\Phi(K)$ be the probability that $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \epsilon$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the dynamical variables of the two Lorenz oscillators and $\epsilon > 0$. We find that $\Phi(K)$ remains constant (near zero) for $K < K_0$ but it increases continuously as K is increased from K_0 and approaches asymptotically to unity as $K \to K_+$. To our knowledge, existing works on intermittent synchronization focus exclusively on the transition from the synchronous state near K_+. Transition to intermittent synchronization at K_0, when one of the originally null Lyapunov exponent becomes negative, is interesting but has not been noticed before.

Figures 1a)-c) show typical results of searching for phase synchronization in a system of two coupled chaotic Lorenz oscillators, where the time evolution of the phase difference $\Delta\phi(t)$ or the frequency difference $\Delta\omega(t)$ are plotted for a number of values of the coupling parameter, and a)-c) correspond to three methods for computing the phase (see figure caption). The system equations are: $\dot{x}_{1,2} = \sigma_{1,2}(y_{1,2} - x_{1,2}) + K(x_{2,1} - x_{1,2})$, $\dot{y}_{1,2} = 28x_{1,2} - y_{1,2} - x_{1,2}z_{1,2}$, $\dot{z}_{1,2} = -(8/3)z_{1,2} + x_{1,2}y_{1,2}$, where $\sigma_1 \approx \sigma_2$ are parameters. For $\sigma_1 = 10$ and $\sigma_2 = 11$, we find that $K_0 \approx 1.3$ and $K_+ \approx 3.9$. If the oscillators were phase coherent, we would expect $\Delta\phi(t)$ to increase with time for $K < K_0$, to remain bounded with 2π for $K_0 < K < K_+$, and to become approximately zero for $K > K_+$ (after adjusting a possible time lag [17]). Figures 1a)-c)

([1])For synchronization in stable cycle oscillators, the notion of isochrone is useful [2]. To define an isochrone, say we fix a point $\boldsymbol{x}^*$ on the cycle and consider all points in its vicinity that are attracted to it under the dynamics. In an N-dimensional phase space, these points form an $(N-1)$-dimensional hypersurface, which is an isochrone that rotates with the same velocity as a point on the cycle. For a Poincaré map defined on an isochrone, all points have the same return time. The existence of an isochrone thus indicates synchronizability. For a fully unstable cycle, isochrones also exist, which can be defined using the inverse time. For saddle cycles or periodic orbits that have both stable and unstable manifolds, isochrones cannot be defined [2]. Since chaotic attractors typically possess stable and unstable manifolds, it may be difficult to apply the notion of isochrones to studying synchronization.

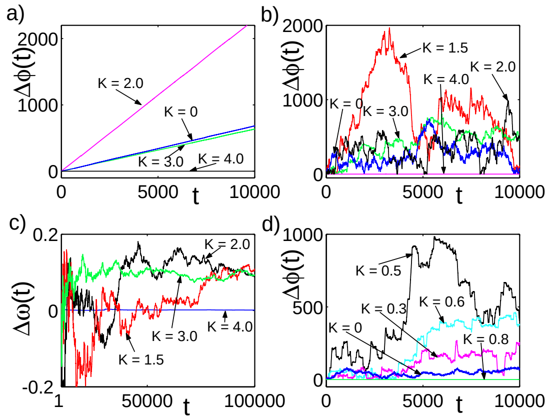

Fig. 1 – For the system of two coupled, slightly nonidentical Lorenz oscillators, time evolution of the phase difference $\Delta\phi(t)$, where the phases are computed using a) the filtering method [5], b) the Poincaré return-time method [6], and c) the frequency method [7]. Panel d) shows the phase difference of two coupled identical Lorenz oscillators, where the coupling is applied to the z-variables. In this case, $K_0 \approx 0.2$ and $K_+ \approx 0.72$. The phases are calculated by using Poincaré return-time method. For all values of the coupling parameter that we examined, no phase synchronization can be observed except in the trivial sense for $K > K_+$, where the dynamical variables of the oscillators are synchronized.

reveal, however, a complete lack of these behaviors. The observation is that there appears to be no signature of phase synchronization for $K > K_0$, in sharp contrast to coupled Rössler type of oscillators. We note that phase synchronization (in the trivial sense) does occur, but only for $K > K_+$ when the dynamical variables of the coupled Lorenz oscillators become synchronized. The failure to observe nontrivial phase synchronization seems to persist for other choices of parameter values, *even when the Lorenz attractors are set to be identical*. Figure 1d) shows the phase difference between two coupled identical Lorenz oscillators ($\sigma_{1,2} = 10$) for a number of values of the coupling parameter, where the coupling occurs in the z-variables. In this case, $K_0 \approx 0.2$ and $K_+ \approx 0.72$, and the phases are calculated by using the Poincaré return-time method. No phase synchronization is found for $K_0 < K < K_+$. This suggests that fundamentally, phase synchronization cannot occur in coupled Lorenz type of chaotic oscillators.In order to explain the observed lack of phase synchronization as described above, we make use of a recent result on the dynamics of the neutral direction [18], as it corresponds to the phase dynamics of a chaotic oscillator. For small coupling, the influences of oscillators on each other can be regarded as small chaotic (or random) perturbations. Thus it is insightful to analyze, for a single chaotic Lorenz attractor, whether the neutral direction can be preserved under random perturbations. We will show below that, because of the double-scroll structure of the Lorenz attractor, the amplitude dynamics couples to the phase dynamics in such a way that the neutral direction is typically destroyed by arbitrarily small perturbations. As an observable consequence, the null Lyapunov exponent becomes positive as the perturbation amplitude is increased from zero.

Figure 2 shows, schematically, a double-scroll chaotic attractor in the phase space, where the left- and right-hand side scrolls are denoted by “L” and “R”, respectively. A typical trajectory visits both scrolls in time. Switchings occur in the region denoted by “S” in which there is an unstable steady state [19]. Motion restricted to one scroll can thus be regarded

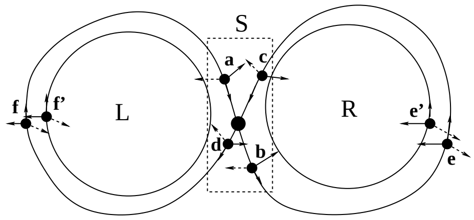

Fig. 2 – Schematic illustration of a double-scroll chaotic attractor and the associated dynamical consistency, *e.g.*, point $\boldsymbol{a}$ ($\boldsymbol{c}$) can only go to $\boldsymbol{b}$ ($\boldsymbol{d}$).

as transiently chaotic. A key feature to notice is that in the deterministic case, the way that switchings occur must be *consistent* with natural dynamics. For instance, a trajectory moving to point $\boldsymbol{a}$ near the switching region must go to point $\boldsymbol{b}$ after the switching. It cannot go to point $\boldsymbol{c}$. Similarly, under the dynamics point $\boldsymbol{c}$ can only move to point $\boldsymbol{d}$.

The idea in ref. [18] is that random perturbations can disturb the dynamical consistency and consequently destroy the neutral direction. To see how, we note that, depending on the location of a trajectory, the effect of perturbation can be quite different. When the trajectory is not in the switching region, a random perturbation can change its position, say from point $\boldsymbol{e}$ ($\boldsymbol{f}$) to point $\boldsymbol{e}'$ ($\boldsymbol{f}'$) or vice versa. As shown in fig. 2, perturbations at such locations will have little effect on the local eigenspace. Taking a pair of original and perturbed points ($\boldsymbol{e}$, $\boldsymbol{e}'$) as an example, we see that the original eigenvector in the neutral direction (direction of the flow) at $\boldsymbol{e}$ remains to be a neutral direction at the perturbed point $\boldsymbol{e}'$. There can, of course, be small deviations from the neutral direction, but they will be averaged out as the trajectory moves in region "R". This is the reason why the neutral direction associated with Rössler type of chaotic attractors can persist under random perturbations. When a trajectory is in the switching region "S", perturbations of arbitrarily small amplitude can alter the local eigenspace *in a significant way*. For instance, when the trajectory is at point $\boldsymbol{a}$, a small perturbation can kick it to point $\boldsymbol{c}$. Since the local eigenspaces of the two points are distinct, the neutral eigenvector at $\boldsymbol{a}$, when carried over by the trajectory perturbed to $\boldsymbol{c}$, will not be in the neutral direction at $\boldsymbol{c}$. The vector typically will have a component in the unstable direction at $\boldsymbol{c}$ and its length will consequently be stretched exponentially. Thus, the length of the neutral vector on the attractor, when it is perturbed in the switching region as described, will generally increase exponentially, causing the originally null Lyapunov exponent to become positive. Also note that random perturbations that move the trajectory from $\boldsymbol{a}$ to $\boldsymbol{c}$ are in fact *inconsistent* with the deterministic dynamics because, without the perturbation, the trajectory would move passing point $\boldsymbol{b}$.

It can be argued [18] that, as a result of the inconsistent perturbations in the switching region, one originally null exponent *vs.* the amplitude of the random perturbation, which is proportional to the coupling parameter K for $K > 0$, obeys the following algebraic scaling law: $\lambda_0 \sim K^{-\alpha}$, where $\alpha = 2$ for Lorenz type of chaotic attractors. *Relevant to phase synchronization here is that the neutral direction is affected in such a way that the stability of the phase dynamics along this direction is also controlled by the chaotic amplitude dynamics.* For small coupling, the phase dynamics is necessarily chaotic, as the corresponding Lyapunov exponent is positive. As K is increased further, the enhanced interaction between the two oscillators means that the newly created positive Lyapunov exponent will decrease and eventually becomes negative at K_0. Because of the role played by the amplitude dynamics in controlling the stability of the phase dynamics, which persists for any $K > 0$, we see that *nontrivial*

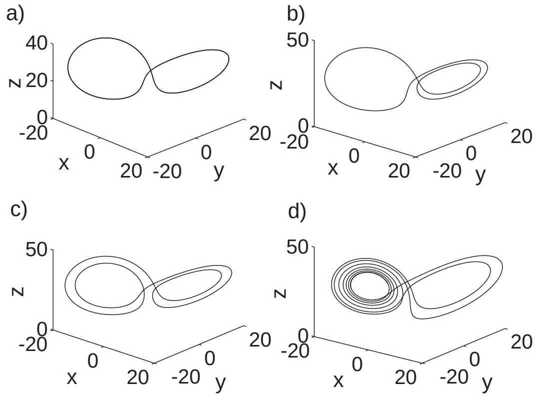

Fig. 3 – Some typical unstable periodic orbits of low periods from the chaotic Lorenz attractor. The lengths and frequencies of the orbits are a) $L = 2$, $\omega = 8.06$, b) $L = 3$, $\omega = 8.18$, c) $L = 4$, $\omega = 8.32$, and d) $L = 9$, $\omega = 9.02$, where for each orbit, its length is the number of cycles contained in it, the frequency [7] is $\omega = 2\pi L/T$, and T is the time it takes to complete the orbit in terms of the natural, dimensionless time unit in the Lorenz equations.

phase coherence, in the presence of chaotic amplitude dynamics, cannot occur. This is fundamentally different from the phase dynamics in coupled Rössler type of chaotic oscillators, for which random perturbations resulted from chaotic coupling dynamics must be consistent and, as such, the originally null Lyapunov exponents can remain zero for small coupling. In fact, for such systems the phase dynamics can be shown to be separable from the amplitude dynamics, with the latter acting as an additive noise term [1].

It is often said that unstable periodic orbits are the "skeleton" of a chaotic invariant set [20] in the sense that there is an infinite number of those orbits embedded in the set and a chaotic trajectory can be regarded as consisting of segments of visits to various periodic orbits. It is therefore of interest to examine the structure of some representative unstable periodic orbits from the chaotic Lorenz attractor. Our point is that, for a double-scroll chaotic attractor, almost all unstable periodic orbits have a similar double-scroll structure with a switching region as in fig. 2. Thus, even for an individual orbit, phase synchronization will be difficult to achieve in the presence of small random perturbations, which can be inconsistent in the switching region. We may imagine that there exists a double-scroll chaotic attractor, for which some of the embedded unstable periodic orbits have a single-scroll structure. In this case, we would expect to observe temporary phase locking when two trajectories spend a long time in the same scroll. The phase locking time would last longer if the percentage of single-scroll periodic orbits were higher. (In the extreme case, if all periodic orbits correspond to a proper rotation, as for the phase-coherent Rössler oscillator, phase synchronization time in a coupled system can be infinite.) However, we find in our numerical search that no periodic orbit from the chaotic Lorenz attractor possesses a single-scroll structure. Some of the unstable periodic orbits of low period are shown in fig. 3, which we computed by using the procedure in ref. [21].

The analysis presented above can be viewed as an extension and application of the scaling theory developed in ref. [18] to phase synchronization. In particular, the emphasis of ref. [18] was, for coupled chaotic Lorenz oscillators, on how an originally null Lyapunov exponent becomes positive as the coupling parameter is increased from zero. The scaling law derived there is applicable only to the regime where the coupling parameter is slightly above zero.

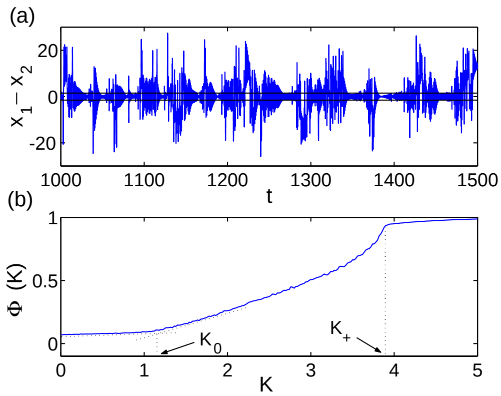

Fig. 4 – For the system of two coupled Lorenz oscillators, (a) intermittent synchronization for $K = 2.5 > K_0$ (but $< K_+$), (b) probability of synchronization $\Phi(K)$ *vs.* K.

In contrast, the focus of the present work is on the parameter regime near K_0 and beyond, where an originally null Lyapunov exponent becomes negative, so that one would expect phase synchronization. In fact, we realized that the same dynamical mechanism, *i.e.*, the dynamics in the switching regime, is responsible for coupling the chaotic amplitude dynamics with the phase dynamics even when one null exponent becomes negative (for $K > K_0$). To our knowledge, a physical explanation based on this mechanism for the inability of coupled Lorenz oscillators to exhibit phase synchronization has not been reported before.

While we have explained why nontrivial phase synchronization cannot occur in coupled Lorenz type of chaotic oscillators, what possible physical consequences can be expected as K is increased through K_0? Our analysis indicates that, because the chaotic amplitude dynamics imposes inconsistent perturbations on the neutral direction as a trajectory passes through the switching region, a certain degree of amplitude coherence develops as an originally null exponent becomes negative for $K > K_0$. It is thus possible that signatures of chaotic amplitude synchronization, or synchronization between the dynamical variables, can appear. Figure 4(a) shows such a situation for $K = 2.5 > K_0$ (but $< K_+$, so that complete synchronization between oscillators has not yet set in), where we observe time intervals during which the chaotic oscillators tend to be synchronized. To quantify this behavior of intermittent synchronization, we compute $\Phi(K)$, the probability of synchronization as a function of K. For $K < K_0$, no synchronization can be expected, so we have $\Phi(K) \approx 0$. As K is increased through K_0, intermittent synchronization occurs, so we expect $\Phi(K)$ to increase. After complete synchronization is achieved for $K > K_+$, we have $\Phi(K) \to 1$. Figure 4(b) shows $\Phi(K)$ *vs.* K, which exhibits the behavior so described. Note that the transition to intermittent synchronization occurs at K_0 when one of the originally null Lyapunov exponents becomes negative. This phenomenon has not been observed previously as all existing works on intermittent synchronization focus on the transition from the synchronous chaos near K_+. Our analysis indicates that it is natural for coupled Lorenz type of chaotic oscillators to develop intermittent synchronization even in the weak-coupling regime, where only an originally zero Lyapunov exponent is slightly negative.

For phase-coherent chaotic oscillators, intermittent synchronization (as shown in fig. 4) can occur in the lag synchronization regime [17], where the dynamical variables of the coupled oscillators are synchronized but with a time lag. For a system of two coupled oscillators, this usually occurs in the parameter regime $K > K_+$, where amplitude coherence sets in. The intermittent synchronization that we emphasize here for the coupled Lorenz oscillators occurs in the weak-coupling regime where there are still two positive Lyapunov exponents and, hence, there is no lag synchronization in this regime.

In summary, we have presented numerical results and an analysis, which indicate that nontrivial phase synchronization cannot occur in coupled Lorenz type of chaotic oscillators. Instead, in the parameter regime where one of the originally null Lyapunov exponent becomes negative so that one would expect to observe phase synchronization, intermittent synchronization in the dynamical variables occurs. Phase synchronization, besides its theoretical appeals, is becoming a popular tool for applications such as analysis of biomedical time series [2–4]. Our results suggest that there may be fundamental limitations to this approach, if the underlying process is governed by a Lorenz type of chaotic attractor. It is interesting to note that the Lorenz oscillator is the first known system to exhibit a chaotic attractor, and this type of double-scroll attractors is expected to occur commonly in applied dynamical systems.

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