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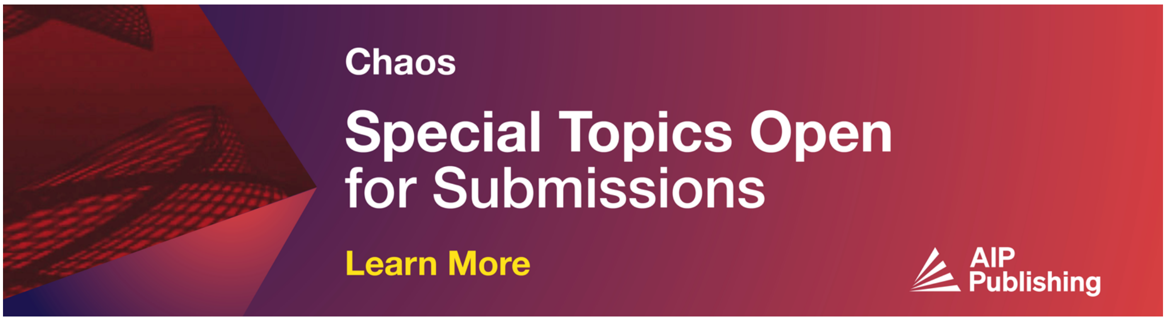

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Global phase-space approach to rate-induced tipping: A brief review

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ABSTRACT

In nonautonomous dynamical systems, rate-induced tipping (R-tipping) is a critical transition triggered by the rate of change of a time-varying parameter, rather than its absolute value. In recent years, there is a growing interest in R-tipping due to its relevance to significant problems of current interest, such as potential, catastrophic collapse of various ecosystems induced by climate change. This brief review provides an overview of the basic concept, theory, and real-world implications of R-tipping from a global phase-space point of view. The key quantity underlying the global approach is the probability of R-tipping defined with respect to initial conditions in the phase space. A recently discovered scaling law governing this probability and the rate of parameter change is introduced, which has so far been restricted to a class of high-dimensional, complex, and empirical ecological networks: pollinator–plant mutualistic networks. Issues such as prediction of tipping and protection of ecosystems from R-tipping are discussed.

Once upon a time, there was a donkey grazing near a river. A flash flood occurred and the donkey began to run toward the high ground some distance away. If the flood front was sufficiently slow, the donkey would reach the high ground in time and survive. However, if the flood front was too fast, the donkey would be swept into the water and would not survive. Clearly, in addition to the speed of the flood front, the survival of the donkey also depends on its initial position: near to or far away from the river. For any specific position, there is a critical speed of the flood, below which there is survival but above which death or extinction occurs. This is an intuitive picture of the phenomenon of rate-induced tipping, or R-tipping. In nonlinear dynamics, if the system is autonomous and bistable with two coexisting steady-state attractors, one corresponding to survival and another to extinction, each with its own basin of attraction, the basin boundary is stationary, i.e., it does not move in time. When a parameter changes with time so that the system becomes nonautonomous, the basin boundary will move with time. Whether a specific initial condition will lead to system survival or extinction depends on its location in the phase space as well as the "speed" at which the basin boundary moves. For an initial condition originally in the basin of the survival attractor, if it moves more "slowly" than the basin boundary, it will be swept into the basin of the extinction attractor, leading to a tipping of its fate. For any such initial condition, there exists a critical speed of the movement of the basin boundary, or a critical rate of the parameter change, above which R-tipping will occur. For a given rate of parameter change, some initial conditions in the phase space will experience R-tipping, while some others will not by remaining in the basin of the survival attractor, in spite of the time variation of the parameter. The probability of R-tipping can then be defined for any given rate of parameter change. How does this probability depend on the rate? Characteristically, there are two possible scenarios. If, as the rate becomes nonzero, this probability increases from zero but slowly, a safe interval will arise in the rate in which the probability remains at some near-zero value—a desired situation. However, if the probability grows immediately and drastically as soon as the rate of parameter change increases from zero, no such safe interval or region will exist. In this case, in order to keep the probability of R-tipping diminishingly small, practically the rate of parameter change must be reduced to zero. Unfortunately, in nonautonomous dynamical systems, the dependence of the probability of R-tipping on the rate of parameter change tends to follow the second scenario. This brief review presents

mathematical reasoning establishing such a dependence, or the scaling law, which so far has been computationally verified only for a class of complex, high-dimensional empirical ecological networks: plant–pollinator mutualistic networks. One implication for such ecological networks is dire: in order to protect ecosystems from climate change, the rate of adverse parameter change, such as the rate of emissions of chemicals into the atmosphere, must be kept at near zero values.

I. INTRODUCTION

A tipping point is a critical threshold within a complex dynamical system where a small perturbation can lead to significant and often irreversible system changes.[1–33] Due to its implications for system stability and resilience, the phenomenon of tipping has garnered substantial attention across various scientific disciplines including climate science,[1,11,30] neuroscience,[6,14] and engineering.[7] In autonomous dynamical systems, a tipping point is the result of parameters reaching a critical level or of the influence of noise, which are referred to as bifurcation-induced tipping or noise-induced tipping,[11,32] respectively. A common dynamical mechanism for tipping is a backward saddle-node bifurcation where, as the bifurcation parameter increases through a critical point, a stable steady state and an unstable one coalesce and disappear together. Assume that the stable steady state corresponds to a "healthy" or "survival" state of the system. Before the bifurcation, there is bistability where the "healthy" steady state and another steady state associated with the "collapse" or "extinction" of the system coexist. After the bifurcation, the "healthy" steady state no longer exists, leaving the "collapse" state as the only attractor of the system. The backward saddle-node bifurcation thus leads to a tipping, after which the system functions are destroyed.

Dynamical systems in the real world are often of the nonautonomous type, due to the ubiquitous variations of the system's parameters with time. For example, influenced by global climate change, the parameters of many ecosystems can drift with time, making them nonautonomous. When a parameter of the system changes with time, its rate of change is of particular concern—one example is the rate of carbon-dioxide emission into the atmosphere, which has been increasing steadily in recent decades. From the point of view of tipping, one might tempt to think that the time rate of change of a parameter may not be important, insofar as its value has not reached the threshold for tipping. More specifically, consider the two cases where a bifurcation parameter of the system changes slowly or fast, respectively, and suppose that the value of the parameter is still far away from a tipping point from the bifurcation point of view. In which case is a tipping event more likely? This question was addressed in a series of seminal works,[6,11,34] where the rate of the parameter change acts as a "super parameter" of the system and can induce tipping. This is known as rate-induced tipping or R-tipping.

Most previous studies on R-tipping were from a near-equilibrium perspective, focusing on initial conditions near the system's steady state. This approach is particularly effective if the system is always near an equilibrium point. In the real world, the state that an ecological system is in depends not only on the deterministic dynamics, but more importantly, on random and unexpected influences such as demographic stochasticity[35] and large-scale stochastic geographical and climatic events.[36] As a result, an ecosystem can be far from a stable equilibrium but may still be in its basin of attraction if the perturbations are not strong enough to push the system across the basin boundary. In fact, for a high-dimensional ecosystem, the probability that it is found far from equilibrium and is in a transient state can be quite appreciable.[37,38] The non-equilibrium initial conditions can significantly influence a system's response to rapid parameter changes, rendering essential and important considering initial conditions from a larger region of the phase space rather than from the vicinity of some equilibrium state.[31,39]

In a previous study, the global approach was employed to demonstrate that different initial states of a ocean circulation can lead to significantly different outcomes under rapid climate changes.[40] This study focused on the Atlantic Meridional Overturning Circulation (AMOC), revealing that certain initial ocean conditions make the system more prone to tipping, while others allow for more gradual transitions. Similarly, an experimental study in thermoacoustic systems revealed that R-tipping can occur when the system is preconditioned with some specific initial states,[41] highlighting the role of non-equilibrium dynamics in R-tipping in that even initial conditions far from equilibrium can trigger a tipping event. In ecological and climate systems, a recent work showed that R-tipping depends not only on the speed of parameter changes but also on the unstable state, basin boundaries, and transient dynamics,[32] demonstrating that R-tipping is sensitive to initial conditions, with different conditions having different critical rates of change.

In this brief review, we explore the intricate dynamics of tipping points with a focus on R-tipping. In Sec. II, we provide a brief historical overview of different types of tipping phenomena, highlighting the distinct characteristics of R-tipping compared to other forms of tipping such as bifurcation and noise-induced tipping. In Sec. III, the necessity of the global phase-space approach beyond the conventional near-equilibrium analysis to understand R-tipping is described. Sections IV and V review the key findings of Ref. 31, including a scaling law between the probability of R-tipping and the rate of parameter change, which were established for a particular class of high-dimensional ecological networks. In Sec. VI, a discussion on the implications of the scaling law for these empirical ecological networks is presented and potential topics for further research of R-tipping in complex systems are suggested.

II. TYPES OF TIPPING IN NONLINEAR DYNAMICAL SYSTEMS

A tipping point is a critical threshold where a system experiences a sudden and irreversible shift from one stable state to another due to internal or external drivers. Tipping points have been extensively studied in various fields, including climate science, biology, economics, and engineering. They can be classified into three main types: bifurcation-induced tipping (B-tipping), noise-induced tipping (N-tipping), and rate-induced tipping (R-tipping). Each type arises from different mechanisms, and their understanding is crucial in predicting and controlling system transitions.

A. Bifurcation-induced tipping (B-tipping)

The study of B-tipping dates back to the discovery of dynamical systems and bifurcation theory in the late 19th and early 20th centuries, pioneered by Henri Poincaré (1854–1912). Bifurcation theory explores how small, continuous changes in system parameters can result in qualitative shifts in its overall behavior, such as saddle-node (fold), transcritical, and pitchfork bifurcations.[42] Among these, the saddle-node bifurcation is particularly relevant to tipping, as it involves the collision and subsequent annihilation of a stable (node) and unstable (saddle) equilibrium point.[43,44] When a saddle-node bifurcation occurs, the system experiences a sudden, discontinuous shift to an alternative stable state—the hallmark of B-tipping. In contrast, other bifurcation types such as the transcritical or pitchfork typically lead to smoother transitions, where the system evolves more gradually between states without abrupt changes.

B-tipping has been studied across a wide range of disciplines from ecology and climate science to neuroscience and engineering. One of the first fields to adopt B-tipping was ecology.[4,16,29,38,45,46] In the 1970s and 1980s, ecological models began incorporating the idea that slow changes in the environmental conditions could lead to sudden, irreversible shifts in ecosystems.[47,48] For example, a lake might gradually become more eutrophic (nutrient-rich), but at a critical threshold, it tips from a clear state to a turbid, algae-dominated state[49,50] In the 1990s and early 2000s, the study of critical transitions in the Earth's climate system began to expand rapidly.[51] Climate systems are highly nonlinear, and many components (such as ice sheets, ocean circulation patterns, and vegetation cover) were shown to exhibit tipping points.[12,52–57] The idea of climate tipping points was popularized,[58] where the major components of the Earth's system were identified that are susceptible to B-tipping. For example, the AMOC could slow down and collapse if fresh water from melting ice caps reduces its salinity past a critical threshold.[40,59–61]

In physiology and medicine, B-tipping was also applied to the study of epilepsy and other neurological disorders.[62–66] In engineering, it was studied for controlling mechanical systems and power grids.[67–71]

B. Noise-induced tipping (N-tipping)

The concept of noise influencing system behavior was first proposed through the phenomenon of stochastic resonance, associated with which noise can actually enhance the detection of weak periodic signals in nonlinear systems.[72] Despite the seemingly disruptive nature of noise, if a dynamical system exhibits a stochastic resonance, some optimal level of noise can improve the system's sensitivity to small external forces.[73] In nonlinear dynamical systems, noise can induce a critical transition such as a crisis[74] or make a non-chaotic system chaotic.[75,76] In biological sciences, it was demonstrated that noise can also play a beneficial role in promoting coexistence in ecological systems[77–80] and enhancing signals in neural systems.[81] In slow–fast excitable systems, a variety of noise-induced phenomena can arise.[82]

Noise-induced tipping, or N-tipping, is referred to as a sudden shift in a system's state caused by external or internal stochastic fluctuations without requiring any smooth changes in system parameters.[83,84] Differing from B-tipping driven by a deterministic shift in some system parameters leading to a critical transition, N-tipping is triggered by random perturbations that can destabilize a system and push it toward a new state, even in the absence of a bifurcation.[83] In climate science, the importance of stochastic fluctuations such as volcanic eruptions or fluctuations in solar radiation in triggering abrupt large-scale climate transitions has been widely recognized.[53,85–88] In ecology, N-tipping was deemed as a potential driver of sudden transitions.[89–91] While deterministic changes in the environmental conditions can cause B-tipping, random environmental fluctuations such as seasonal changes, storms, or fires can also push an ecosystem from one state to another, even in the absence of a parameter crossing a bifurcation threshold. N-tipping has also been explored in neuroscience and engineering, particularly in the context of brain dynamics and disorders.[81,92–95]

C. Rate-induced tipping (R-tipping)

R-tipping is a ubiquitous phenomenon in natural and man-made systems.[30,33,96–98] It is particularly relevant to phenomena induced by the global climate change.[99,100] Rapid environmental changes, such as accelerated global warming, can induce tipping events in various components of the Earth's climate system, including ice sheets, ocean currents, and ecosystems.[30] For example, the rapid melting of Arctic sea ice due to increasing temperatures can trigger feedback mechanisms that further accelerate ice loss and contribute to global climate instability.[101,102] Ecosystems, characterized by their complex interactions and nonlinear dynamics, are also susceptible to R-tipping.[32,103] Rapid changes in environmental conditions, such as deforestation, pollution, or climate change, can push ecosystems past critical thresholds, leading to regime shifts.[32] These shifts can result in significant biodiversity loss and altered ecosystem services.[39] For instance, coral reefs are vulnerable to rapid increases in sea temperature and acidification, which can cause widespread coral bleaching and degradation.[104] Engineering systems such as power grids, transportation networks, and industrial processes are not immune to R-tipping.[30] These systems often operate under dynamic conditions where parameters such as the load, demand, or operational settings can change with time.[105] If the rates of these changes surpass their corresponding critical thresholds, system failures, cascading outages, or catastrophic breakdowns can occur.

The phenomenon of R-tipping was first conceived in 2008 when the critical-rate hypothesis was proposed, suggesting that the rate at which environmental changes occur can be more significant than the magnitude of those changes in determining the long-term ecosystem states.[34] The concepts of critical ramping rate and rate-dependent tipping were subsequently introduced.[6,11,18] Insights into R-tipping can be gained through the bifurcation diagrams of the corresponding autonomous system.[18,106] Another approach is constructing asymptotic series expansions to characterize R-tipping.[107] The interplay between noise and the rate of parameter change in triggering a tipping event was studied.[108,109] R-tipping was also investigated in cases where the quasi-static attractor is not necessarily an equilibrium state but periodic,[110] chaotic[25] in multi-dimensional,[111–113] discrete-time dynamical systems,[114] and spatiotemporal dynamical systems.[115] Further, early warning signals, which are a critical indicator for detecting tipping in dynamical systems, have been adapted to R-tipping.[116–118]

The R-tipping phenomenon can be better appreciated from a global perspective with a special focus on mutualistic networks.[31] These complex real-world ecological networks, e.g., the plant–pollinator networks from different geographical regions of the world. The key quantity of interest is the probability of R-tipping in these systems. In particular, for a given rate of parameter change, a large number of initial conditions sampled from a relevant region of the phase space can be examined so that the probability of R-tipping can be defined as the fraction of initial conditions that lead to tipping at a specific rate. For mutualistic networks, there are two potential scenarios for this dependence at the opposite extremes, as illustrated schematically in Fig. 1. One is that the probability increases slowly from zero as the rate increases. Such a "benign" dependence might be our hope, e.g., for an ecosystem, as this means there is a certain tolerance of the system against even fast parameter changes, leaving room for opportunities for control and mitigation. The opposite scenario is that the probability of R-tipping grows drastically and immediately as the rate of parameter change increases from zero. In this case, the system has no tolerance against any rate value, even if it is small. The implication is dire: in order to avoid R-tipping, the rate of parameter change must be reduced to practically near zero values. Which scenario of R-tipping do real-world dynamical systems tend to follow? The answer, unfortunately, is likely to be the second scenario for mutualistic networks.[31] Quantitatively, this scenario can be characterized by a general scaling law between the probability of R-tipping and the rate of parameter change. In the following section, a mathematical theory will be introduced to understand the scaling law, with numerical support from real-world plant–pollinator networks. The overall message is that, to prevent R-tipping from occurring in ecosystems and climate systems, the time rate of parameter change, such as the rate of emissions of harmful chemicals into the environment, must be significantly suppressed and kept at near zero.

FIG. 1. Probability of R-tipping vs the time rate of parameter change: two possible scenarios. The first one, as indicated by the black dashed curve, is that the R-tipping probability increases from zero slowly with the rate, where there is a "safe region" or "safe interval" of the rate in which the probability remains at some near zero value, thereby providing opportunities of control or intervention to reduce the rate of parameter change. The second scenario, as illustrated by the solid blue curve, is that the R-tipping probability grows dramatically and immediately as the rate of parameter change increases from zero. In this case, there is no safe region: in order to keep the probability near zero, the rate of parameter change must practically be reduced to zero. Mathematical reasoning and numerical evidence from empirical complex ecological networks tend to support the second scenario for nonautonomous dynamical systems in the real world.

III. NECESSITY OF A GLOBAL PHASE-SPACE APPROACH TO R-TIPPING

A. R-tipping as a global dynamical phenomenon

In real-world dynamical systems, parameters are not stationary but constantly change with time. For example, in climate systems, factors such as temperature and CO_2 emissions are not static but increase over time, influencing the system's stability with potentially severe consequences that can lead to the possible collapse of the system. To gain a qualitative understanding of the phenomenon of R-tipping in nonautonomous dynamical systems from a global perspective, we compare it with bifurcation-induced tipping in autonomous dynamical systems.

For bifurcation-induced tipping, we consider the scenario where, in the parameter regime of interest, two stable steady states (or attractors) coexist: a "left" state corresponding to "normal" functioning, as highlighted by yellow in Fig. 2(a), and a "right" or "abnormal" state. Each state has its own basin of attraction. If external factors cause an increase in a bifurcation parameter of the system, a tipping point can occur through a backward saddle-node bifurcation, which is reached when the "normal" fixed point disappears, leaving the "abnormal" state as the sole attractor. As a result, the basin of attraction for the original "normal" state is absorbed into that of the "abnormal" state. This means that, once the system crosses this tipping point, it inevitably transitions to the "abnormal" state, with no possibility of returning to the "normal" state through small perturbations. This type of bifurcation highlights the critical threshold beyond which a system undergoes an irreversible transition to a drastically different state.

The R-tipping scenario is illustrated in Fig. 2(b). Rapid variations of a parameter can trigger a critical transition even if the parameter values remain within some safe bounds. Consider the setting where, in the range of parameter variations, the system exhibits bistability with two coexisting stable steady-state attractors, represented by yellow and blue, respectively, in Fig. 2(b). Each state has its own basin of attraction, as indicated by green and purple arrows, respectively. During the transition, the basin of attraction of the left state (yellow) expands, while the basin of attraction of the right state (blue) shrinks. In this case, the final state of the system is determined by both the initial condition and the speed at which the parameter changes. In particular, rapid parameter changes can push the system past some critical threshold, causing it to settle into a different stable state than it would under slower, more gradual parameter changes.

Most previous studies of R-tipping focused on low-dimensional dynamical systems from a near equilibrium point of view, emphasizing the behavior of specific initial conditions in the vicinity of a stable equilibrium point and their trajectories to address issues such as the critical rate for tipping. Commonly, R-tipping was conceived as an abrupt change in the system behavior occurring at a specific rate of change of a bifurcation parameter.[18] However,

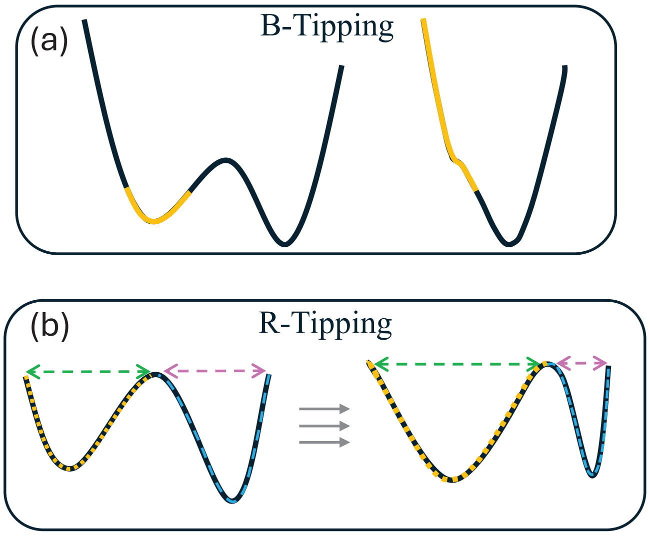


FIG. 2. Schematic comparison of bifurcation-induced tipping and R-tipping in a simple potential model. (a) Bifurcation-induced tipping: static changes in the control parameter lead to the disappearance of one steady state of the system. (b) R-tipping: the state of the system is determined by both the initial condition and the speed at which the parameter changes.

real-world systems are inherently dynamic and often operate under nonideal conditions:[38,119] they are constantly influenced by external disturbances, fluctuating environmental conditions, and inherent variability where disturbances and stochastic variations are the norm rather than the exception.[35,87] For example, ecological systems are affected by changes in species interactions and environmental factors all the time, which, in turn, influence population dynamics and stability.[36–38] In these contexts, species may face varying levels of predating, competition, and resource availability, leading to complex and unpredictable behaviors. The dynamic nature of these systems means that they often drift away from the equilibrium point, making them susceptible to tipping events triggered by rapid changes in external conditions. This variability necessitates the study of R-tipping from a global perspective, taking into account the entire relevant phase space rather than focusing solely on the near-equilibrium dynamics surrounding the equilibrium point.

More generally, initial conditions away from an equilibrium point can influence the system's response to rapid parameter changes, making it essential to consider a broader range of initial conditions in R-tipping studies. For example, in ecological systems, the natural variability in species populations and environmental conditions invalidate the assumption of initial conditions being near some equilibrium. Similarly, in engineering systems, operational parameters can drift due to external shocks or internal variability, necessitating a comprehensive analysis that accounts for a wide range of the initial states. In fact, the integration of diverse initial conditions into R-tipping research is not merely an academic exercise, but a practical necessity for accurately assessing and managing the risks associated with critical transitions in natural and engineering systems.

B. Dependence of critical rate for R-tipping on initial conditions

A consequence of a global analysis is that the critical rate for R-tipping depends on the initial condition. This dependence can be conveniently analyzed in the general setting of bistability in a two-dimensional phase space in which two steady-state attractors coexist: survival and extinction, as illustrated by the filled green and red circles, respectively, in Fig. 3. To gain insights, we begin by considering two extreme cases: $r = 0$ and $r \to \infty$. For $r = 0$, there is no change in the parameter p with time: it stays at the initial value p_1. For this relatively small parameter value, the system is mostly "healthy" so the basin of attraction of the survival attractor is "larger." In this case, the majority of the initial conditions in the phase space will land the system in the survival attractor, so the basin boundary Σ_1 between the two basins is located closer to the extinction attractor, as shown in Fig. 3(a). In the opposite extreme $r \to \infty$, for any $t > 0$, the system is already at the larger parameter value p_2 so the system is in a decayed environment that is hostile to system's normal functioning. In this case, the majority of the initial conditions in the phase space will lead to extinction and the basin boundary Σ_2 is closer to the survival attractor, as shown in Fig. 3(b). The phase-space structures for these two extreme cases, as illustrated in Figs. 3(a) and 3(b) for $r = 0$ and $r \to \infty$, respectively, implies that for any finite value of r, say r^*, the basin boundary Σ^* must lie in between Σ_1 and Σ_2, as shown in Fig. 3(c).

We now provide a mathematical reasoning for the dependence of the critical rate for R-tipping on the initial condition, which provides a geometric principle to determine the critical rate for any given initial condition. In particular, in Fig. 3(c), all initial conditions on the basin boundary Σ^* have the critical rate r^*. To see this, consider two parameter values in the vicinity of r^*: $r^* \pm \delta r$, where δr is infinitesimal. For $r = r^* - \delta r < r^*$, the basin boundary is located slightly closer to the extinction attractor than Σ^*, as shown in Fig. 3(d). In this case, all initial conditions on Σ^* belong to the basin of the survival attractor. For $r = r^* + \delta r > r^*$, the basin boundary is located slightly closer to the survival attractor than Σ^*, so all initial conditions on Σ^* now belong to the basin of the extinction attractor. As a result, an infinitesimal increment in the rate from r^* will cause all initial conditions on Σ^* to switch their fate: from survival to extinction, indicating that r^* is the critical rate of R-tipping for all the initial conditions on the basin boundary Σ^*. For a different value of the rate, say $r^\dagger$, the location of the basin boundary in the phase space is different, so the initial conditions on this boundary will have the critical rate $r^\dagger$, which is different from r^*. It is therefore apparent that, from the point of view of the whole phase space, the concept of critical rate may not be meaningful as there is an uncountably infinite number of critical rate values for R-tipping, depending on the initial condition.

IV. SCALING LAW OF R-TIPPING PROBABILITY

In our recent work,[31] a scaling law governing the dependence of the probability of R-tipping on the rate of parameter change in mutualistic networks was uncovered. To explain the scaling law, we assume that a parameter of the system, denoted as $p(t)$, increases at the linear rate r from an initial value p_1 at time $t = 0$ to a final

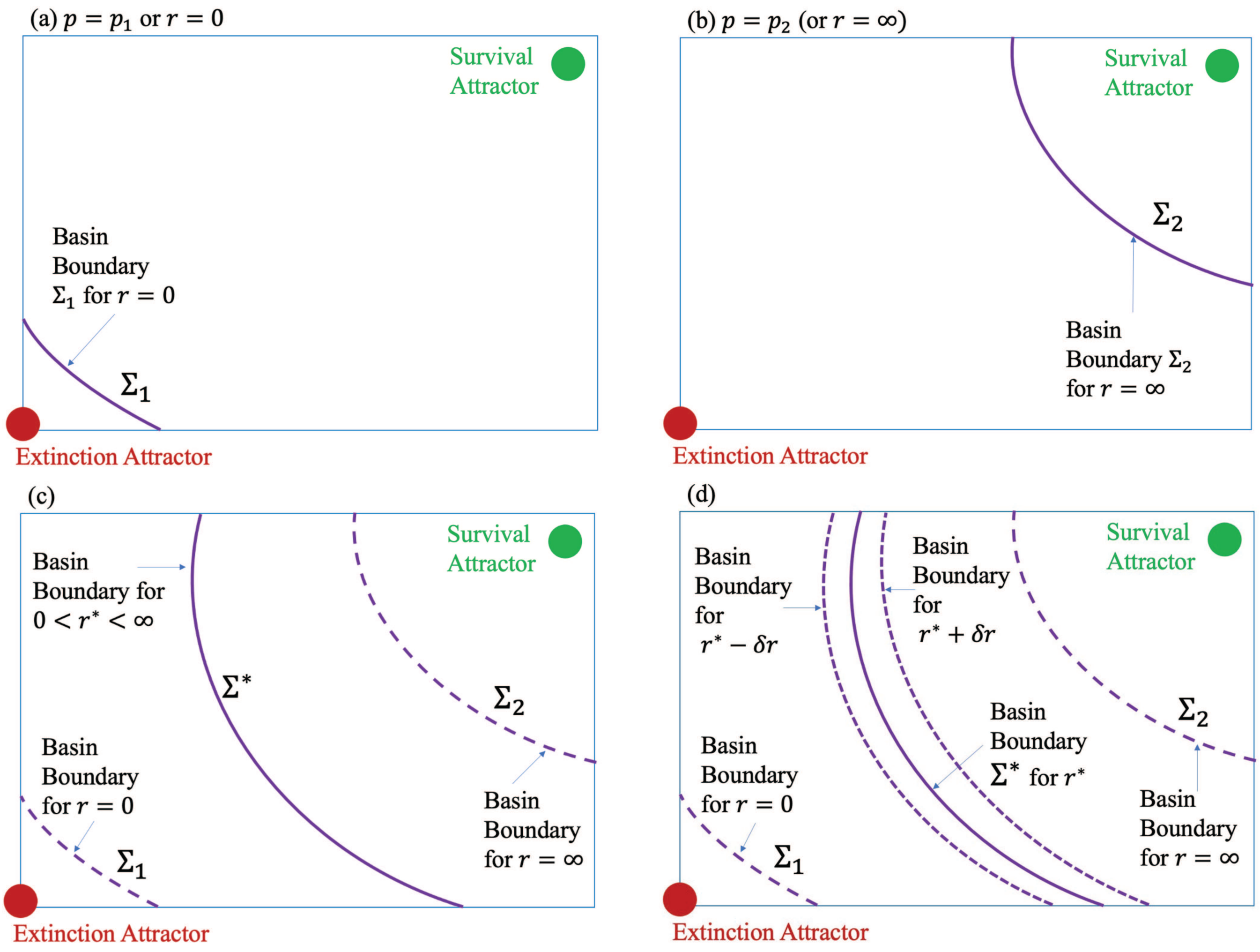


FIG. 3. Phase-space structure for different values of the rate parameter and dependence of the critical rate on initial condition. The nonautonomous bistable system has a time-varying parameter $p(t)$ that increases at the linear rate r from an initial value p_1 at time $t = 0$ to a final value $p_2 > p_1$ during the time interval $[0, T]$, where p_1 (p_2) corresponds to a health (deteriorated) environment. There are two attractors throughout the parameter variation: a survival and an extinction attractor. (a) The extreme case $r = 0$ [$p(t) = p_1$], where the basin boundary Σ_1 is located closer to the extinction attractor. (b) The opposite extreme case $r \to \infty$, where basin boundary Σ_2 is located closer to the survival attractor. (c) The location of the basin boundary Σ^* for any finite rate value r^*, which is in between Σ_1 and Σ_2. (d) The basin boundaries for rate values $r^* \pm \delta r$ with infinitesimal δr, revealing that all initial conditions on Σ^* share r^* as the critical rate value for R-tipping.

value $p_2 > p_1$ during the time interval $[0, T]$ with $T = (p_2 - p_1)/r$. We assume a qualitative correspondence between the environmental condition and $p(t)$: a smaller value of p indicates a relatively more healthy state of the system. In ecosystems, such a parameter could be, e.g., the decay rate of some species, which tends to increase as the environment deteriorates with time. Let $\Phi(r)$ be the probability of R-tipping. The scaling law is

$$\Phi(r) \sim \exp\left[-C\frac{(p_2 - p_1)}{r}\right], \tag{1}$$

where $C > 0$ is a constant. This scaling law has the following features: (1) the probability $\Phi(r)$ is an increasing function of r, (2) the derivative $\Phi'(r)$ is a decreasing function of r, and (3) the maximum rate of increase in $\Phi(r)$ occurs for $r \gtrsim 0$. The second feature indicates that the rate at which $\Phi(r)$ increases slows down as r increases, eventually approaching zero as $r \to \infty$. The third feature emphasizes the fact that, insofar as the parameter varies with time, even with a very small rate of change, the chance of R-tipping can be nonzero and large. This has alarming real-world implications: a slow change in the parameter could still precipitate a system collapse with catastrophic consequences, so simply slowing down the rate of parameter change might not be sufficient to prevent tipping. Instead, the rate of change must be reduced to a nearly zero value to prevent R-tipping.

To derive the scaling law (1), we consider two different rates of parameter change: r_1 and $r_2 > r_1$. In an autonomous system, the boundary is the stable manifold of an unstable fixed point, denoted as $\mathbf{f}$, as indicated by the filled green circle in Fig. 4. When the system becomes nonautonomous with a time-varying parameter $p(t)$, the unstable fixed point and the basin boundary become time dependent, so we write $\mathbf{f}(t)$ or $\mathbf{f}(p)$. We focus on an infinitesimal neighborhood of $\mathbf{f}(t)$, where the basin boundary is approximately straight, as shown in Figs. 4(a) and 4(b) for $r = r_1$ and $r = r_2$, respectively. For the two cases, the parameter variation occurs within the interval $[p_1, p_2]$ in the time interval $[0, T_1]$ and $[0, T_2]$, respectively. Since $r_2 > r_1$, we have $T_2 < T_1$. In each case, the arrowed dashed line through $\mathbf{f}(t)$ indicates the direction along which it moves in the phase space as the parameter changes over time. The solid green (at $t = 0$) and blue (for $t = T_1$ or T_2) line segments through $\mathbf{f}(t)$ represent the boundaries separating the extinction from the survival basin. Before the parameter variation starts ($p = p_1$), initial conditions below the solid green lines belong to the basin of the extinction attractor, while those above belong to the survival attractor. After the parameter variation ends ($p = p_2$) the initial conditions below the solid blue lines belong to the extinction basin, and those above belong to the survival basin.

During the parameter variation, $\mathbf{f}(t)$ shifts from the green circle's position to the blue circle's position, causing its stable manifold (the basin boundary) to move accordingly. Initial conditions in the light-green shaded area, which initially belong to the survival attractor's basin for $p = p_1$, will be exponentially stretched along $\mathbf{f}(t)$'s unstable direction and compressed in the stable direction, forming a long blue rectangle. Since $T_1 > T_2$, the blue rectangle for $r = r_1$ is longer and thinner compared to that for $r = r_2$. Due to $\mathbf{f}(t)$'s movement and the changing basin boundary as the parameter varies, a portion of the long rectangle (dark shaded blue) now falls within the basin of the extinction attractor. The initial conditions in the original green rectangle that evolve into this dark shaded blue region are those that switch from the survival to the extinction attractor due to the time-dependent parameter change, experiencing R-tipping, as indicated by the red rectangle within the green area in Fig. 4(b). The fraction of such initial conditions for any given rate r determines the R-tipping probability. Denoting the fraction of R-tipping initial conditions by $d(0)$ and the distance between the basin boundaries along $\mathbf{f}(t)$'s unstable direction by Δ, we have

$$d(T) = \Delta = d(0)\exp(\lambda T), \tag{2}$$

where λ is the unstable eigenvalue of $\mathbf{f}(t)$, which is assumed to remain approximately constant in the course of time variation of the parameter. Substituting $T = (p_2 - p_1)/r$ into Eq. (2) leads to the scaling law (1).

In the derivation of the scaling law (1), the assumption that the initial conditions are uniformly distributed in the phase-space region was employed.[31] This assumption is speculative and at the best approximate as there are no empirical data on how the initial conditions are distributed in typical real-world ecological systems. A heuristic justification is that an ecosystem is under constant bombardment of stochastic perturbations of different scales, such as small-scale demographic stochasticity and large-scale climatic events. Under such perturbations, an ecosystem is pushed away from a stable equilibrium into different regions of the phase space along a variety of directions at all times. In this sense, a uniform distribution is a crude but not unreasonable assumption. How different initial-condition distributions may modify the scaling law is a question that warrants further efforts.

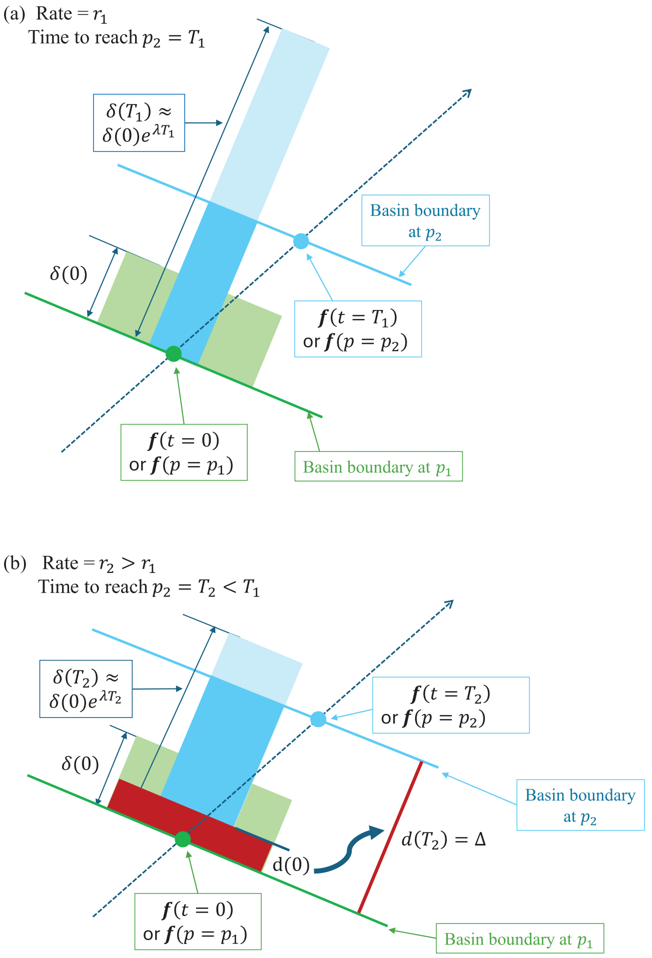


FIG. 4. A geometrical argument leading to the scaling law (1). See text for details. This figure is based on Fig. 5 in Ref. 31.

V. SCALING LAW OF R-TIPPING IN COMPLEX ECOLOGICAL NETWORKS

The derivation of the scaling law (1) is based on a two-dimensional phase-space structure. Real-world systems are high-dimensional. Can the scaling law (1) be expected to hold in high-dimensional nonautonomous dynamical systems? One class of such systems that was used[31] to test the scaling law (1) is complex plant-pollinator mutualistic networks,[16,20,23,90,120–127] where a species in the plant group benefits from interacting with some species in the pollinator group, and vice versa, as illustrated in Fig. 5(a). The structures of over 100 empirical mutualistic pollinator–plant networks from many geographical regions in the world have been well documented

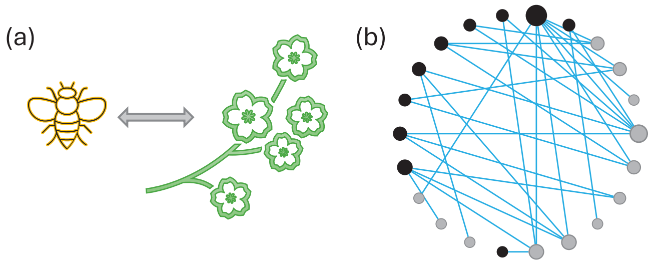


FIG. 5. Illustration of mutualistic networks. (a) Mutualistic interaction between a plant and a pollinator species. (b) Network structure of an empirical mutualistic network from the Web of Life database.

(Web of Life database, www.Web-of-Life.es). One example of such a network is shown in Fig. 5(b). At the present, a detailed computational model for these empirical mutualistic networks is available.[120] In spite of their high dimensionality, due to the nature of steady-state dynamics associated with tipping, a mutualistic network can be reduced effective to a two-dimensional system[20] through some conventional dimension-reduction methods, providing a feasible paradigm for numerically testing the scaling law (1).

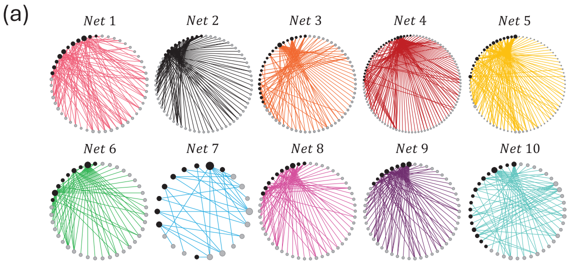
The dynamics of a plant–pollinator mutualistic network, comprising N_A pollinator and N_P plant species, are described by a system of $N = N_A + N_P$ nonlinear differential equations of the Holling type in terms of the species abundances,[120]

$$\dot{P}_i = P_i\left(\alpha_i^P - \sum_{l=1}^{N_P}\beta_{il}^P P_l + \frac{\sum_{j=1}^{N_A}\gamma_{ij}^P A_j}{1+h\sum_{j=1}^{N_A}\gamma_{ij}^P A_j}\right), \tag{3}$$

$$\dot{A}_j = A_j\left(\alpha_j^A - \kappa_j - \sum_{l=1}^{N_A}\beta_{jl}^A A_l + \frac{\sum_{i=1}^{N_P}\gamma_{ji}^A P_i}{1+h\sum_{i=1}^{N_P}\gamma_{ji}^A P_i}\right), \tag{4}$$

where P_i and A_j are the abundances of the ith and jth plant and pollinator species, respectively, $i = 1, \ldots, N_P$ and $j = 1, \ldots, N_A$. The equations account for intrinsic growth rates, intraspecific and interspecific competitions, and mutualistic interactions, with specific parameters characterizing the dynamics, such as the pollinator decay rate κ, intrinsic growth rate $\alpha^{P(A)}$, and the half-saturation constant h. The mutualistic interactions are further quantified through parameters $\gamma_{ij}^{P(A)} = \xi_{ij}\gamma_0/K_i^{\tau}$ that depend on whether a mutualistic interaction exists (ξ_{ij}), the general interaction parameter γ_0, and the degree of the plant species K_i.

To introduce the rate change of a parameter, three different scenarios were considered[31] where negative environmental impacts can lead to: (1) a linear increase in the species decay rate, (2) a linear decrease in the mutualistic interaction strength, and (3) simultaneous linear changes in both parameters over time. Simulations were conducted using ten empirical mutualistic pollinator–plant networks, where the intervals of parameter variations were selected to ensure the occurrence of bistability. The structures of the ten networks are illustrated in Fig. 6(a), and their structural parameters and origin are listed in Fig. 6(b).



(b)

Table I. Ten empirical mutualistic networks from the Web of Life database

Network	Country	N_P	N_A	κ-interval
Net 1	Canary Islands	11	38	[0.90, 0.93]
Net 2	South Africa	9	56	[0.70, 0.99]
Net 3	Argentina	21	45	[0.75, 0.93]
Net 4	Argentina	23	72	[0.87, 0.96]
Net 5	New Zealand	18	60	[0.90, 0.95]
Net 6	USA	7	33	[0.70, 0.99]
Net 7	Acores	10	12	[0.74, 0.88]
Net 8	Denmark	10	40	[0.80, 0.93]
Net 9	Denmark	8	42	[0.70, 0.95]
Net 10	Greenland	17	26	[0.77, 0.98]

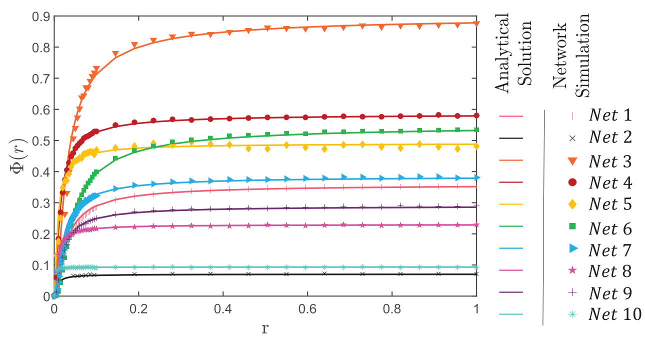


FIG. 6. Scaling law of R-tipping for ten empirical mutualistic networks. The time-varying parameter is the pollinator decay rate κ. (a) The network structures. (b) Structural parameters and origins of the ten empirical networks. (c) R-tipping probability $\Phi(r)$ vs the rate r of parameter change. The dots are the results from direct numerical simulations of the high-dimensional networks with an ensemble of random initial conditions from a substantial volume of the high-dimensional phase space. The solid curves are the theoretical fits of the scaling law (1). Other parameter values in the computational model are fixed: $\alpha = 0.3$, $\beta = 1$, $h = 0.4$, $\gamma^P = 1.93$, and $\gamma^A = 1.77$. This figure is based on Fig. 5 in Ref. 31.

For the high-dimensional mutualistic network given by Eqs. (3) and (4), the probability $\Phi(r)$ of R-tipping can be calculated, as follows. One first sets $r = 0$ and solves Eqs. (3) and (4) numerically for a large number of random initial conditions chosen uniformly from the whole high-dimensional phase space. Next, a large number (e.g., 10^5) initial conditions resulting in trajectories that approach the high stable steady state in which no species becomes extinct are determined. The rate r is then increased from zero. For each fixed value of r and for each of the selected 10^5 initial conditions, whether or not the final state is the high stable state is checked. If yes, there is no R-tipping for the particular initial condition. However, if the final state becomes the extinction state, R-tipping has occurred for this value of r. The probability $\Phi(r)$ can be approximated by the fraction of the number of initial conditions leading to R-tipping out of the 10^5 initial conditions.

Figure 6(c) presents examples of the scaling law of the probability of R-tipping $\Phi(r)$ with the rate r where, for each network, the species decay rate κ varies with time linearly across the interval specified in Fig. 6(b). It can be seen that the R-tipping probabilities for all ten networks exhibit qualitatively similar behavior: as the rate of parameter change increases from zero, the probability initially rises rapidly and then plateaus at an approximately constant value, as predicted by the scaling law (1). The final value of the R-tipping probability depends on the specific structural characteristics of the network.

VI. DISCUSSION

Investigating R-tipping is important for understanding how complex systems respond to the constant changes in their parameters due to environmental effects. There are two approaches to R-tipping in nonautonomous dynamical systems: near-equilibrium and global. The near-equilibrium approach focuses on the behavior near some equilibrium point, taking advantage of the corresponding autonomous system to determine the conditions under which the system transitions from one state to another. This approach has provided insights into the R-tipping mechanism.

This brief review aims to introduce the global approach to the nonlinear-dynamics community, which offers a comprehensive and holistic picture, particularly for real-world systems that are typically open, dynamic, and under the influences of stochastic disturbances. Due to these effects, in the phase space, the dynamical trajectory of the system can hardly be confined near some equilibrium point but rather, it can wander in an extended region of the phase space. Under such circumstances, a more effective approach to studying R-tipping is to move beyond near-equilibrium initial conditions and explore a broader range of initial conditions within the relevant phase-space region. Consequently, the concept of a single critical rate for R-tipping becomes less applicable, as it varies with the initial condition, leading to an uncountably infinite number of possible critical values. The scaling law (1), derived through a geometric argument and validated in a number of complex empirical plant-pollinator mutualistic networks, suggests a dire consequence of the rate of parameter changes for these ecological networks: insofar as it is not zero, the probability of R-tipping can be substantial.

A topic of current interest is data-driven prediction of tipping in nonlinear and complex dynamical systems. Traditional methods often rely on detailed knowledge of the system's dynamics, which may not always be available. A promising approach is machine learning, especially recurrent neural-network architectures capable of identifying dynamical patterns and predicting critical transitions by learning from data.[128–136] To develop effective machine-learning models for predicting tipping, two considerations are essential. First, the availability of training data is crucial. While traditional classification problems require data from both sides of the critical transition, this is impractical since post-transition data are not available. Training must be based on data from the pre-critical regime, necessitating the development of models that can extrapolate from this information. Second, the machine-learning architecture must be capable of Self-organizing evolution, mimicking the inherent dynamics of the target system. The recently developed adaptable reservoir-computing[129–133] provides a possibility.

A significant problem to which R-tipping is relevant is the possible collapse of the AMOC[40,137,138] that plays a crucial role in maintaining moderate temperature conditions in Western Europe by transporting warmer, upper waters northward and returning colder, deeper waters southward.[138,139] While studies have indicated a tendency for AMOCs to weaken in the last 30 years,[140,141] at the present the AMOC is still in a stable state that ensures the continuation of these critical ocean flows. However, due to the increasing human influence on the climate change, a potential halt of this circulation signifying a collapse of the AMOC represents a shift to another stable steady state within the underlying dynamical system, which is characteristic of R-tipping. Recent research based on simplified stochastic dynamical system models suggests that the AMOC may be on the course toward collapse, with a tipping point potentially occurring as early as 2025.[102] Estimating the probability of the collapse due to R-tipping based on measurement data is an urgent but open problem.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Shirin Panahi: Conceptualization (supporting); Formal analysis (supporting); Investigation (equal); Software (lead); Visualization (lead); Writing – original draft (equal). **Ying-Cheng Lai:** Conceptualization (lead); Formal analysis (lead); Funding acquisition (lead); Investigation (equal); Methodology (lead); Project administration (lead); Supervision (lead); Writing – original draft (equal); Writing – review & editing (lead).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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