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ABSTRACT

A problem in nonlinear and complex dynamical systems with broad applications is forecasting the occurrence of a critical transition based solely on data without knowledge about the system equations. When such a transition leads to system collapse, as often is the case, all the available data are from the pre-critical regime where the system still functions normally, making the prediction problem challenging. In recent years, a machine-learning based approach tailored to solving this difficult prediction problem, adaptable reservoir computing, has been articulated. This Perspective introduces the basics of this machine-learning scheme and describes representative results. The general setting is that the system dynamics live on a normal attractor with oscillatory dynamics at the present time and, as a bifurcation parameter changes into the future, a critical transition can occur after which the system switches to a completely different attractor, signifying system collapse. To predict a critical transition, it is essential that the reservoir computer not only learns the dynamical "climate" of the system of interest at some specific parameter value but, more importantly, discovers how the system dynamics changes with the bifurcation parameter. It is demonstrated that this capability can be endowed into the machine through a training process with time series from a small number of distinct, pre-critical parameter values, thereby enabling accurate and reliable prediction of the catastrophic critical transition. Three applications are presented: predicting crisis, forecasting amplitude death, and creating digital twins of nonlinear dynamical systems. Limitations and future perspectives are discussed.

Critical transitions are a ubiquitous phenomenon in a variety of complex dynamical systems. For example, in an ecosystem, as the environmental conditions continuously deteriorate due to global climate change, some key parameters underlying the system dynamics can drift through a critical point and tipping leading to mass extinction can arise. In an electrical power-grid network, as a parameter characterizing the load increases through a critical value, voltage collapse and a large-scale blackout can occur. In engineering, an infrastructure such as a bridge can suddenly collapse as some intrinsic parameter varies through a critical value. In social, economical, and political systems, a large institution or even a government can collapse as the result of accumulated deterioration of certain internal and/or external conditions. To be able to predict or anticipate the potential occurrence of a critical transition is of broad interest. For complex systems in the real world, it is not possible to write down the mathematical equations governing their dynamics. Even for those for which an approximate set of the governing equations can be obtained through mathematical modeling, the inevitable existence of errors, disturbances, and noise can defy any prediction from the mathematical model. Predicting critical transitions should then be done based on data. A realistic situation is that, typically, data can be measured from a system but only when it is functioning—no data can be obtained after its collapse. Because of this realistic constraint, a useful prediction framework must rely on data that can be collected while the system is still in a normal operational regime. The difficulty is that, for fixed parameters in the pre-critical regime, no sign of

any collapse can be detected from the measured dataset, making predicting critical transitions a challenging problem in the field of nonlinear and complex dynamical systems. In recent years, a machine-learning framework for anticipating critical transitions has begun to emerge, where a major approach is based on reservoir computing, a recurrent neural-network architecture that has been investigated extensively for predicting the dynamical evolution of nonlinear and chaotic systems for specific parameter values. To enable a reservoir computer to predict critical transitions, it must be endowed with the ability, through training, of assessing the dependence of the system dynamics on changes in the parameters. That is, a well-trained reservoir computing machine must not only learn the dynamical "climate" of the target system for any fixed parameter set, but more importantly, to learn how the "climate" changes with the parameter. Adaptable reservoir computing was developed to meet these requirements, with demonstrated success in predicting critical transitions in a variety of low-dimensional dynamical systems or high-dimensional systems with a simple parameter structure. Adaptable reservoir computing also provides the foundation for creating digital twins of nonlinear dynamical systems. This Perspective is devoted to introducing adaptable reservoir computing, explaining its working, and presenting some representative examples of successful prediction of different types of critical transitions. Limitations and future perspectives are also discussed.

I. INTRODUCTION

In real-world applications, it is often the case that a precise mathematical model describing the underlying system is not available, so understanding and predicting its dynamical behaviors must be based on observation and measured data. The reality raises a challenging problem in nonlinear dynamics: how to anticipate potential future collapse of the system with only data collected in the past? More specifically, suppose that, at the present, the system is in a precritical parameter regime of normal functioning, but a bifurcation parameter of the system changes slowly with time. At certain points in the future, as the parameter increases through a critical value, a transition can occur that places the system in a catastrophic state of collapse. Nearly half a century of research in nonlinear dynamics revealed that such critical transitions are in fact quite common,[1,2] such as a crisis[3] at which a chaotic attractor is suddenly destroyed or an amplitude death[4,5] at which the normal oscillatory behavior halts. The question is that given measured data as the only available information about the system, how can a critical transition that can potentially occur in the future be anticipated with confidence?

Developing a model-free, purely data-driven framework to predict the future state of a nonstationary complex dynamical system is pertinent to some of the most pressing problems in modern times. For example, due to global warming and climate change, some natural systems may have already passed the so-called tipping point and are in a transient state awaiting a catastrophic collapse to occur. A reliable and accurate assessment that the system has already passed the critical transition point to a transient state would send a clear message to policy makers and the general public that actions must be taken immediately to avoid the otherwise inevitable catastrophic collapse. A data-driven prediction paradigm also has applications in engineering systems that can be vulnerable to catastrophic failures due to a variety of internal and external factors. While external factors such as extreme weather can be foreseen, it is much harder to detect intrinsic dynamical effects that can potentially lead to system collapse. An example is aircraft landing on an aircraft carrier, where internal maneuvers such as reducing the fuel mixture are necessary. However, this can lead to engine flameout that can be catastrophic. Developing methodologies to detect precursors and predict the possible occurrence of engine flameout in advance so that appropriate control can be applied to significantly reduce the probability of engine failure is of considerable engineering interest.

There have been two approaches to addressing the problem of model free, data-driven prediction of critical transitions in nonlinear and complex dynamical systems. The first is the natural one of finding the governing equations of the system from data. A pioneering effort was made by Crutchfield and McNamara[6] who articulated the method of finding the equations governing the deterministic dynamics of the system by estimating the information contained in the data through the classical Takens delay-coordinate embedding theory. Later, methods based on approximating the invariant density,[7] least-squares estimates,[8] or sparse optimization[9–18] such as compressive sensing[19–24] were proposed. When the system equations have been found, an accurate prediction of the dynamical evolution of the system into the future becomes possible through some appropriate computational analysis of the intrinsic bifurcations. However, this "equation-finding" approach has limitations and are not generally applicable. For example, the sparse-optimization approach, by its name, applies only to systems with a simple structure in the sense that the velocity field can be represented by a small number of power-series expansion or Fourier-series terms. Another challenge is that real-world systems are complex and subject to internal fluctuations and external disturbances, so precise knowledge of the system equations is generally not available. Even if an approximate set of the equations has been found, the underlying system can be fundamentally nonlinear and is likely to exhibit sensitive dependence on initial conditions, parameter variations, stochastic fluctuations, and perturbations, rendering ineffective any equation-based prediction methods.

The second, more general and modern approach to predicting critical transitions is machine learning, which is model-free and fully data-driven.[25–27] The basic idea is that, if a neural network can be trained to capture not only the "dynamical climate" of the system of interest for specific parameter values but also how the "climate" changes with parameters, it may then become possible for the neural network to produce the correct evolution of the system into the future. To enable a machine-learning architecture to anticipate critical transitions, there are four basic requirements. First, the "complexity" of the neural network must surpass that of the target system of interest, i.e., the neural network should be capable of encapsulating the diverse behaviors within the system, enabling accurate prediction even in the presence of uncertainties or variations. This can be achieved by making the dimension of the dynamical neural network significantly higher than that of the target system. Second, since the target system is a continuously evolving dynamical system in that a change in the initial state can often significantly affect the state of the system some time later, it is necessary that the neural network should possess certain intrinsic memory properties, which can be achieved by a recurrent structure of the neural network. Third, the machine-learning architecture must be such that it is capable of self-evolution, because the original system is a self-evolving dynamical system. These three requirements can be met by reservoir computing.[28–30] Indeed, recent years have witnessed a growing interest in exploiting reservoir computing for predicting the state evolution of nonlinear dynamical systems.[31–50] Importantly, to endow a reservoir computer with the ability to prognosticate critical transitions, it must be trained to learn how the dynamical behavior of the target system changes with a bifurcation parameter. The fourth requirement is then that the neural machine be capable of reproducing the characteristic changes in the dynamics of the original system as a parameter changes. To achieve this, an input parameter channel was introduced to reservoir computing, leading to the machine-learning architecture of the adaptable reservoir computing.[25–27]

The focus of this Perspective is on adaptable reservoir computing. Section II describes the basics of its architecture for anticipating critical transitions in dynamical systems. In Sec. III, the power of adaptable reservoir computing in predicting crisis is demonstrated. Section IV presents another application: predicting amplitude death. Section V is devoted to reservoir-computing based digital twin of nonlinear dynamical systems, demonstrating that not only critical transitions but the whole global bifurcation diagram can be reconstructed using adaptable reservoir computing in a fully data-driven manner. A discussion and future perspective is offered in Sec. VI.

II. ADAPTABLE RESERVOIR COMPUTING

The core of reservoir computing[28–30,51] is a complex network of artificial but nonlinear neurons, in which the neurons interact with each other according to its topology, producing a recurrent neural network as interactions can propagate forward and backward in the network. The idea and principle of exploiting reservoir computing for predicting the state evolution of chaotic systems were first laid out about two decades ago.[28,30] There are two major types of reservoir computing systems: echo state networks[28] and liquid state machines.[29] The training of an echo state network is associated with supervised learning underlying recurrent neural networks, where the basic principle is to drive a large neural network of a random or complex topology, the reservoir network, with the input signal. Each neuron in the network generates a nonlinear response signal. Linearly combining all the response signals with a set of trainable parameters yields the output signal.

There are three main components in reservoir computing: an input layer, a hidden layer with a high-dimensional and complex neural network (the reservoir network), and an output layer, as illustrated in Fig. 1(a). The input layer maps the available, typically low-dimensional time series data into the high-dimensional state space of the reservoir network and the output layer projects the high-dimensional dynamical evolution of the neural network state back into low-dimensional time series (readout). Training is administered to adjust the elements of the projection matrix of the output layer to minimize the difference between the output and the true input time series. Because of the nature of the recurrent neural network, the elements of the input reservoir network matrices are

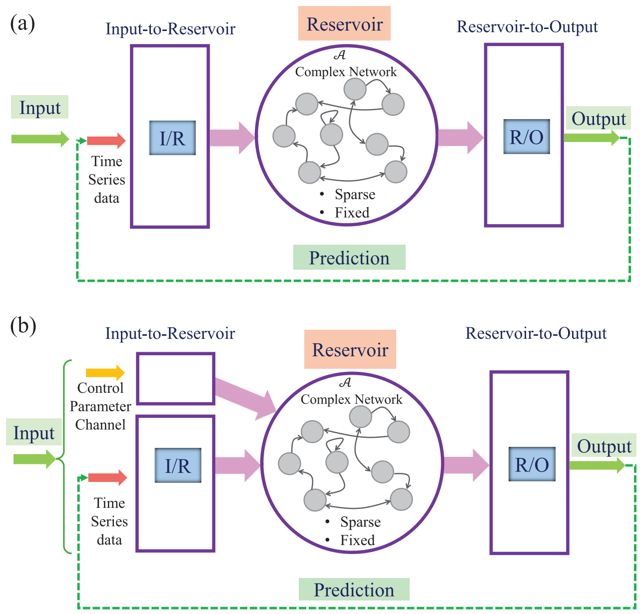

FIG. 1. Reservoir computing architecture. The neural machine consists of three layers: an input layer, a hidden layer, and an output layer. The input-to-reservoir is characterized by the input matrix W_{in} whose elements are randomly chosen. The hidden layer consists of N neurons interacting with each other according to the weighted matrix $\mathcal{A}$ characterizing the complex network, whose elements are also random and pre-chosen. The reservoir-to-output is governed by the output matrix W_{out}, whose elements are determined by training (supervised learning) with the available data. The training process is of the open-loop type because of the data input. After training, the output variables are connected to the input, generating a closed-loop, self-evolving dynamical system. (a) Conventional reservoir computing for predicting the state evolution of the target system. (b) Adaptable reservoir computing for predicting critical transitions with parameter variations, where the specific value of the bifurcation parameter associated with the input time-series data is also input to each and every neuron in the hidden-layer reservoir network, making the neural-network dynamics dependent upon the parameter.

chosen *a priori* and fixed during the training and prediction phases so as to achieve highly efficient learning. In terms of hardware realization, reservoir computing can be implemented using electronic, time-delay autonomous Boolean systems[31] or high-speed photonic devices.[32]

Reservoir computing has been applied to forecasting critical transitions. A relatively earlier work[52] proposed a data-driven method to predict noise-induced critical transitions in multiscale nonlinear dynamical systems by using the conventional architecture in Fig. 1(a), where a deep version of the echo-state network was used to predict the short-term state evolution of nonstationary dynamical systems. It was demonstrated that rare events induced by noise can be forecasted. Since such a transition is induced by noise without involving any parameter change of the target system, it is not necessary for the hidden-layer network to be “aware” of the pertinent parameter value, rendering applicable the conventional reservoir-computing architecture in Fig. 1(a).

To predict critical transitions due to parameter changes, it is essential that the reservoir neural network is trained to generate parameter-dependent dynamics. The idea of a parameter-aware recurrent neural network was first proposed in an early work,[53] where a neural network was trained using time series from different parameter values of the logistic and Hénon maps. It was demonstrated that, with a "fixed weight neural network," changing input in the parameter channel alone can make the neural network produce various dynamical behaviors of the target system for different parameter values. More recently, adaptable reservoir computing was developed to address the problem of predicting critical transitions.[25,26,47,54,55]

Additional works on reservoir computing with a parameter channel are Refs. [44–46]. In particular, in Ref. 44, a dynamical learning approach was articulated for reservoir computing, where an error feedback loop and a context feedback loop were added to the standard reservoir-computing structure. It was shown that, after training, the modified reservoir-computing system has the ability to learn the dynamics that were different from those of the training set with a small amount of data. The process was named "dynamical learning" due to its relation with the automatic adaptation of fixed-weight neural networks with the error and context feedback loops. As a concrete example, it was demonstrated that the framework is capable of predicting the Hopf bifurcation in the Lorenz system.[44] In Ref. 45, reservoir computing with a parameter channel was used to predict the occurrence of periodic windows and other regimes of transitions in nonstationary chaotic systems with or without dynamical noise. In Ref. 46, the approach was applied to the Lorenz system to predict Hopf, saddle-node, and pitchfork bifurcations, where training was carried out based on the normal forms of the bifurcations. Transient chaotic behavior of the reservoir neural network was also demonstrated,[46] where the predicted trajectory of the Lorenz attractor behaves chaotically for a while and then begins to fall into a fixed point. A systematic study was carried out,[25,26] demonstrating the ability of reservoir computing to capture and predict transient chaotic dynamics triggered by a crisis. More recently, this approach was extended[56] to scenarios where knowledge about the known parameter variations is unavailable and the observed pre-transition motion is confined to a smaller state space subset than the post-transition motion.

In adaptable reservoir computing, an additional input channel is designated to simultaneously input the specific parameter value associated with the input time-series data to each and every neuron in the network, as illustrated in Fig. 1(b). The training process rendering the reservoir computer adaptable to parameter changes in the target system can be explained using the schematic illustration in Fig. 2. Let p be the bifurcation parameter of the target system. As p varies, a critical point occurs at p_c, where the system functions normally for $p < p_c$ and it exhibits a transient toward collapse for $p > p_c$. Training of the reservoir machine is done based on time series taken from a small number of parameter values in the normal regime, e.g., $p_1 < p_2 < p_3 < p_c$. For each parameter value, adequate training is required in the sense that the machine is able to predict correctly and accurately the oscillatory behavior at the same parameter value for a reasonable amount of time. Suppose that, currently, the system functioning is normal and it operates at the parameter value $p_0 < p_c$. In the prediction phase, suppose a parameter change $\Delta p > 0$ has occurred. The new parameter value $p_0 + \Delta p$ is then fed into the reservoir machine through the parameter channel. The prediction is deemed successful if the machine generates normal oscillations for $p_0 + \Delta p < p_c$ but exhibits a transient toward collapse for $p_0 + \Delta p > p_c$.

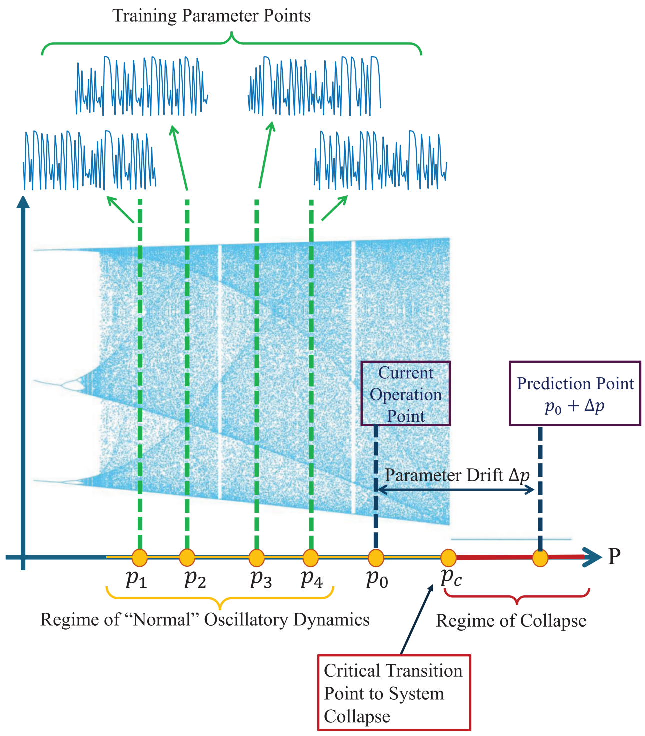

FIG. 2. Illustration of adaptable machine-learning scheme for prediction of system collapse. Training of the neural machine is done in the pre-transition regime for a small number of bifurcation parameter values (as indicated by the four vertical dashed green lines), where the system is in the normal regime of operation and generates oscillatory time series. The critical transition to collapse occurs at p_c. The target system currently operates at p_0. Prediction is done for $p = p_0 + \Delta p$, where $\Delta p > 0$ is a parameter drift. Depending on whether the value of $p_0 + \Delta p$ is below or above the transition, a properly trained machine shall be able to predict either a normal oscillatory behavior or a transient followed by system collapse, respectively.

III. PREDICTING CRISIS AND TRANSIENT CHAOS

In nonlinear and complex dynamical systems, a catastrophic collapse is often preceded by transient chaos. For example, in electrical power systems, voltage collapse can occur after the system enters into the state of transient chaos.[57] In ecology, slow parameter drift caused by environmental deterioration can induce a transition into transient chaos, after which species extinction follows.[58,59] A common route to transient chaos is the global bifurcation termed crisis,[3] at which a chaotic attractor is destroyed through collision with its own basin boundary, generating a chaotic transient.

A closely related problem is to determine if the system is already in a transient state—the question of "how do you know you are in a transient?." In nonlinear dynamics, this is a difficult question because the underlying system can be in a long transient in which all measurable physical quantities exhibit essentially the same behaviors as if the system were still in a sustained state with a chaotic attractor. Applying the traditional method of delay coordinate embedding[60] to such a case would yield estimates of dynamical invariants such as the Lyapunov exponents and fractal dimensions,[2,61] but it would give no indication that the system is already in a transient and so an eventual collapse is inevitable. Developing a predictive framework solely based on data, without relying on models, is valuable for addressing some current challenges in the real world.

The adaptable reservoir-computing scheme illustrated in Figs. 1(b) and 2 was applied to predicting crisis and transient chaos in a number of low-dimensional chaotic systems such as the logistic map and the Lorenz oscillator[26] as well as an electrical power model and a chaotic food chain.[25] It was also demonstrated[26] that this approach is capable of predicting critical transitions in situations where the sparse optimization approach to finding the governing equations fails. One example is the Ikeda–Hammel–Jones–Moloney optical cavity system described by a two-dimensional map.[62–64] For this classical and well-studied chaotic system,[65] the map equations have such a form that their power-series or Fourier expansions contain an infinite number of terms, which violates the sparsity condition. In fact, the Ikeda–Hammel–Jones–Moloney map is a convenient example used to explain the failure of the sparse optimization approach in an intuitive way. For data-driven prediction of critical transitions in such a system, machine learning is a viable approach. Indeed, it was demonstrated[26] that the critical transition of crisis can be anticipated by using the adaptable reservoir-computing scheme.

The classical Ikeda–Hammel–Jones–Moloney map describes the dynamics of a laser pulse propagating in a nonlinear cavity, which is described by[62–64]

$$z_{n+1} = \mu + \gamma z_n \exp\left(i\kappa - \frac{i\nu}{1+|z_n|^2}\right), \tag{1}$$

where z is a complex dynamical variable, the dimensionless laser input amplitude is a convenient bifurcation parameter that can be experimentally controlled, γ is the coefficient of the reflectivity of the partially reflecting mirrors of the cavity, κ is the laser empty cavity detuning, and ν measures the detuning due to the presence of a nonlinear medium in the cavity. As explained, the basic requirement of any sparse optimization technique for finding the system equations is *sparsity*: when the system equations are expanded into a power or a Fourier series, it must be that only a few terms are present so that the coefficient vectors to be determined from data are sparse.[9,66] At the present, a sparse representation of Eq. (1) is not available, rendering infeasible the data-driven approach of equation finding to predicting critical transitions.

In Ref. 26, the values of the system parameters in Eq. (1) were set as $\gamma = 0.9$, $\kappa = 0.4$, and $\nu = 6.0$, with μ being the bifurcation parameter. The system exhibits a boundary crisis[67] at $\mu = 1.0027$, as shown by the black vertical dashed line in Fig. 3(a). The dynamical behaviors for $\mu < \mu_c$ and $\mu > \mu_c$ are shown in Figs. 3(b) and 3(c), respectively. There is a chaotic attractor for $\mu < \mu_c$, and transient chaos leads to an escape of the system out of the previous operation region for $\mu > \mu_c$. For each selected value of μ, the training and validation lengths were $t_{train} = 800$ steps and $t_{validating} = 15$ steps, respectively. During validation, the adaptable reservoir-computing

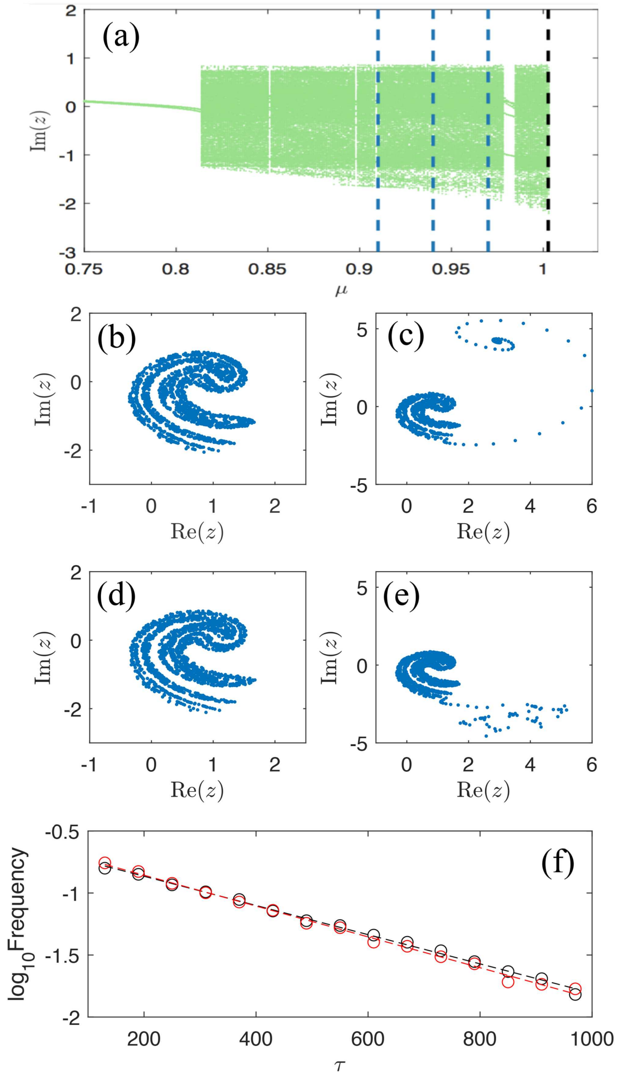

FIG. 3. Predicting crisis and transient chaos in a two-dimensional optical-cavity map—the classical Ikeda–Hammel–Jones–Moloney map. (a) A typical bifurcation diagram. The vertical black dashed line indicates the crisis point $\mu_c = 1.0027$. The three vertical blue dashed lines specify the three values of the bifurcation parameter μ used for training the adaptable reservoir-computing machine: $\mu = 0.91, 0.94, 0.97$. (b) and (c) Typical behaviors of the system in the sustained and transient chaos regimes, respectively, for $\mu = 0.99 < \mu_c$ and $\mu = 1.01 > \mu_c$. (d) and (e) Predicted dynamical behaviors by adaptable reservoir computing for the same values of μ as in (b) and (c), respectively. The machine predicts correctly the system collapse for $\mu > \mu_c$. (f) Predicted (red) exponential transient lifetime distributions for $\mu = \mu_c + 0.02$ and the ground truth (black). With permission from Kong *et al*., J. Phys. Complex. **2**, 035014 (2021). Copyright 2021, IOP Publishing.

scheme was able to predict the system evolution for more than five Lyapunov times with a small relative error.

To make the reservoir computer adaptable, training was conducted using time series from three values of the bifurcation parameter: $\mu = 0.91$, 0.94, and 0.97, as shown in Fig. 3(a) by the three vertical blue dashed lines. After training, a parameter change $\Delta\mu$ was applied and a test was conducted to determine if the machine could generate transient chaos. As shown by an exemplary pair of machine-generated attractors in Figs. 3(d) and 3(e), the reservoir computer can successfully generate transient chaos in the transient regime, even though it was trained only with time-series data from three chaotic attractors.

Note that, in Fig. 3(a), there is a periodic window. A pertinent question is whether a properly trained reservoir computing machine is able to predict periodic windows. In Ref. 26, the goal was to predict crises and transient chaos, where multiple time series from a limited number of distinct values of the bifurcation parameter, all within the regime of attracting chaos, were utilized to train the reservoir-computing machine, but the generation of the entire bifurcation diagram or the prediction of periodic windows was not addressed. It was demonstrated that a reservoir computer so trained is able to accurately reproduce the dynamical behavior within the transient chaos regime of the target system. Furthermore, the results indicate that predicting system collapse as triggered by parameter drifts leading the system into a transient regime is possible. It is worth noting that the training parameters can also be chosen from a periodic window in the bifurcation diagram of the system where there is transient chaos. In Ref. 27, it was reported that a single or parallel reservoir computer trained as a digital twin of the target system can generate its global bifurcation behaviors including periodic windows, which was demonstrated using a chaotic CO_2 laser system, a phytoplankton model with seasonal variations, and the Lorenz-96 climate network.

The statistical characteristics of the generated transient dynamics were demonstrated through a comparison of the transient lifetime distribution generated by the reservoir and the ground truth. For example, the input of the parameter channel of the adaptable reservoir computer was set to be $\mu = \mu_c^* + 0.02$, where μ_c^* is the critical point calculated from each realization of the reservoir machine. In total, 50 stochastic realizations of the adaptable reservoir computer and 400 random initial conditions for each realization were used, and the transient lifetimes of these 20 000 trials were recorded. The result shown in Fig. 3(f), where the distribution from RC (marked in red) is quite close to that of the real system with $\mu = \mu_c + 0.02$ (marked in blue), demonstrated the power of the adaptable reservoir-computing approach to predicting transient chaos.

We emphasize that the reason to use chaotic data to train the reservoir computer lies in making the neural network to "learn" the phase-space structure of the target system adequately, as a chaotic trajectory will cover a large portion of the phase space. A reservoir computer with a parameter channel, when trained using data from a small number of parameter values from the regime of sustained chaos, is capable of predicting the critical transition from sustained to transient chaos. This parameter channel enables the machine to sense, detect, and recognize the changes in the chaotic time-series data and to "teach" the neural network to establish an association between a specific value of the bifurcation parameter and the corresponding dynamical behavior. That is, training using data taken from different values of the parameter in the chaotic regime enables the neural network not only to learn the dynamical climate of the target system but also how the climate changes with the parameter. It is also worth noting that, in general, oscillatory time series can be used to train a reservoir computer for tasks such as anticipating amplitude death. For example, it was demonstrated that, using data from a small number of parameter values in the oscillatory regime, transition to amplitude death can be anticipated[54] (Sec. IV).

IV. PREDICTING AMPLITUDE DEATH OF NONLINEAR OSCILLATORS

In nonlinear dynamical systems, amplitude death is a phenomenon by which the oscillatory behaviors of the state variables halt suddenly and completely.[4,5] A typical route to amplitude death is the drift of a system parameter through a critical point at which a bifurcation from oscillations to a steady state occurs. In biological and physiological contexts, oscillations are essential to maintaining the normal functions of the system. For example, in biomedicine, normal physiological conditions are associated with oscillations, while the system's settling into a steady state is often viewed as the onset of pathological conditions or is associated with death. Because of the relevance of the phenomenon of amplitude death to physical, chemical, biological, and physiological systems,[68–72] it has been studied extensively in the past.

As amplitude death is undesired in real-world systems, it is of interest to be able to predict its occurrence while the underlying system is still in the regime of normal functioning. The formulation of the prediction problem is similar to that of predicting a crisis. More specifically, suppose a control or bifurcation parameter has been specified and the system is currently in the parameter regime in which the dynamical variables exhibit normal and "healthy" oscillations, where oscillatory time series from a number of parameter values in this regime have been measured. Suppose the bifurcation parameter begins to drift toward a regime that the system has never been in, i.e., no information is available about the system dynamics in the new parameter territory. For a given amount of parameter change, how can the occurrence of amplitude death be predicted with confidence? If the system equations are known, this prediction problem is trivial, as it can be solved by a simple computational bifurcation analysis. However, in real-world applications, often the only available information is the oscillatory time series collected when the system is in a healthy regime. The problem is challenging because of the requirement to predict the catastrophic behavior based on the presently accessible information, which indicates that the system should and would be completely normal by all measures. In fact, if one measures the dynamical variables of the system, the resulting time series are healthy in the sense that they all exhibit oscillations, giving no traceable sign that a catastrophic event such as amplitude death would occur upon some amount of parameter drift or a perturbation. Adaptable reservoir computing provided a viable solution to this problem.[54]

The following Stuart–Landau oscillator system was used to demonstrate how adaptable reservoir computing can be exploited to predict amplitude death,[54]

$$\begin{aligned} \dot{z}_1 &= \left(1 + i\omega_1 - |z_1|^2\right) z_1 + \varepsilon\,(z_2 - z_1), \\ \dot{z}_2 &= \left(1 + i\omega_2 - |z_2|^2\right) z_2 + \varepsilon\,(z_1 - z_2), \end{aligned} \tag{2}$$

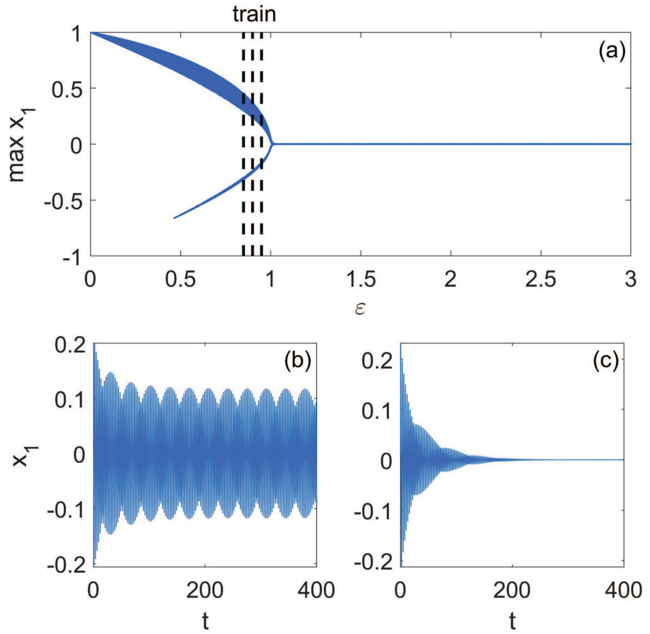

FIG. 4. Oscillations and amplitude death in the system of coupled Stuart–Landau oscillators. (a) A bifurcation diagram with the coupling parameter ε. The system is oscillatory for $0 < \varepsilon < 1$ and amplitude death occurs for $\varepsilon > 1$. (b) An oscillatory time series for $\varepsilon = 0.99$. (c) Amplitude death preceded by a short transient for $\varepsilon = 1.02$. Other parameter values are $\omega_1 = 2.0$ and $\omega_2 = 7.0$. With permission from Xiao *et al.*, Phys. Rev. E **104**, 014205 (2021). Copyright 2021, APS.

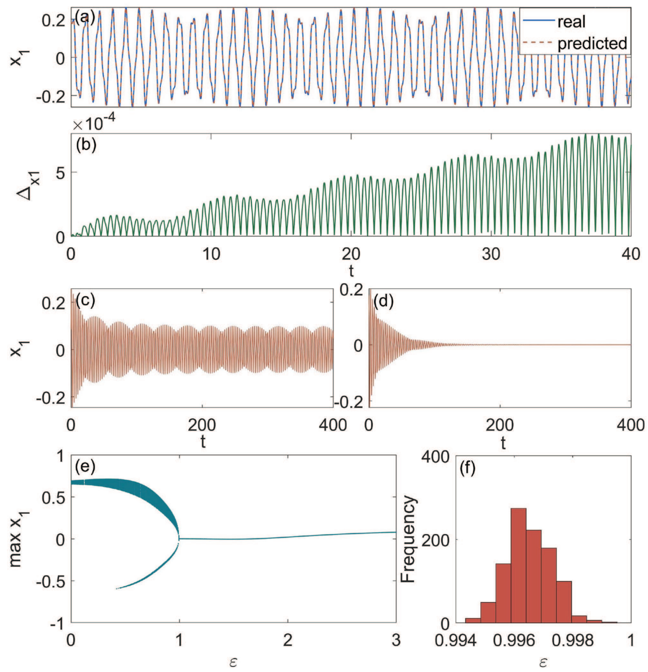

FIG. 5. Predicting amplitude death in the system of coupled Stuart–Landau oscillators. (a) Predicted and real time series for $\varepsilon = 0.95$. (b) The difference between the predicted and real time series for $\varepsilon = 0.95$. (c) Predicted state of oscillation for $\varepsilon = 0.99$. (d) Predicted amplitude death preceded by a transient for $\varepsilon = 1.02$. (e) Predicted bifurcation diagram. (f) The distribution of the predicted transition point to amplitude death from 1000 random realizations of reservoir machine and initial conditions. With permission from Xiao *et al.*, Phys. Rev. E **104**, 014205 (2021). Copyright 2021, APS.

where $z_i = x_i + iy_i$ $(i = 1, 2)$ are complex variables and ω_1 and ω_2 are parameters. Without coupling, i.e., $\varepsilon = 0$, both oscillators have an unstable fixed point at $z^*_{1,2} = 0$. A previous study[73] revealed that amplitude death occurs for $\varepsilon > 1$ and $\Delta\omega > 2\sqrt{2\varepsilon - 1}$, where $\Delta\omega \equiv |\omega_1 - \omega_2|$ represents the mismatch between the two oscillators. Depending on the amount of mismatch, the system dynamics can be quite different. Figure 4(a) shows a bifurcation diagram of the dynamical variable x_1. It can be seen that the system is in an oscillatory state for $\varepsilon \in (0, 1)$, and amplitude death occurs for $\varepsilon > 1$. Figures 4(b) and 4(c) show an oscillatory time series for $\varepsilon = 0.99$ and amplitude death (with a short transient) for $\varepsilon = 1.02$, respectively.

The adaptable reservoir computer was trained[54] for three values of the bifurcation parameter: $\varepsilon = 0.85, 0.9$, and 0.95, as indicated by the three vertical dashed lines in Fig. 4(a). Figures 5(a) and 5(b) show an example of the predicted time series $x_1(t)$ and the difference between the predicted time series and the ground truth, respectively. The reservoir computer predicts correctly that the system is in an oscillatory state for $\varepsilon < 1$, as exemplified in Fig. 5(c) for $\varepsilon = 0.99$. For $\varepsilon > 1$, the machine predicts successfully amplitude death, as demonstrated in Fig. 5(d) for $\varepsilon = 1.02$. Figure 5(e) shows a machine predicted bifurcation diagram, which agrees with the real diagram in Fig. 4(a). Figure 5(f) shows a distribution of the predicted transition point from 1000 random realizations of the reservoir machine, where all predictions are close to the true transition point $\varepsilon^* = 1$.

In general, having data from parameter values close to the critical point can lead to higher prediction accuracies. As the training parameter values move away from the critical point, prediction error will increase.[27] The proximity of parameter selection to the critical value depends on the specific characteristics of the system and the prediction task at hand. In real-world applications, it can happen that the parameter values from which observations are taken cannot be freely chosen. In this case, large prediction errors may arise.

V. DIGITAL TWINS OF NONLINEAR DYNAMICAL SYSTEMS

Digital twins are virtual replicas of real systems, exhibiting the dynamic attributes, appearance, and behavior of their real counterparts. The significance of digital twins in nonlinear dynamics can be appreciated through the main question addressed in this Perspective: how can a critical transition be anticipated from measured time series collected from the system when it is in a "healthy" operating regime? It is essentially a model-free prediction task, which assumes that the only accessible information about the system is measured time series and the system equations are unknown. Here, a healthy system is referred to as "normal" functioning of a system in the sense that the dynamical variable has a finite mean value, in spite of the statistical fluctuations (e.g., an ecosystem). The fundamental question revolves around determining, at a point where all measurements or observations show no signs of any "abnormal" system behavior, how one can distinguish whether the system will operate in its functional state or undergo a catastrophic collapse.

Developing digital twins has diverse applications in simulation, integration, testing, monitoring, and maintenance.[74] There is a growing interest in digital twins across various areas such as scientific research, engineering, industry, healthcare, and defense systems.[75] The inception of digital twins can be traced back to their initial application in predicting the structural lifespan of aircraft.[76] In the realm of medicine and healthcare, digital twins are used to provide patient-specific treatment[77] or have been used to provide a qualitatively new Earth system to facilitate the green transition.[78] Agriculture is another field that can benefit from digital twins as they are able to provide real-time monitoring and optimizing crop growth, track soil conditions, and manage irrigation.[79] In dynamical systems, digital twins can be used to forecast future states and preempt emergent, potentially catastrophic behaviors.[27,80] Here, we provide a brief overview of the machine-learning approaches to digital twins tailored to nonlinear dynamical systems.

A basic requirement for a digital twin of a nonlinear dynamical system is to accurately generate the bifurcation behaviors. This entails not only capturing the original system's "dynamical long-term statistical properties" but also faithfully reflecting how these long-term statistical properties change with variations in bifurcation or control parameters. Adaptable reservoir computing[25,45] has been used to address this problem. As an intuitive illustration, we use the one-dimensional logistic map,[81]

$$x(n+1) = Ax(n)\big(1 - x(n)\big), \tag{3}$$

where $0 \le x(n) \le 1$ is the dynamical variable in discrete time and $A > 0$ is the bifurcation parameter. For $A \in [3.5\ 4]$, the system exhibits chaotic behavior, as exemplified in Fig. 6(a).

We train the digital twin with time series from four values of A, all in the chaotic regime: $A = 3.8, 3.85, 3.9$, and 3.95. The size of the random reservoir network is 400. For each value of A in the training set, the training and validation lengths are $t = 1250$ and $t = 15$, respectively, where the latter corresponds to approximately five Lyapunov times. The warming-up length is $t = 20$. The results exemplified in Fig. 6 indicate that adaptable reservoir computing provides an effective approach to creating a digital twin of the logistic map.

A high-dimensional example is the Lorenz-96 climate system, which is described by m coupled first-order nonlinear differential equations under a sinusoidal driving force $f(t)$,

$$\dot{x}_i = x_{i-1}\big(x_{i+1} - x_{i-2}\big) - x_i + f(t), \tag{4}$$

where $i = 1, \ldots, m$ is the spatial index. Under the periodic boundary condition, the m nodes constitute a ring network, where each node is coupled to three neighboring nodes. For $m = 6$ and $f(t) = Asin(\omega t) + F$ with $\omega = 2$ and $F = 2$, a digital twin was trained[27] with time series from four values of A, all in the chaotic regime: $A = 2.2, 2.6, 3.0$, and 3.4.

Figures 7(a1) and 7(a2) exemplify chaotic and quasiperiodic dynamics of the Lorenz-96 system for $A = 2.2$ and $A = 1.6$, respectively. Figures 7(b1) and 7(b2) show the corresponding dynamical

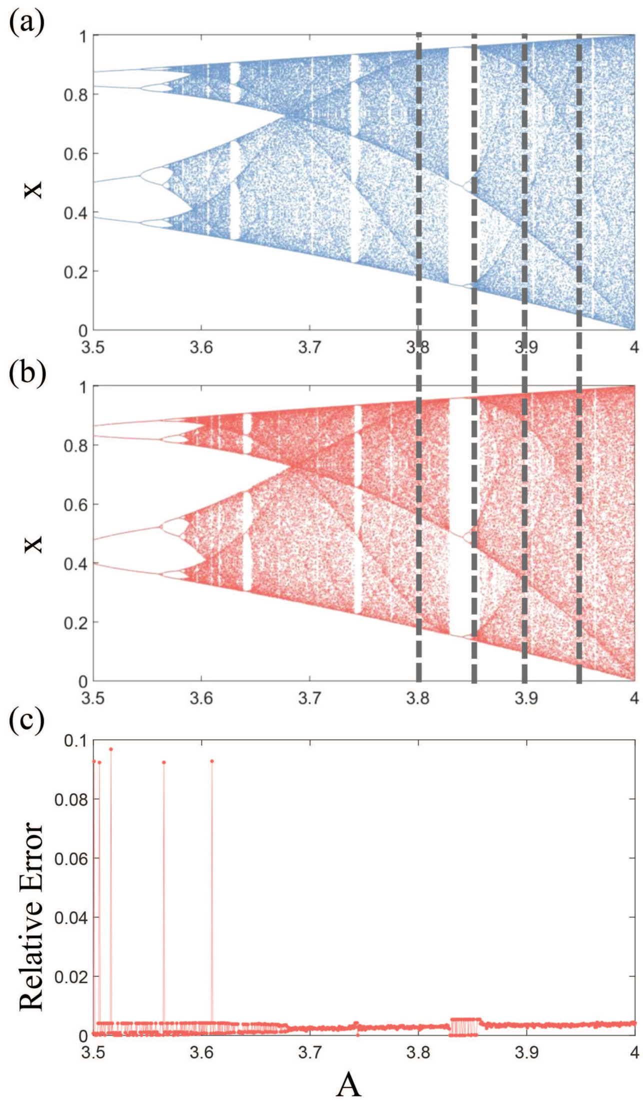

FIG. 6. Performance of an adaptable reservoir-computer based digital twin of the classical logistic map in terms of the bifurcation diagram. (a) and (b) True and digital-twin generated bifurcation diagrams, where the four vertical black dashed lines indicate the values of driving amplitudes $A = 3.8$, 3.85, 3.9, and 3.95, from which the training time series data are obtained. The excellent agreement between the two bifurcation diagrams attests to the capability of the digital twin in accurately replicating the dynamic behaviors of the chaotic logistic map in a wide parameter range, despite being trained only on data from a small number of parameter values in the chaotic regime. (c) The relative error between the two bifurcation diagrams.

behaviors generated by the digital twin. Figures 7(c) and 7(d) present the ground truth and digital-twin generated bifurcations, respectively, with their relative error shown in Fig. 7(e). The digital twin not only accurately reproduces the expected dynamical behavior

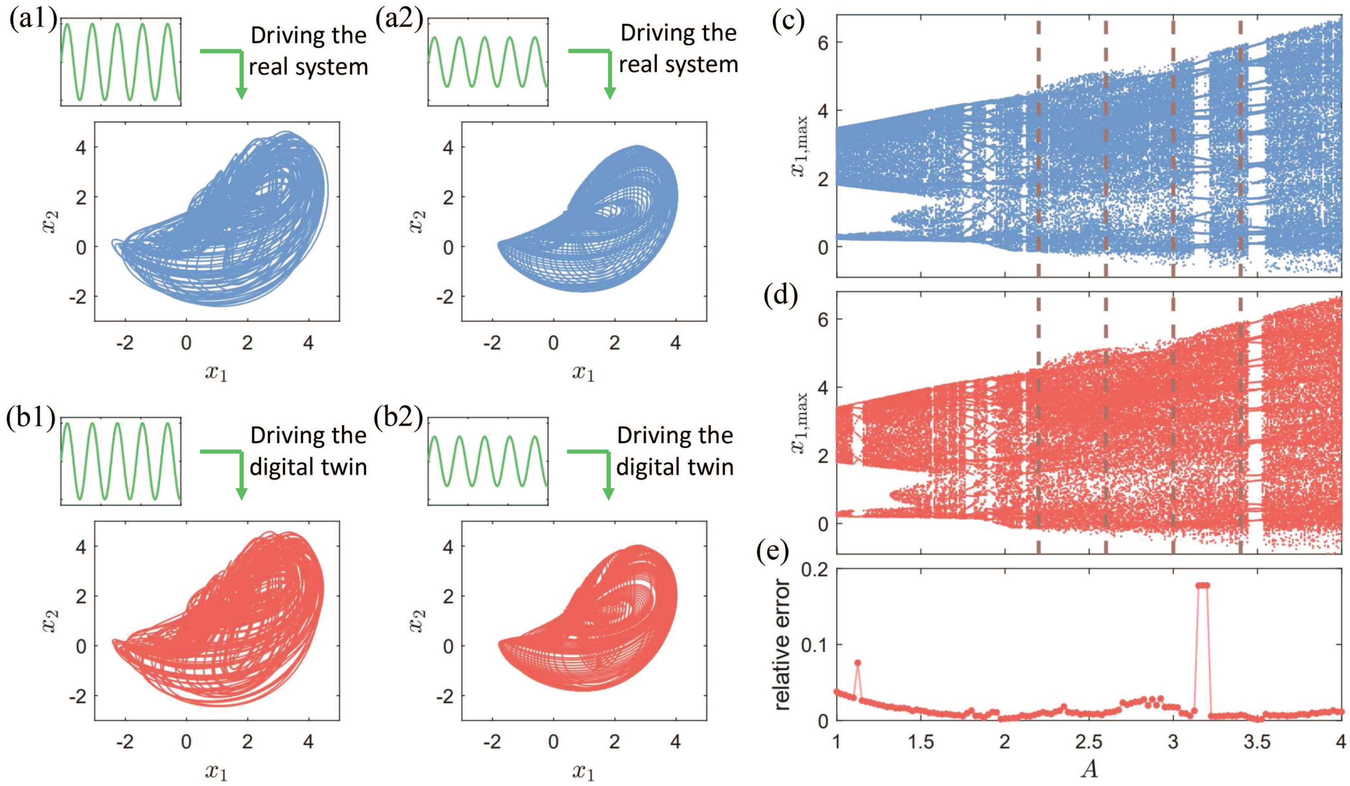

FIG. 7. Digital twin for the Lorenz-96 climate system. (a1 and a2) Ground truth simulated chaotic and quasiperiodic dynamics in the system for $A = 2.2$ and $A = 1.6$, respectively. The periodic forcing signals $f(t)$ are illustrated in green. (b1 and b2) The corresponding dynamics of the digital twin under the same driving signal $f(t)$. Training of the digital twin is conducted using time series from the chaotic regime. (c) and (d) True and digital-twin generated bifurcation diagrams, where the four vertical red dashed lines indicate the values of the driving amplitude A from which the training time series data are obtained. The agreement between the two bifurcation diagrams attests to the ability of the digital twin to reproduce the distinct dynamical behaviors of the target climate system in different parameter regimes, even with training data only in the chaotic regime. (e) Relative error of the spanned regions. With permission from Kong *et al.*, Chaos 33, 033111 (2023). Copyright 2023, AIP Publishing.

within the chaotic regime where it was trained but also has the ability to predict the correct dynamics beyond the trained parameter range.

VI. DISCUSSION AND FUTURE PERSPECTIVE

Data-driven prediction of critical transitions in nonlinear dynamical systems involves two main approaches: mathematical modeling and machine learning. The former relies on finding an approximate set of the governing equations, but it is effective only when underlying equations have a simple mathematical structure in the sense of sparsity: they contain a small number of power-series or Fourier-series terms only. The machine-learning approach is generally applicable, as it relies on a high-dimensional recurrent neural network to capture the dynamical climate of the target system, regardless of how complex the governing equations might be.

Some basic considerations in developing an effective machine-learning scheme to predict critical transitions in dynamical systems are as follows.

The first is the available data for training. Naively, one may attempt to regard machine-learning prediction of critical transitions as a traditional binary classification problem in computer science: with or without such a transition. To solve a classification problem, data from both below and above the critical point are required for training the neural network. However, from a practical perspective, this is an ill-defined problem: if a critical transition had occurred, the system would have collapsed and no measurement data could have been available anymore. Because of this difficulty, predicting critical transitions is physically meaningful only if it is done using data collected while the system is still normally functioning. Developing any machine-learning approach must take into account this realistic constraint.

The second factor is that the neural network architecture should be capable of self-dynamical evolution, i.e., it can evolve without external input or driving, as the target system is, in general, a self-evolving dynamical system with memories: a small perturbation to the initial condition can lead to a large change in the system state after a long time, as in a chaotic system. In addition, the complexity of the neural network must surpass that of the target system. These considerations lead to the choice of recurrent neural-network architecture, e.g., reservoir computing that is fully capable of self-dynamical evolution after training.

The third consideration is that the neural machine needs to be trained to possess the ability to generate parameter-dependent dynamical evolution. Since all the available training data are from the pre-critical regime of the target system, in order to predict the critical transition, the neural machine should be trained to possess the same intrinsic dynamical climate of the target system, i.e., producing distinct dynamics for different values of the parameter. As described, this challenge can be met by making the neural network "parameter aware," and a straightforward way to realize this is to design an input parameter channel: the value of the parameter is fed to all neurons in the network simultaneously with the time-series data from this parameter value. Existing works[25,27] demonstrated that, with such a combined input configuration of both the time series data and the parameter value, even if the number of distinct combinations is small (e.g., as few as three), the neural machine is capable of learning the dynamical climate of the target system. A theoretical understanding of this remarkable phenomenon has not been available.

The recent adaptable reservoir-computing machine was articulated according to the three ingredients.[25,27] A distinct feature, as justified above, is an additional input channel to accommodate the value of the bifurcation parameter as a special label for the time-series data that the machine is being trained on. Through such parameter-dependent training with data from a few distinct values of the bifurcation parameter in the normal functioning regime of the target system, the reservoir-computing machine establishes a "regression" between the dynamical behavior of the target system and the bifurcation parameter, enabling the machine to capture the "long-term statistical properties" of the dynamics of the target system and making accurate prediction outside the training parameter regime possible. While training is conducted with data from a few distinct values of the bifurcation parameter, individual prediction from a single reservoir can have large errors. However, the average behavior of a large number of independent reservoir machines, trained using the same set of data, usually produces quite accurate predictions. Not only can the critical transition point be predicted, but the exponential distribution of transient lifetimes beyond the transition, a key characteristic of transient chaos, can also be accurately assessed. In fact, the scaling relation between the average lifetime and the parameter difference from the critical point can be predicted by adaptable reservoir computing, even in the presence of observational noise. It is worth noting that the scaling law holds in the post-critical regime of transient chaos, to which the neural machine has not been previously exposed. The ability of the machine-learning scheme to reproduce such scaling relations in unseen conditions underscores its robustness and utility in predicting statistical characteristics of transient chaos.

A properly trained adaptable reservoir computer is in fact a digital twin of the target system: not only can it predict critical transitions, but a global bifurcation diagram can be reproduced. It was demonstrated that the reservoir-computing based digital twin can faithfully reproduce the dynamics of the target system in a wide parameter interval, even with training data from a small number of distinct parameter values. The digital twin is particularly effective for low-dimensional nonlinear dynamical systems[27] as well as for high-dimensional spatiotemporal coupled systems with a simple parameter structure, e.g., the Lorenz-96 climate model. Here, the term "simple" means that the bifurcation parameter is additive and uniform for all the spatial sites involved. Note that the Lorenz-96 system belongs to the broad class of systems under an external, time-dependent forcing. For such systems, the digital twin can be quite effective in anticipating critical transitions or regime shifts in the target system, providing early warnings of a potential catastrophic collapse. In scenarios where direct measurements of the target system are impractical or costly, the digital twin offers a means to assess the dynamical evolution of the system. The privilege of digital twins also extends to various qualitative capabilities such as providing long-term continual forecasts for nonlinear dynamical systems subject to nonstationary external driving, even with sparse state updates.[27] Furthermore, it can extrapolate the presence of hidden variables within the system and reproduce/predict their dynamical evolution. The digital twin is adaptable to external driving with different waveforms, and it can extend predictions of global bifurcation behaviors to systems of varying sizes. This versatility underscores the potential of digital twins based on adaptable reservoir computing in diverse applications involving dynamical systems under changing external conditions.

For adaptable reservoir computing, a challenge is to construct digital twins for spatiotemporal dynamical systems with a somewhat sophisticated parameter structure. For example, consider the Kuramoto–Sivashinsky equation: $u_t + au_{xxxx} + b(u_{xx} + uu_x) = 0$, where $u(x, t)$ is a scalar field and a and b are two parameters. Unlike the forcing parameter in the Lorenz-96 model, both parameters here are not additive to the equation but are multiplicative as they are directly associated with terms that involve the field and its derivatives. To our knowledge, efforts in exploiting adaptable reservoir computing to create digital twins for such spatiotemporal dynamical systems have not been successful.

The critical transitions discussed in this Perspective have one common feature: before the critical point is reached, the system exhibits oscillatory behaviors, chaotic or periodic. This feature facilitates training of the reservoir computer. There exists an important class of critical transitions for which oscillatory time series data are not available: a tipping point.[82–107] The standard definition of a tipping point is a transition from a normal or healthy stable steady state to another that can often be catastrophic. For example, in ecological systems, the normal stable steady state or fixed point corresponds to coexistence of species with healthy abundance, and a tipping point means the transition to a massive extinction state—another fixed point of the system. Conceivably, adaptable reservoir computing can be used to anticipate a tipping point, but a challenge is that the available time-series data for training are typically not oscillatory. A possible solution is exploiting dynamic noise in the system, which generates time series with random fluctuations about the deterministic steady-state value. While noise can potentially degrade the quality of prediction, it can benefit the training process by enabling the neural network to explore a larger region of the phase space, potentially revealing hidden features or dynamics that might be obscured in noise-free conditions. In fact, dynamical noise and/or measurement noise in the data can benefit the training process, sometimes significantly, through a stochastic-resonance type of mechanism.[108] Moreover, an optimal level of noise can prevent overfitting and promote generalization, allowing the reservoir computer to adapt to varying environmental conditions and data distributions.

Considerable efforts were devoted to anticipating tipping by identifying early warning indicators or signals.[83,109–113] As a bifurcation parameter changes so that the system approaches a tipping point, the dynamics are governed by a few normal modes that define the qualitative aspects of the new state. Recently, a deep-learning algorithm was developed, which was capable of providing early warning signals, classifying bifurcations, predicting the tipping point, and identifying the normal form and scaling behavior of the dynamics near the tipping point.[112] Subsequently, a deep learning classifier was developed to provide an early warning signal for discrete-time bifurcations and abrupt changes in dynamics cardiology, ecology, and economic systems.[114] Deep learning has also been used to predict critical transitions such as bifurcations, tipping points, and hysteresis in ecological systems,[115] where various machine-learning methods such as recurrent neural networks, transformers, and encoder-decoder were discussed. Without being parameter adaptable, these methods were not effective in predicting qualitative shifts in dynamic behavior in critical-transition scenarios. The findings of this work underscore the significance of dynamical learning and control parameter channels in enhancing the performance of conventional machine-learning methods to generate diverse, previously unseen dynamical behaviors within the target system.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Shirin Panahi: Conceptualization (supporting); Data curation (lead); Investigation (lead); Methodology (lead); Writing – original draft (equal). **Ying-Cheng Lai:** Conceptualization (lead); Funding acquisition (lead); Methodology (equal); Project administration (lead); Writing – original draft (equal); Writing – review & editing (lead).

DATA AVAILABILITY

The data and relevant computer codes that support the findings of this study are available from the corresponding author upon reasonable request.

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