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Ling-Wei Kong ; Ying-Cheng Lai

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Ling-Wei Kong[1] and Ying-Cheng Lai[1,2,a)]

AFFILIATIONS

[1]School of Electrical, Computer and Energy Engineering, Arizona State University, Tempe, Arizona 85287, USA
[2]Department of Physics, Arizona State University, Tempe, Arizona 85287, USA

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[a)]**Author to whom correspondence should be addressed:** ylai1@asu.edu

ABSTRACT

In the classic Kuramoto system of coupled two-dimensional rotators, chimera states characterized by the coexistence of synchronous and asynchronous groups of oscillators are long-lived because the average lifetime of these states increases exponentially with the system size. Recently, it was discovered that, when the rotators in the Kuramoto model are three-dimensional, the chimera states become short-lived in the sense that their lifetime scales with only the logarithm of the dimension-augmenting perturbation. We introduce transverse-stability analysis to understand the short-lived chimera states. In particular, on the unit sphere representing three-dimensional (3D) rotations, the long-lived chimera states in the classic Kuramoto system occur on the equator, to which latitudinal perturbations that make the rotations 3D are transverse. We demonstrate that the largest transverse Lyapunov exponent calculated with respect to these long-lived chimera states is typically positive, making them short-lived. The transverse-stability analysis turns the previous numerical scaling law of the transient lifetime into an exact formula: the "free" proportional constant in the original scaling law can now be precisely determined in terms of the largest transverse Lyapunov exponent. Our analysis reinforces the speculation that in physical systems, chimera states can be short-lived as they are vulnerable to any perturbations that have a component transverse to the invariant subspace in which they live.

In spatiotemporal nonlinear dynamical systems possessing a high degree of symmetry, such as a system of coupled, completely identical oscillators with translational symmetries, spontaneous symmetry breaking can lead to a class of dynamical states characterized by the simultaneous coexistence of ordered (coherent) and disordered (incoherent or random) groups of oscillators. Such states were first discovered by Umberger, Grebogi, Ott, and Afeyan more than three decades ago in their numerical simulations of the system of coupled Duffing oscillators. Approximately a decade later, Kuramoto and Battogtokh found these states in the celebrated Kuramoto model of globally coupled two-dimensional (2D) rotators and was subsequently mathematically analyzed and coined the name "chimera" by Abrams and Strogatz. An analytic theory for the dynamical behaviors associated with the two interacting populations of oscillators (effectively a chimera state) was developed by Montbrió, Kurths, and Blasius in 2004. Since then, there has been extensive research on chimera states. By its very nature, a chimera state must be transient as the mutual interactions between the coherent and incoherent groups of oscillators will eventually destroy their coexistence. In the classical Kuramoto model of coupled, strictly 2D rotators, the lifetime of the chimera states tends to increase exponentially with the system size; therefore, for reasonably large systems, these states can be regarded as sustained or long-lived in any practical sense. Nevertheless, certain types of perturbations, such as noise or an augmentation in the dimension of the oscillators (e.g., a perturbation rendering the rotators in the Kuramoto model 3D), can turn the long-lived chimera states into short-lived in the sense that their lifetime scales only with the logarithm of the magnitude of the perturbation. This article introduces transverse-stability analysis to understand the dynamical mechanism leading to the short-lived chimera states. Detailed analysis of the generalized Kuramoto model of coupled 3D rotators reveals that the long-lived chimera states in the classical Kuramoto model of 2D rotators occur in an invariant subspace of the full phase space, and they are typically unstable with respect to perturbations that are transverse to the subspace. The analysis turns the previous numerical scaling law governing the average transient lifetime of the short-lived chimera states with respect to the perturbation into an exact formula, with the previously undetermined

proportional scaling constant precisely identified, thereby leading to a deep understanding of the dynamical origin of the short-lived chimera states.

I. INTRODUCTION

Approximately 35 years ago, Umberger, Grebogi, Ott, and Afeyan at the University of Maryland began to simulate a class of spatiotemporal dynamical systems described by a chain of coupled nonlinear Duffing oscillators.[1] What they found was that, even when all the oscillators are identical, they can exhibit characteristically distinct collective behaviors. In particular, it was observed that the collective motion of a group of oscillators can be quite coherent, while the complementary group can exhibit incoherent collective dynamics. A system of coupled identical nonlinear oscillators possesses a high degree of symmetry; therefore, in a general sense, the emergence of the coexistence of a dynamically coherent group and an incoherent group of oscillators is characteristic of spontaneous symmetry breaking that is responsible for the occurrence of a vast variety of observable phenomena in the physical world, most notably in condensed matter physics. Thirteen years later, a similar phenomenon was uncovered[2] in the classical Kuramoto model of coupled, completely identical two-dimensional rotators: the coexistence of two groups of oscillators: one exhibiting synchronous (regular) and another of asynchronous (irregular or random) motions. Subsequently, in 2004, the phenomenon was mathematically analyzed and given the name "chimera."[3,4] About the same time, the problem of synchronization of two interacting populations of oscillators was studied by Montbrió, Kurths, and Blasius,[5] where the setting is precisely one producing chimera states and the authors developed comprehensive analytic theory to uncover and understand various dynamical behaviors, such as different types of bistability, higher-order entrainment, and the existence of states with unconventional stability properties. (A similar treatment was published four years later[6].) Since then, the subject of chimera states has become an active area of research in nonlinear dynamics and complex systems.[7–63]

Intuitively, a chimera state cannot last forever due to the mutual interactions among the oscillators. To see this, note that after a chimera state has emerged, the groups of coherent and incoherent oscillators continue to interact with each other due to the coupling among the oscillators from the two groups. Two distinct scenarios can arise. First, while the coupling originated from the oscillators in the coherent group to those in the incoherent group is regular, the coupling from the latter to the former has a random component. Subject to continuous random perturbations, the oscillators in the coherent group will gradually become incoherent, and there will come a time after which all oscillators become incoherent, destroying the chimera state. The second scenario is somewhat opposite to the first where, due to the regular coupling from the coherent oscillators, the incoherent oscillators will gradually become more coherent, leading to full coherence among all the oscillators in the system and making the chimera state disappear. In either scenario, a chimera state cannot sustain indefinitely, implying the fundamentally transient nature of the chimera states. Indeed, it was demonstrated that the chimera states in the Kuramoto model are typically transient.[15] Of physical interest is then how long a chimera state can last. In the same work,[15] it was found that the average lifetime of the chimera states follows an exponential scaling law with an exponent that increases with the system size; therefore, for any reasonably large systems, these states can sustain for such a long time that they can be regarded as "permanent" in any practical sense. We note that it had been known for a long time that transient behaviors characterized by a combined exponential-algebraic scaling law occur in other contexts of dynamical systems,[64] e.g., "superpersistent" chaotic transients.[65–69]

A physical issue, thus, concerns about the robustness of chimera states against external perturbations. Depending on the type of perturbations, two cases can arise: a chimera state can be robust or fragile; for example, in the standard setting of the classic Kuramoto model of identical coupled 2D rotators where the chimera states are practically permanent, it was found previously that the states are robust against perturbations to the structure of the underlying lattice or networks.[26,56,70] In an all-to-all coupled network, the chimera states can persist even when there is substantial link removal.[26] The phenomenon of self-organization and adaptation of chimera states was also uncovered,[60] providing a dynamical mechanism for these states to survive in response to perturbations. In contrast, perturbations of a different nature can make the chimera states fragile. For example, noise can significantly reduce the average lifetime of the chimera states.[71] More recently, it was found[61] that disturbing the 2D nature of the individual Kuramoto rotators can have a devastating effect on the chimera states. In particular, note that in the classic Kuramoto model, each uncoupled oscillator is described by a single dynamical variable: the rotational or phase angle characterizing the rotation on a unit circle in the plane. Now, imagine a 3D rotator characterized as a point moving on the surface of a unit sphere, where two independent angle variables are needed to describe the rotator and the unperturbed phase oscillators correspond to 2D rotation confined to the equator. It was demonstrated that arbitrarily weak dimension-augmenting perturbations making the phase oscillators 3D can drastically reduce the average lifetime of the chimera states.[61] In particular, a scaling law was uncovered, where the lifetime scales only with the logarithm of the magnitude of the perturbation, meaning that reducing the perturbation strength by many orders of magnitude will only lead to an incremental increase in the lifetime of the chimera states. The previous findings,[61,71] thus, indicated that, when the classical Kuramoto system is subject to certain noise or dimension-augmenting perturbations, the originally long-lived chimera states effectively become *short-lived*.

In this paper, we articulate a general framework to elucidate the dynamical mechanism leading to short-lived chimera states. To be concrete, we consider a networked system of N identical phase-coupled oscillators, where each oscillator is described by two independent dynamical variables so that the phase-space dimension of the whole system is $2N$. Assume that the system has a symmetry, e.g., a reflection symmetry, which generates an N-dimensional dynamically invariant subspace. That is, any initial condition taken from this subspace leads to dynamical state evolution or trajectories confined to the same subspace. Now, assume that long-lived chimera states can arise in the invariant subspace, which occur, e.g., when the dynamics in the subspace are described by the classical Kuramoto model of 2D rotators. Consider perturbations that are transverse to the invariant subspace. If a chimera state in the invariant subspace is unstable in response to such a perturbation so that the dynamical trajectory "escapes" away from the subspace, the "chimera" nature of the state can be quickly destroyed in the sense that its lifetime scales with the logarithm of the perturbation strength. The largest transverse Lyapunov exponent for the chimera state can then be defined, and its positiveness makes the state fragile and short-lived. As a concrete physical example, we consider a generalized model of N coupled 3D rotators, demonstrate the existence of an N-dimensional invariant subspace that hosts the classical Kuramoto dynamics, and demonstrate the general positiveness of the transverse Lyapunov exponent. These results provide a setting that allows the phenomenon of short-lived chimera states and their lifetime to be understood in a more satisfactory manner. In particular, the transverse stability analysis has turned the previous numerical scaling law for the average lifetime of the short-lived chimera states into an exact formula, with the proportional scaling constant determined by the largest transverse Lyapunov exponent.

II. EMERGENCE OF SHORT-LIVED CHIMERA STATES: A GENERAL SETTING

The paradigmatic system exhibiting chimera states, the classical Kuramoto model of N identical 2D rotators coupled on a circle, has a translational symmetry: the system is invariant upon any spatial displacement that is the integer multiple of the distance between two adjacent oscillators. Chimera states are the result of spontaneous breaking of this translational symmetry. To study the transverse stability of chimera states, it is necessary to augment the phase-space dimension of the system. This can be done by designating an invariant subspace that hosts the classical N-dimensional Kuramoto system with chimera states and introducing a subspace transverse to the invariant subspace. In particular, let θ_i ($i = 1, \dots, N$) be the set of angle variables describing the 2D rotators in the Kuramoto system. The dynamical state in the invariant subspace can be denoted as an N-dimensional column vector,

$$\mathbf{X} \equiv (\theta_1, \theta_2, \dots, \theta_N)^T, \tag{1}$$

where $\mathbf{X} \in \mathcal{R}^N$ and $()^T$ denotes the transpose. The system equation governing the evolution of $\mathbf{X}$ is

$$d\mathbf{X}/dt = \mathbf{F}(\mathbf{X}), \tag{2}$$

where $\mathbf{F}(\mathbf{X})$ denotes the nonlinear vector field. A transverse subspace can be introduced into the system, e.g., by imposing a reflection symmetry with respect to the invariant subspace. In particular, let $\mathbf{Y} \in \mathcal{R}^N$ be the set of dynamical variables in the transverse subspace, which can be, e.g., a set of N independent angle variables, and the invariant subspace is defined as $\mathbf{Y} = \mathbf{0}$. The dimension of the full phase space is now $2N$, and the system equations are

$$d\mathbf{X}/dt = \mathbf{F}(\mathbf{X}) + \text{high order terms of } \mathbf{Y}, \tag{3}$$

$$d\mathbf{Y}/dt = \mathbf{G}(\mathbf{X}, \mathbf{Y}), \tag{4}$$

where $\mathbf{G}(\mathbf{X}, \mathbf{Y})$ is a nonlinear vector function of both $\mathbf{X}$ and $\mathbf{Y}$. Since $\mathbf{Y} = \mathbf{0}$ is the invariant subspace, $\mathbf{G}(\mathbf{X}, \mathbf{Y})$ must satisfy

$$\mathbf{G}(\mathbf{X}, \mathbf{0}) = \mathbf{0}.$$

Assume $\mathbf{X}$ exhibits a chimera state. Its transverse stability can be determined by solving the following linear variational equation governing the evolution of the infinitesimal transverse vector $\delta\mathbf{Y}$,

$$d\delta\mathbf{Y}/dt = \mathcal{J}(\mathbf{X}, \mathbf{Y} = \mathbf{0}) \cdot \delta\mathbf{Y}, \tag{5}$$

where $\mathcal{J}(\mathbf{X}, \mathbf{Y} = \mathbf{0})$ is the Jacobian matrix defined as

$$\mathcal{J}(\mathbf{X}, \mathbf{Y} = \mathbf{0}) \equiv \frac{\partial \mathbf{G}}{\partial \mathbf{Y}}|_{\mathbf{Y}=\mathbf{0}}. \tag{6}$$

Note that the partial derivatives in Eq. (5) are evaluated at the invariant subspace $\mathbf{Y} = \mathbf{0}$. In addition, since the Jacobian matrix $\mathcal{J}(\mathbf{X}, \mathbf{Y} = \mathbf{0})$ contains $\mathbf{X}$, it is necessary to integrate Eqs. (2) and (5) together to obtain $\delta\mathbf{Y}(t)$.

The exponential growth rate of the length of the vector $\delta\mathbf{Y}(t)$ is the largest transverse Lyapunov exponent, denoted as $\Lambda_\perp$. For $\Lambda_\perp < 0$, the chimera state in the invariant subspace is transversely stable, and it will be a permissible state of the whole system because $\mathbf{Y} = \mathbf{0}$ remains to be a solution of Eqs. (3) and (4). However, if $\Lambda_\perp$ is positive, the chimera state in the invariant subspace is transversely unstable. In this case, $\mathbf{Y} = \mathbf{0}$, and thus, the chimera state in the invariant subspace is no longer a solution of the whole system. The original chimera state is destroyed.

From a practical point of view, if a chimera state is transversely unstable, it will become short-lived, which can be understood, as follows. Suppose the system is initialized in the chimera state in the invariant subspace, and an infinitesimally small perturbation of magnitude δ_0 is applied to the state in the transverse direction. The perturbation will grow as $\delta(t) = \delta_0 e^{\Lambda_\perp t}$, as schematically illustrated in Fig. 1. If $\delta(t)$ is still infinitesimal, the observable state of the system will remain approximately a chimera state. Let ε be a small threshold in $\delta(t)$ beyond which the state of the system can no longer be considered being confined to the vicinity of the invariant subspace. The time it takes for this to occur is proportional to $|\ln \delta_0|$. On average (with respect to an ensemble of initial conditions in the invariant subspace), the lifetime of the chimera state is given by

$$\langle \tau \rangle = \frac{C(\epsilon) - \ln \delta_0}{\Lambda_\perp}, \tag{7}$$

where $C(\epsilon)$ is a constant determined by an empirical threshold ϵ to distinguish between a chimera and a non-chimera state. The observation is that, even if the transverse perturbation δ_0 is reduced by many orders of magnitude, the average lifetime of the chimera state will be prolonged only incrementally. In this sense, the chimera state is short-lived.

It should be noted that the concept of transverse Lyapunov exponents is fundamental to a number of phenomena in nonlinear dynamics, such as riddling[72,73] and bubbling[74] bifurcations, synchronization[75] (where the largest transverse Lyapunov exponent is called the master-stability function[76,77]), and on–off intermittency.[78–81]

III. SHORT-LIVED CHIMERA STATES IN A 3D KURAMOTO MODEL

A. Kuramoto network of coupled 3D rotators and calculation of the transverse Lyapunov exponent

In Ref. 82, a high-dimensional Kuramoto model of N coupled oscillators was introduced, with the differential equation governing

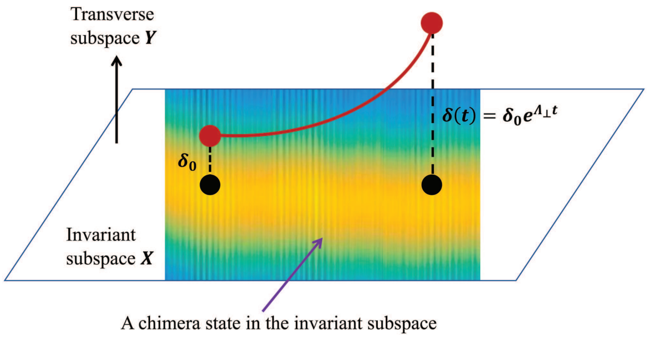

FIG. 1. Dynamical mechanism making a chimera state short-lived. The phase space of the system is the direct product between an invariant subspace **X** and a transverse subspace **Y**, where any initial condition taken from the invariant subspace will result in a dynamical state confined to it. There is a chimera state in the invariant subspace. Initially, the system is in the chimera state, and a small perturbation of magnitude δ_0 is applied in the transverse direction. If the largest transverse Lyapunov exponent $\Lambda_\perp$ is positive, the perturbation will grow exponentially in time, destroying the chimera state in a finite time and making the original state short-lived. The average lifetime of the chimera state depends on the logarithm of δ_0.

the ith oscillator given by

$$\frac{d\boldsymbol{\sigma}_i}{dt} = \frac{K}{N}\sum_{j=1}^{N}\left[\boldsymbol{\sigma}_j - \left(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_i\right)\boldsymbol{\sigma}_i\right] + \mathbf{W}_i \cdot \boldsymbol{\sigma}_i, \tag{8}$$

where the N oscillators are arranged on a ring; the state of the ith oscillator is described by the D-dimensional unit vector $\boldsymbol{\sigma}_i$; $\mathbf{W}_i$ is a real $D \times D$ antisymmetric matrix whose elements are determined by the natural frequency of the oscillator, which provides a constant bias to the dynamics of $\boldsymbol{\sigma}_i$; and K is the coupling parameter. In 2D, the unit vector is $\boldsymbol{\sigma}_i = (\cos\theta_i, \sin\theta_i)$, and the matrix $\mathbf{W}_i$ is

$$\mathbf{W}_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix};$$

therefore, Eq. (8) reduces to the classic Kuramoto model. In the context of chimera states, the oscillators are identical; therefore, the matrix does not depend on the oscillator: $\mathbf{W}_i \equiv \mathbf{W}$ for $i = 1, \ldots, N$.

We focus on the 3D Kuramoto model as it is relevant to physical phenomena, such as flocking and swarming. In 3D, the unit vector is given by

$$\boldsymbol{\sigma}_i = (\cos\gamma_i\cos\theta_i, \cos\gamma_i\sin\theta_i, \sin\gamma_i)^T, \tag{9}$$

where $-\pi/2 \le \gamma_i \le \pi/2$ and $0 \le \theta_i < 2\pi$ are the polar (latitudinal) and azimuthal (longitudinal) angles of the ith rotator, respectively. Unlike in 2D where a rotation can be described by a scalar quantity—the frequency, in 3D, a rotation vector $\boldsymbol{\omega}_i$ is required. Again, for chimera states, the oscillators are identical; therefore, $\boldsymbol{\omega}_i \equiv \boldsymbol{\omega}$ for $i = 1, \ldots, N$. In this case, the product $\mathbf{W} \cdot \boldsymbol{\sigma}_i$ in Eq. (8) becomes[82]

$$\mathbf{W} \cdot \boldsymbol{\sigma}_i = \boldsymbol{\omega} \times \boldsymbol{\sigma}_i. \tag{10}$$

Equation (8) describes a system of globally (all-to-all) coupled oscillators. A more realistic scenario is that the strength of coupling between any two oscillators decreases with their distance. In Ref. 61, the following generalized model was introduced:

$$\frac{d\boldsymbol{\sigma}_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} G\left(i-j\right)\left[\mathbf{T}\cdot\boldsymbol{\sigma}_j - \left(\left(\mathbf{T}\cdot\boldsymbol{\sigma}_j\right)\cdot\boldsymbol{\sigma}_i\right)\boldsymbol{\sigma}_i\right] + \mathbf{W}\cdot\boldsymbol{\sigma}_i, \tag{11}$$

where $G\left(i-j\right)$ is a distance-dependent coupling function and $\mathbf{T}_i$ is a $D \times D$ isometric matrix with a phase lag. A simple form of $G\left(i-j\right)$ is[61]

$$G\left(i-j\right) = 1 + A\cos\left[2\pi\left(i-j\right)/N\right], \tag{12}$$

with $0 \le A \le 1$ being a coupling parameter.

As described in Sec. II, the general setting of our study of chimera-state lifetime requires an invariant subspace in the full phase space. From the 3D Kuramoto model [Eq. (8) for $D = 3$], a convenient choice is to make the longitudinal angle at the equator of each 3D rotator defined by zero latitudinal angle ($\gamma_i = 0$) as a coordinate of the invariant subspace so that it hosts a chimera state from the classical 2D Kuramoto model. This requirement can be met by properly choosing the isometric matrix $\mathbf{T}$ as

$$\mathbf{T} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{13}$$

where the upper left two-by-two block represents a 2D rotation by the angle α about the z axis and the choice of the element $T_{33} = -1$ introduces a reflection symmetry about the equator, thereby guaranteeing that $\gamma_i = 0$ for all $i = 1, \ldots, N$ constitutes an invariant subspace. (The parameter α was introduced in the original derivation of the classical Kuramoto model—see Appendix A.)

To explicitly demonstrate the existence of the invariant subspace defined by $\gamma_i = 0$ ($i = 1, \ldots, N$), we make use of the position-dependent order parameter

$$\boldsymbol{\rho}_i = N^{-1}\sum_{j=1}^{N} G\left(i-j\right)\boldsymbol{\sigma}_j, \tag{14}$$

based on which Eq. (11) can be written as[61]

$$d\boldsymbol{\sigma}_i/dt = \mathbf{T}\boldsymbol{\rho}_i - \left(\mathbf{T}\boldsymbol{\rho}_i \cdot \boldsymbol{\sigma}_i\right)\boldsymbol{\sigma}_i + \mathbf{W}\cdot\boldsymbol{\sigma}_i. \tag{15}$$

In the spherical coordinate, the order parameter $\boldsymbol{\rho}_i$ can be written as

$$\boldsymbol{\rho}_i = R_i\left(\cos\Gamma_i\cos\Theta_i, \cos\Gamma_i\sin\Theta_i, \sin\Gamma_i\right)^T, \tag{16}$$

where R_i is the magnitude and Γ_i and Θ_i are the latitudinal and longitudinal angles, respectively, of $\boldsymbol{\rho}_i$. By choosing a reference frame according to the rotation vector $\boldsymbol{\omega}$, the natural-frequency term $\mathbf{W}\cdot\boldsymbol{\sigma}_i$ can be eliminated, leading to

$$d\boldsymbol{\sigma}_i/dt = \mathbf{T}\boldsymbol{\rho}_i - \left(\mathbf{T}\boldsymbol{\rho}_i \cdot \boldsymbol{\sigma}_i\right)\boldsymbol{\sigma}_i. \tag{17}$$

In the spherical coordinates, Eq. (17) can be explicitly expressed as (Appendix B)

$$\frac{d\theta_i}{dt} = -R_i\cos\left(\Gamma_i\right)\sin\left(\theta_i - \Theta_i + \alpha\right)/\cos\left(\gamma_i\right), \tag{18}$$

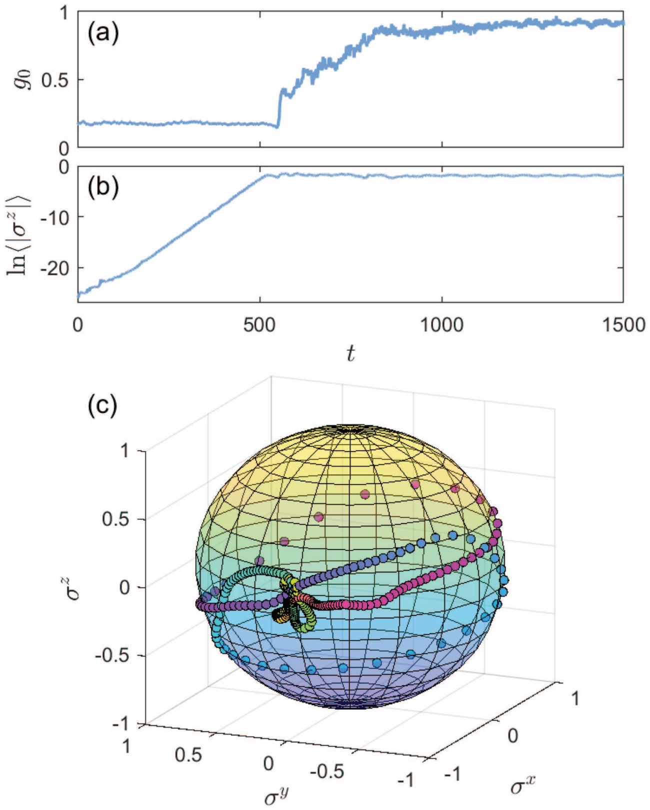

FIG. 2. An example of a short-lived chimera state in the generalized Kuramoto model of coupled 3D rotators. (a) Time evolution of g_0, the relative size of the group of coherent rotators for $N = 256$, $A = 0.7$, and $\alpha = \pi/2 - 0.04$, where $g_0 \approx$ constant between 0 and 1 signifies the occurrence of a chimera state. The lifetime of this chimera state is $\tau \approx 500$. (b) Exponential growth of the latitudinal components σ^z of the rotators from a random set of infinitesimal initial values. The average absolute value of the z-components $\langle|\sigma^z|\rangle$ reaching the order of magnitude of unity marks the complete destruction of the chimera state. (c) The final "quasi-coherent" state of the system in the spherical representation after the destruction of the chimera state, where all rotators are relatively coherent. Each dot represents the state $\sigma = (\sigma^x, \sigma^y, \sigma^z)$ of one rotator. Since $|\sigma| = 1$, all the rotators are on the spherical surface $\mathbb{S}^2$. The colors of the dots are assigned according to their spatial locations in the ring structure defined in Eq. (12), which determines their relative distances in the coupling term. Dots with similar colors are closer to each other and have stronger coupling among them. At a specific time point, most rotators cluster in a small region on the spherical surface, while other rotators form several rings surrounding the sphere. Each rotator does not always belong to the cluster or the rings but travels between them repeatedly. Another possible final state in this system is a globally synchronized state.

$$\frac{d\gamma_i}{dt} = -R_i\left[\sin(\gamma_i)\cos(\Gamma_i)\cos(\theta_i - \Theta_i + \alpha) + \cos(\gamma_i)\sin(\Gamma_i)\right]. \tag{19}$$

For $\gamma_i = 0$ $(i = 1, \ldots, N)$, we have $\Gamma_i = 0$; therefore, the right-hand side of Eq. (19) is zero and $\gamma_i(t) = 0$ is a solution of the system. As a result, $\gamma_i = 0$ $(i = 1, \ldots, N)$ is an invariant subspace in which the dynamics are governed by

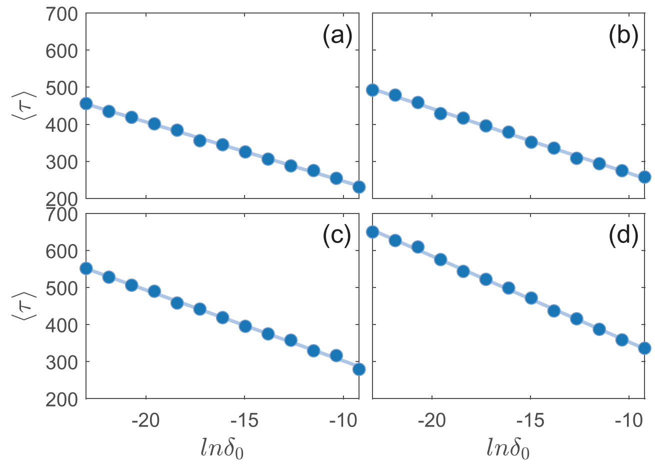

FIG. 3. Dependence of the average lifetime of the transient chimera states on the magnitude of the transverse perturbation. (a)–(d) The relation between $\angle\tau\rangle$ and $\ln\delta_0$ for $A = 0.5$, 0.6, 0.7, and 0.8, respectively. $N = 400$ and $\alpha = \pi/2 - 0.05$. The plot is robustly linear in all cases.

$$\frac{d\theta_i}{dt} = -R_i\sin(\theta_i - \Theta_i + \alpha), \tag{20}$$

which is the classical 2D Kuramoto model in terms of the order parameter and permits long-lived chimera states.

Suppose a chimera state has arisen in the invariant subspace, the corresponding transverse Lyapunov exponents can be calculated from the following set of variational equations (Appendix C):

$$\frac{d\delta\gamma_i}{dt} = -R_i\cos(\theta_i - \Theta_i + \alpha)\delta\gamma_i - \frac{1}{N}\sum_{k=1}^{N} G(i-k)\delta\gamma_k \tag{21}$$

for $i = 1, \ldots, N$, where the dynamical variables θ_i and Θ_i are obtained by integrating Eq. (20). The exponential growth rate of the magnitude of the perturbation vector gives the largest transverse Lyapunov exponent $\Lambda_\perp$.

B. Demonstration of short-lived chimera states

We use the relative size g_0 of the coherence region in the space to determine if a chimera state has occurred.[49] When the system exhibits a chimera state, the value of g_0 will be approximately constant with time. Emergence of fluctuations in the time evolution of g_0 signifies deterioration and destruction of the chimera state. Figure 2(a) presents one example for $N = 256$, $A = 0.7$, and $\alpha = \pi/2 - 0.04$, where the system is initiated from a 2D equatorial chimera state defined by $\gamma_i = 0$ $(i = 1, \ldots, N)$. A small random perturbation $\delta_0 = 10^{-10}$ is then applied. To apply this perturbation to γ_i, we first generate $\delta\gamma_i$ $(i = 1, \ldots, N)$ by i.i.d. standard Gaussian distribution. We then renormalize these rotator-wise perturbations $\delta\gamma_i$ following

$$\delta_0 = \sqrt{\sum_{i=1}^{N}(\delta\gamma_i)^2}. \tag{22}$$

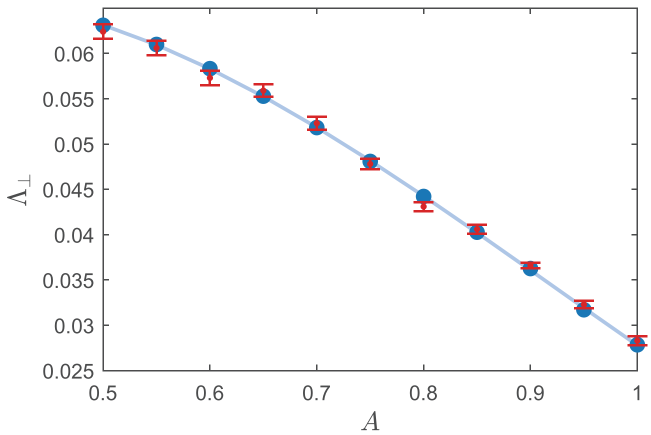

FIG. 4. Validation of the exact formula for the average lifetime of the transient chimera states. Equation (7) stipulates that the proportional coefficient between $\langle\tau\rangle$ and $-\ln\delta_0$ is the inverse of the largest transverse Lyapunov exponent $\Lambda_\perp$, providing a way to calculate $\Lambda_\perp$ from the linear plots in Fig. 3 (red crosses). Alternatively, $\Lambda_\perp$ can be calculated from the variational equation [Eq. (21)] together with the equation governing the chimera state in the invariant subspace (blue filled circles). The values of $\Lambda_\perp$ obtained from the two approaches agree well with each other.

It can be seen from Fig. 2(a) that g_0 remains to be approximately constant for $t \lesssim 500$ before exhibiting significant fluctuations. The lifetime of this chimera state is then estimated to be about 500. The gradual destruction of the chimera state can also be seen from Fig. 2(b), where the latitudinal angles of the rotators grow to their full range in $t \approx 500$. After the disappearance of the chimera state, the system can evolve into two different types of states. One is a completely synchronized state of all the rotators. Another is a relatively coherent yet not completely synchronized state, as exemplified in Fig. 2(c), where many rotators concentrate in a small region on the $\mathbb{S}^2$ spherical surface of $\boldsymbol{\sigma}$, while other rotators form rings around the sphere. Rotators are not fixed to a cluster or a ring. Alternatively, they travel between these two types of regions in time.

To verify the main result [Eq. (7)], we carry out direct numerical simulations to calculate the relation between the average transient lifetime of the chimera state and the logarithm of the magnitude of the transverse perturbation $\ln\delta_0$ for a set of parameter values. Four representative examples are shown in Figs. 3(a)–3(d), for $A = 0.5, 0.6, 0.7$, and 0.8, respectively, and $\alpha = \pi/2 - 0.05$. In all four cases, the relation between $\langle\tau\rangle$ and $\ln\delta_0$ is linear, as predicted. The slope of the linear fit determines the transverse Lyapunov exponent $\Lambda_\perp$. Alternatively, the exponent can be calculated by integrating Eqs. (20) and (21). Figure 4 shows the values of $\Lambda_\perp$ calculated from the two different ways for a number of parameter values, which agree with each other quite well.

IV. DISCUSSION

Research on chimera states in the past two decades revealed the ubiquity of these dynamical states in systems of coupled identical nonlinear oscillators. A vast majority of these works were done using the classical Kuramoto model of 2D rotators, where it was established that the chimera states are "long-lived" in the sense that their lifetime increases exponentially with the system size. Short-lived chimera states do arise by the two known scenarios: (1) there is noise[71] and (2) the dimension of the individual rotators is more than two.[61] In both cases, a scaling law was found, where the average lifetime of the transient chimera states is proportional to the logarithm of the perturbation strength (noise amplitude[71] or the magnitude of the dimension-augmenting perturbation[61]). The scaling relation indicates that even when the perturbation is weakened by many orders of magnitude, the transient lifetime will increase only incrementally. While the scaling law provides a quantitative understanding of the short-lived transient nature of the chimera states, it is what it is—a scaling law with a free proportional constant.

We have reexamined the short-lived chimera states using a generalized Kuramoto model of 3D rotators. The system possesses an invariant subspace in which the dynamics are those of the classical Kuramoto model of 2D rotators in the longitudinal angles so that long-lived chimera states can arise. The realization that any dimension-augmenting perturbation in the latitudinal angles, in fact, transverse to the invariant subspace, renders appropriate an understanding of the short-lived chimera states in terms of their transverse stability characterized by the largest transverse Lyapunov exponent. For the generalized 3D Kuramoto model, we have derived a set of variational equations for calculating this Lyapunov exponent. This transverse-stability analysis has straightforwardly turned the previous numerical scaling law of the average transient lifetime into an exact formula. In particular, the "free" proportional constant in the scaling law can now be determined exactly by the largest transverse Lyapunov exponent.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Ling-Wei Kong: Investigation (equal); Validation (equal). **Ying-Cheng Lai:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

All relevant data and computer codes that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: ORIGIN OF PHASE LAG α IN EQ. (13)

The classical Kuramoto model was derived from the system of coupled one-dimensional complex Ginzburg–Landau equation. In particular, in continuous space and time, the spatiotemporal phase variable $\phi(x,t)$ obeys the following equation:[2]

$$\frac{\partial}{\partial t}\phi(x,t) = \omega - \int G(x-x')\sin\left[\phi(x,t)-\phi(x',t)+\alpha\right]dx', \quad \text{(A1)}$$

where the time scale has been set to normalize the coupling, ω is the rescaled natural frequency, $G(x-x')$ is the coupling function, and the phase constant α is related to the parameters a and b in the complex Ginzburg–Landau equation as

$$\tan\alpha = \frac{b-a}{1+ab} \quad \text{(A2)}$$

for $\alpha(b-a) > 0$.

APPENDIX B: DERIVATION OF EQS. (18) AND (19)

The left-hand side of Eq. (17) is

$$d\boldsymbol{\sigma}/dt = \begin{pmatrix} -\sin(\gamma_i)\cos(\theta_i)d\gamma/dt - \cos(\gamma_i)\sin(\theta_i)d\theta/dt \\ \cos(\gamma_i)\cos(\theta_i)d\theta/dt - \sin(\gamma_i)\sin(\theta_i)d\gamma/dt \\ \cos(\gamma_i)d\gamma/dt \end{pmatrix} \equiv \begin{pmatrix} d\sigma_{i1}/dt \\ d\sigma_{i2}/dt \\ d\sigma_{i3}/dt \end{pmatrix}. \quad \text{(B1)}$$

The right-hand side of Eq. (17) is a vector $\mathbf{v}$ with the following components:

$$v_1/R_i = \cos(\Gamma_i)\left[\cos(\alpha-\Theta_i) - \cos^2(\gamma_i)\cos(\theta_i)\cos(\alpha+\theta_i-\Theta_i)\right] + \sin(\gamma)\cos(\gamma)\sin(\Gamma)\cos(\theta), \quad \text{(B2)}$$

$$\begin{aligned} v_2/R_i = {} & \sin(\gamma_i)\cos(\gamma_i)\sin(\Gamma_i)\sin(\theta_i) \\ & - \cos(\Gamma_i)\left[\cos^2(\gamma_i)\sin(\theta_i)\cos(\alpha+\theta_i-\Theta_i)\right. \\ & \left. + \sin(\alpha)\cos(\Theta_i) - \cos(\alpha)\sin(\Theta_i)\right] \\ & + \sin(\theta_i)(\cos(\alpha-\Gamma_i+\gamma_i)\sin(\theta_i)\sin(\Theta_i) \\ & - \cos(\theta_i)\cos(\Theta_i))\sin(\gamma_i), \end{aligned} \quad \text{(B3)}$$

$$\begin{aligned} v_3/R_i = {} & -\cos(\gamma_i)(\sin(\gamma_i)\cos(\Gamma_i)\cos(\theta_i-\Theta_i+\alpha) \\ & + \cos(\gamma_i)\sin(\Gamma_i)). \end{aligned} \quad \text{(B4)}$$

Using

$$\sin(\theta_i)\frac{d\sigma_{i1}}{dt} - \cos(\theta_i)\frac{d\sigma_{i2}}{dt} = -\cos(\gamma_i)\frac{d\theta_i}{dt}, \quad \text{(B5)}$$

we get Eq. (18). Equating $d\sigma_{i3}/dt$ to v_3 gives Eq. (19).

APPENDIX C: DERIVATION OF EQ. (21)

Consider an infinitesimal variation of the set of transverse dynamical variables: $(\delta\gamma_1,\ldots,\delta\gamma_N)^T$ from the invariant subspace. The differential equation governing the evolution of $\delta\gamma_i$ evaluated at $(\gamma_1,\ldots,\gamma_N) = (0,\ldots,0)$ is

$$\frac{d\delta\gamma_i}{dt} = -R_i[\cos(\theta_i-\Theta_i+\alpha)\delta\gamma_i + \delta\Gamma_i], \quad \text{(C1)}$$

where

$$\delta\Gamma_i = \sum_{j=1}^{N}\frac{\partial\Gamma_i}{\partial\gamma_j}\delta\gamma_j.$$

From the definition of the order parameter [Eq. (14)], we have

$$\tan\Gamma_i = \frac{\frac{1}{N}\sum_{j=1}^{N}G(i-j)\cos\gamma_i}{\left(\frac{1}{N}\sum_{j=1}^{N}G(i-j)\cos\theta_j\right)^2 + \left(\frac{1}{N}\sum_{j=1}^{N}G(i-j)\sin\theta_j\right)^2}. \quad \text{(C2)}$$

We get

$$\frac{\partial\Gamma_i}{\partial\gamma_k} = \frac{G(i-k)}{N\sqrt{Q_1^2+Q_2^2}}, \quad \text{(C3)}$$

where

$$Q_1 \equiv \frac{1}{N}\sum_{j=1}^{N}G(i-j)\cos\theta_j \quad \text{and} \quad Q_2 \equiv \frac{1}{N}\sum_{j=1}^{N}G(i-j)\sin\theta_j.$$

In the invariant subspace, we have

$$\begin{aligned} R_i^2 &= \left(\frac{1}{N}\sum_{j=1}^{N}G(i-j)\cos\gamma_j\cos\theta_j\right)^2 + \left(\frac{1}{N}\sum_{j=1}^{N}G(i-j)\cos\gamma_j\sin\theta_j\right)^2 + \left(\frac{1}{N}\sum_{j=1}^{N}G(i-j)\sin\gamma_j\right)^2 \\ &= \left(\frac{1}{N}\sum_{j=1}^{N}G(i-j)\cos\theta_j\right)^2 + \left(\frac{1}{N}\sum_{j=1}^{N}G(i-j)\sin\theta_j\right)^2 = Q_1^2+Q_2^2; \end{aligned}$$

therefore,

$$\frac{\partial\Gamma_i}{\partial\gamma_k} = \frac{G(i-k)}{NR_i}. \quad \text{(C4)}$$

Substituting this into Eq. (C1) leads to Eq. (21).

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