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Forecasting the future: Is it possible for adiabatically time-varying nonlinear dynamical systems?

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Nonlinear dynamical systems in reality are often under environmental influences that are time-dependent. To assess whether such a system can perform as desired or as designed and is sustainable requires forecasting its future states and attractors based solely on time series. We propose a viable solution to this challenging problem by resorting to the compressive-sensing paradigm. In particular, we demonstrate that, for a dynamical system whose equations are unknown, a series expansion in both dynamical and time variables allows the forecasting problem to be formulated and solved in the framework of compressive sensing using only a few measurements. We expect our method to be useful in addressing issues of significant current concern such as the sustainability of various natural and man-made systems. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4740057]

Nonlinear dynamical systems with one or a few parameters varying slowly with time are of considerable interest in many areas of science and engineering. In such a system, the attractors in the future can be characteristically different from those at the present. To predict the possible future attractors based on available information at the present is thus a well-defined and meaningful problem, which is challenging especially when the system equations are not known but only time-series measurements are available. Here, we extend our recently developed, compressive-sensing based method for predicting catastrophic bifurcations in stationary dynamical systems to time-varying systems. We demonstrate that this framework allows us to reconstruct the system equations and time dependence of parameters based on limited measurements so that the future attractors of the system can be predicted through computation.

I. INTRODUCTION

A dynamical system in the physical world is constantly subject to random disturbance or adiabatic perturbation. Broadly speaking, there are two types of perturbations: stochastic or deterministic. Stochastic disturbances (or noise) can typically be described by random processes and they do not alter the intrinsic structure of the underlying equations of the system. Deterministic perturbations, however, can cause the system equations or parameters to vary with time. Suppose the perturbations are adiabatic, i.e., T_i, the time scale of the intrinsic dynamics of the system is much smaller than T_e, that of the external perturbation. In this case, some "asymptotic states" or "attractors" of the system can still be approximately defined in a time scale that is much larger than T_i but smaller than T_e. When the dynamics in such a time interval is examined, the attractor of the system will depend on time. Often, one is interested in forecasting the "future" asymptotic states of the system. Take the climate system as an example. The system is under random disturbances, but adiabatic perturbations are also present, such as CO_2 injected into the atmosphere by human activities, the level of which tends to increase with time. The time scale for appreciable increase in the CO_2 level to occur (e.g., months or years) is much larger than the intrinsic time scales of the system (e.g., days). The climate system is thus an adiabatically time-varying, nonlinear dynamical system. It is of tremendous interest to forecast what the future attractors of the system might be in order to determine whether it will behave as desired or sustainably. The issue of sustainability is, of course, critical to many other natural and man-made systems as well. To be able to forecast the future states of such systems is essential to assessing their sustainability.

In this paper, we address the following question: given a nonlinear dynamical system whose equations or parameters vary adiabatically with time, but otherwise are completely unknown, can one predict, based solely on measured time series, the future asymptotic attractors of the system? To state the problem in a formal way, we consider time-varying dynamical systems, mathematically described by

$$d\mathbf{x}/dt = \mathbf{F}[\mathbf{x}, \mathbf{p}(t)], \tag{1}$$

where $\mathbf{x}$ is the dynamical variable of the system in the d-dimensional phase space and $\mathbf{p}(t) = [p_1(t), \ldots, p_K(t)]$ denotes K independent, time-varying parameters of the system. We assume, however, that both forms of $\mathbf{F}$ and $\mathbf{p}$(t) are unknown but at time t_M, the end of the time interval during which measurements are taken, time series $\mathbf{x}$(t) for $t_M - T_M \leq t \leq t_M$ are available, where T_M denotes the measurement time window. Our idea is to predict the precise mathematical forms of $\mathbf{F}$ and $\mathbf{p}$(t) based on the available time series at t_M so that the evolution and the likely attractors

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of the system for $t > t_M$ can be computationally assessed and anticipated. We shall establish that this can be accomplished by using the compressive-sensing algorithm[1,2] that has recently been applied to predicting catastrophic bifurcations in time-independent dynamical systems.[3] The predicted form of $\mathbf{F}$ and $\mathbf{p}(t)$ at time t_M would contain errors that in general will increase with time. In addition, for $t > t_M$, new perturbations can occur to the system so that the forms of $\mathbf{F}$ and $\mathbf{p}(t)$ may be further changed. It is thus necessary to execute the prediction algorithm frequently using time series available at the time. In particular, the system could be monitored at all times so that time series can be collected, and predictions should be carried out at t_i's, where $\ldots > t_i > \ldots > t_{M+2} > t_{M+1} > t_M$. For any t_i, the prediction algorithm is to be performed based on available time series in a suitable window prior to t_i.

There is large literature on forecasting nonlinear dynamical systems.[3–10] A conventional approach is to approximate a nonlinear system by a large collection of linear equations in different regions of the phase space to reconstruct the Jacobian matrices on a proper grid[5,6,8] or fit ordinary differential equations to chaotic data.[7] Approaches based on chaotic synchronization[9] or genetic algorithms[10] to system-parameter estimation were also investigated. In most existing works, short-term predictions of a dynamical system can be achieved by employing the classical delay-coordinate embedding paradigm.[11,12] For nonstationary systems, the method of over-embedding was introduced[13] in which the time-varying parameters were treated as independent dynamical variables so that the essential aspects of determinism of the underlying system could be restored. In spite of the previous works, prediction of time-varying dynamical system remains to be a challenging issue. As we shall describe, besides being applicable to time-varying dynamical systems, the principle of our method is drastically different from those of previous ones because we aim to predict globally, in long term, the *exact* forms of both system equations and parameter functions based on time series that are presently available.

The basic strategy upon which our method is based is compressive sensing,[1,2] which has been exploited widely in all kinds of signal-processing problems in different fields of science and engineering. Recently compressive sensing has been applied to predicting catastrophic bifurcations in stationary dynamical systems.[3] The purpose of this paper is to extend the methodology to predicting time-varying dynamical systems.

In Sec. II, we describe our compressive-sensing based method for predicting the parameters and equations of time-varying dynamical systems. In Sec. III, we demonstrate our method through numerical examples. A brief conclusion is presented in Sec. IV. In Appendix, we describe a general solution of compressive sensing.

II. METHOD

The problem of compressive sensing can be described as the reconstruction of a sparse vector $\bar{x} \in R^N$ from linear measurements $\bar{y}$ in the form: $\bar{y} = A\bar{x}$, where $\bar{y} \in R^M$ and A is a $M \times N$ matrix. Accurate reconstruction can be achieved by solving the following convex optimization problem[1,2]

$$\min \|\bar{x}\|_1 \quad \text{subject to} \quad A\bar{x} = \bar{y}, \tag{2}$$

where $\|\bar{x}\|_1 = \sum_{i=1}^{N} |\bar{x}_i|$ is the L_1 norm of vector $\bar{x}$. A general solution procedure of compressive sensing problems is described in Appendix. An extremely attractive feature of compressive sensing is that the number of measurements can be much less than the number of components of the unknown signal: $M \ll N$.

Our goal is to formulate the problem of predicting time-varying dynamical systems in the framework of compressive sensing. To accomplish this, we expand all components of the time-dependent vector field $\mathbf{F}[\mathbf{x}, \mathbf{p}(t)]$ in Eq. (1) into a power series in terms of both dynamical variables $\mathbf{x}$ and time t. The ith component $\mathbf{F}[\mathbf{x}, \mathbf{p}(t)]_i$ of the vector field can be written as

$$\sum_{l_1,\cdots,l_m=1}^{n} [(\alpha_i)_{l_1,\ldots,l_m} x_1^{l_1} \cdots x_m^{l_m} \cdot \sum_{w=0}^{v} (\beta_i)_w t^w] \equiv \sum_{l_1,\ldots,l_m=1}^{n} \sum_{w=0}^{v} (c_i)_{l_1,\ldots,l_m;w} x_1^{l_1} \cdots x_m^{l_m} \cdot t^w, \tag{3}$$

where x_k $(k = 1,\ldots,m)$ is the kth component of the dynamical variable and c_i is the ith component of the coefficient vector to be determined [vector $\bar{x}$ in Eq. (2)]. We assume that the time evolution of each term can be approximated by the power series expansion in time, i.e., $\sum_{w=0}^{v} (\beta_i)_w t^w$. The power-series expansion allows us to cast Eq. (1) into the standard form of compressive sensing, namely, Eq. (2) (see Ref. 3 for details). In principle, if every combined scalar coefficient $(c_i)_{l_1,\ldots,l_m;w}$ associated with the corresponding term in Eq. (3) can be determined from time series for $t \le t_M$, the vector-field component $[\mathbf{F}(\mathbf{x}, \mathbf{p}(t))]_i$ becomes known. Repeating the procedure for all components, the entire vector field for $t > t_M$ can be found.

A fundamental requirement in compressive sensing is that the vector to be determined be sparse. For a dynamical system whose vector field is given by a finite number of power-series terms, such as the classical Lorenz[14] or Rössler systems,[15] this sparsity requirement can be readily satisfied by assuming as many terms as possible in Eq. (3), since the coefficients associated with most terms are zero. For systems whose vector fields contain, e.g., trigonometric functions for which the power-series expansions contain an infinite number of terms, some alternative expansion base, such as the Fourier base, can be used to ensure the sparsity condition.[3]

To explain our method in an intuitive way, we consider a special case where the number of components of the dynamical variables is $m = 3$ (x, y, and z), the order of the power series is $l_1 + l_2 + l_3 \le 2$, and the maximum power of time t in Eq. (3) is $v = 1$, i.e., we include only t^0 and t^1 terms. Focusing on one dynamical variable, say x, the total number of terms in the power-series expansion is 20, as specified in Fig. 1(a). Let the measurements $x(t)$, $y(t)$, and $z(t)$ be taken at times t_1, t_2, …, t_M, as shown in Fig. 1(b). The values of all 20 power-series terms at these time instants can then be obtained, as shown in Fig. 1(c), where we divide all the terms into two blocks according to the distinct powers of the time variable t: t^0 and t^1. The matrix A in Eq. (2) thus

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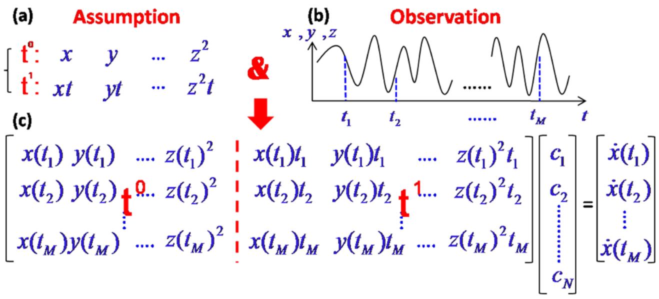

FIG. 1. Illustration of our scheme to map the problem of predicting time-varying nonlinear dynamical systems into the framework of compressive sensing.

consists of these two blocks. (In the general case where higher powers of the time variable are involved, A would contain a corresponding number of blocks.) The components of vector $\bar{y}$ in Eq. (2) are the first derivatives dx/dt evaluated at $t_1, t_2, \ldots, t_M$, which can also be approximated by the measured time series $x(t)$ at these times. As shown in Fig. 1(c), Eq. (3) for this simple example can be written in the form $A\bar{x} = \bar{y}$, where both the matrix A and vector $\bar{y}$ can be determined straightforwardly from measured time series, and the task is to solve for the coefficient vector ($\bar{x}$ or **c**). In general, we assume many terms in the power-series expansion up to some high order n, and the total number of terms in Eq. (3), N, will be quite large. As a result, $\bar{x}$ is high-dimensional but most of its components are zero, ensuring sparsity. However, the number of measurements, M, needs not be as large as N. Another requirement of compressive sensing is the restricted isometric property[1,2] which can be guaranteed by normalizing the matrix A and by using linear-programming based signal-recovery algorithms. To determine the set of power-series coefficients corresponding to a different dynamical variable, say y, we simply replace the measurement vector by $\bar{y} = [\dot{y}(t_1), \dot{y}(t_2), \ldots, \dot{y}(t_M)]^T$. The matrix A, however, remains the same. We see that the problem of forecasting *time-varying* nonlinear dynamical systems fits perfectly into the compressive-sensing paradigm.

III. A NUMERICAL EXAMPLE

As a proof of principle, we take the classical Lorenz chaotic system[14] as an example by incorporating explicit time dependence in a number of additional terms. The modified Lorenz system is given by

$$\begin{aligned}\dot{x} &= -10(x - y) + k_1(t)y,\\ \dot{y} &= 28x - y - xz + k_2(t)z,\\ \dot{z} &= -(8/3)z + xy + [k_3(t) + k_4(t)]y,\end{aligned} \quad (4)$$

where $k_1(t) = -t^2$, $k_2(t) = 0.5t$, $k_3(t) = t$, and $k_4(t) = -0.5t^2$. Suppose that the system equations are unknown but only measured time series $x(t)$, $y(t)$, and $z(t)$ in a finite time interval are available. The number of dynamical variables is $m = 3$ and we choose the orders of the power-series expansions in the three variables according to $l_1 + l_2 + l_3 \leq 3$. The maximum power in the time dependence is chosen to be $v = 2$ so that explicit time-dependent terms t^0, t^1, and t^2 are included. The total number of coefficients to be predicted is then $(v+1)\sum_{i=1}^{3}(i+1)(i+2)/2 = 57$. (Note that, using low-order power-series expansions in both dynamical variables and time is solely for facilitating explanation and presentation of results, while the forecasting principle is the same for realistic dynamical systems where much higher orders may be needed.) Figure 2 shows the predicted coefficient values versus the term index for all three dynamical variables, where in each panel, solid triangles and open circles denote predicted non-zero and zero coefficients, respectively, and the red dashed dividing lines indicate the terms associated with different powers of the time variable, i.e., t^0, t^1, and t^2 (from left to right). The meaning of these results can be explained by using any one of the dynamical variables. For example, for the x-component of the vector field, our prediction algorithm gives only 3 nonzero coefficients. By identifying the corresponding values of the term index, we read that they correspond to the two terms without explicit time dependence: y, x, and the term that contains explicit such dependence: t^2y, respectively. A comparison of the predicted nonzero coefficient values with the actual ones in the original Eq. (4) indicates that the method works remarkably well. Similar results have been found for y and z components of the vector field, where we also find excellent agreements between the predicted and the actual functions.

When the vector field **F**[**x**, **p**(t)] of the underlying dynamical system has been predicted, one can solve Eq. (1) numerically to assess the state variables at any future time and the asymptotic attractors. Figures 3(a) and 3(b) present one example, where a forecasted time series calculated from the predicted vector field is shown, together with the values of the corresponding dynamical variable from the actual Lorenz system at a number of time instants. The two cases

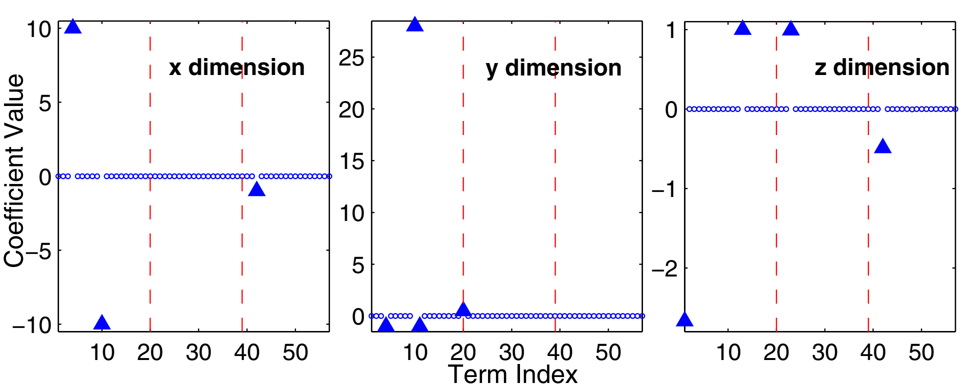

FIG. 2. For the time-varying Lorenz chaotic systems Eq. (4) with $k_1(t) = -t^2$, $k_2(t) = 0.5t$, $k_3(t) = t$, and $k_4(t) = -0.5t^2$, predicted values of coefficients of power-series terms versus the term index for the x-, y-, and z-equations, where solid triangles and open circles denote nonzero and zero coefficients, respectively. Note that, the number M of data points used for prediction is about 50% of the total number N of unknown coefficients in each power-series expansion.

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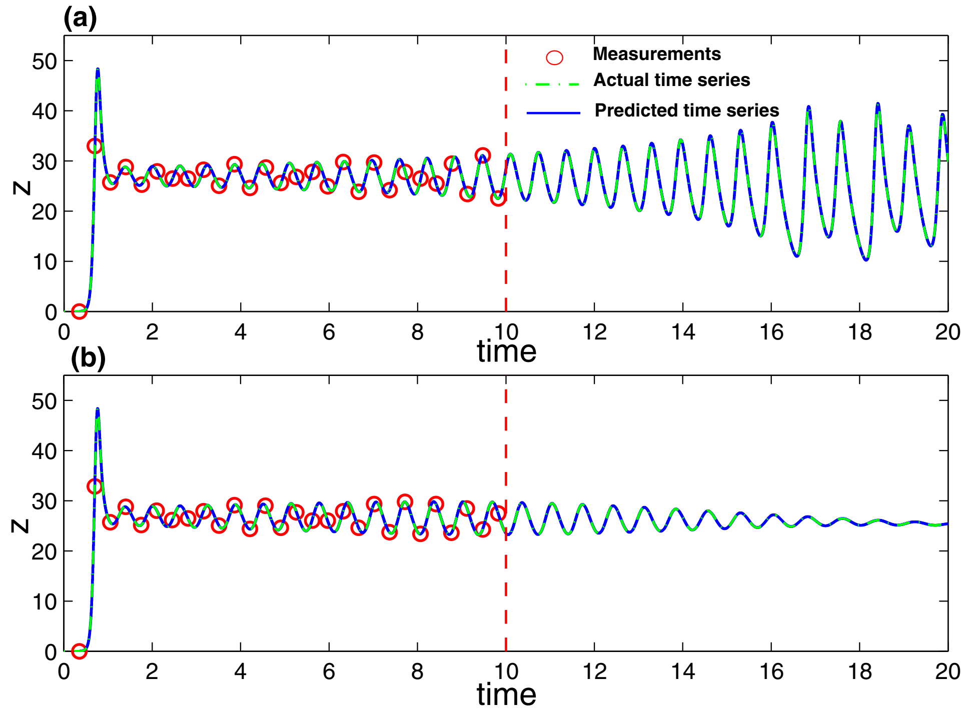

FIG. 3. For the Lorenz system (4), predicted time series and measured values of the dynamical variable $z(t)$ for (a) time-independent case where $k_i(t) = 0$ $(i = 1, \ldots, 4)$ and (b) time-dependent case where $k_1(t) = -0.01t^2$, $k_2(t) = 0.01t$, $k_3(t) = 0.01t$, and $k_4(t) = -0.01t^2$. Red circles denote the measurements used for prediction, while the green dash and blue solid lines represent the actual and predicted time series, respectively. In both panels, $t = 0$ and $t = 10$ correspond to the beginning and end of the measurement window, i.e., t_1 and t_M, respectively.

shown are where the parameter functions $k_i(t)$ $(i = 1, \ldots, 4)$ are all zero and time-varying, respectively. Excellent agreement is again obtained, indicating the power of our method to predict the future states and attractors of time-varying dynamical systems. The interpretation and implication of Figs. 3(a) and 3(b) are the following. Note that $t = 0$ and $t = 10$ correspond to the beginning and end of the measurement time window $[t_1, t_M]$, respectively. For the original classical Lorenz system without time-varying parameters, the asymptotic attractor is chaotic, as can be seen from Fig. 3(a). However, as external perturbations are turned on at $t = 0$, there are four time-varying parameters in the system for $t > 0$. In this case, the attractor becomes a fixed-point, as can be seen from the behavior $z \to$ constant in Fig. 3(b). In both cases,

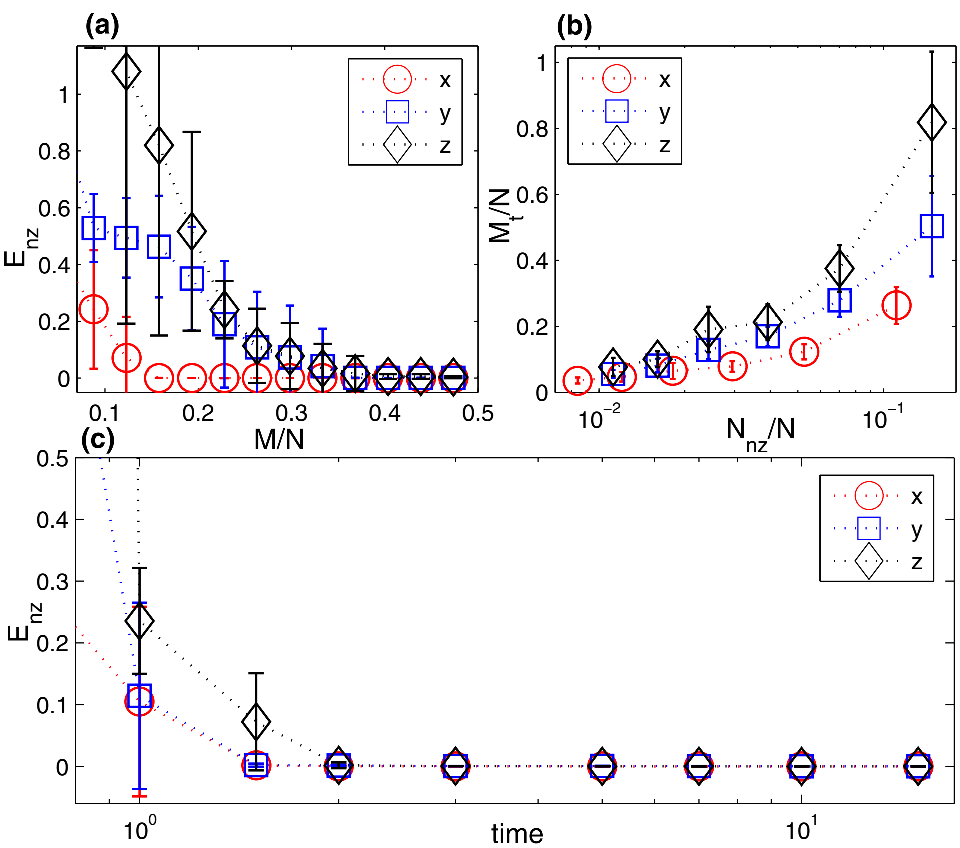

FIG. 4. For the time-varying Lorenz system as in Fig. 2, (a) prediction errors E_{nz} as a function of the ratio M/N, and (b) ratio M_t/N as a function of the ratio N_{nz}/N, where N can be increased by using power-series expansion to higher order (e.g., up to 7). (c) With fixed number of measurement M, prediction errors E_{nz} as a function of the length of measurement window. The error bars are obtained from 20 independent realizations of the prediction algorithm. The prediction errors E_z associated with nonexistent terms show similar behaviors.

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by using limited amount of measurements, namely, available time series in the window $[t_1, t_M]$, we obtain quite accurate forecasting results. The result exemplified in Fig. 3(b) is especially significant, as it indicates that the future state and attractors of time-varying dynamical systems can be accurately predicted with only limited data availability.

We now determine and characterize the prediction errors. Two types of errors can be defined: one for non-zero (existent) terms in the power-series expansion and another for zero (non-existent) terms. For each existent term, a relative error can be defined, which is the ratio of the absolute difference between the predicted and true values to the true value. For non-existent terms the absolute errors are meaningful. Taking the average of errors over all the corresponding terms, we obtain E_{nz} and E_z, the prediction error for existent and non-existent terms, respectively. Figure 4(a) shows, for the time-varying Lorenz system, E_{nz} versus the ratio of the number M of measurements to the total number N of terms to be predicted. For all dynamical variables, we observe that, as M exceeds a threshold value M_t, E_{nz} becomes effectively zero, where M_t can be defined quite arbitrarily, e.g., the minimum number of measurements required to achieve $E_{nz} = 10^{-3}$. The data requirement for accurate prediction can then be assessed by examining how M_t depends on the sparsity of the coefficient vector to be predicted, which can be defined as the ratio of the number N_{nz} of the nonzero terms to the total number N of terms to be predicted. Note that, N or the ratio N_{nz}/N can be adjusted by varying the order of the assumed power series. From Fig. 4(b), we see that, as N_{nz}/N is decreased (e.g., by increasing N) so that the vector to be predicted becomes more sparse, the ratio M_t/N also decreases. In particular, for the smallest value of N_{nz}/N examined, where $N = 357$, only about 5% of the data points are needed for accurate prediction, despite the time-varying nature of the underlying dynamical system. Figure 4(c) shows the prediction errors with respect to different length of the measurement window for a fixed number of data points. We see that, when the length exceeds a certain (small) value so that the time series extends to the whole attractor in the phase space, E_{nz} approaches zero rapidly.

Dynamical systems are often driven by time-periodic forces, such as the classical Duffing system.[16] In such a case, it is necessary to explore alternative bases of expansion with respect to the time variable other than power series to ensure the sparsity condition. A realistic strategy to choose a suitable expansion base is to make use of the basic physics underlying the dynamical system of interest. Insofar as an appropriate base can be chosen so that the coefficient vector to be predicted is sparse, the methodology proposed and elaborated in this paper is applicable.

IV. CONCLUSION

We have articulated a compressive-sensing based approach to forecasting the state and attractors of time-varying nonlinear dynamical systems. Our central idea is to expand the vector field of the underlying system in both dynamical and time variables in a suitable base to ensure that the vector constituting the coefficients of all terms in the expansion is sparse. The main achievement of this paper is a demonstration that the future states and asymptotic attractors of time-varying dynamical systems can be accurately forecasted based on limited time series. Because of the ubiquity of nonlinear dynamical systems and common encounters with time-dependent external perturbations in the real world, forecasting the future behavior of such systems is of significant value to science and engineering, and beyond. Our work represents a step forward in this direction.

ACKNOWLEDGMENTS

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APPENDIX: SOLUTION OF COMPRESSIVE SENSING METHOD

Generally, the problem of compressive sensing is described by reconstructing a vector $\bar{x} \in \mathbb{R}^N$ from linear measurements $\bar{y}$ about $\bar{x}$ in the form

$$\bar{y} = A\bar{x}, \tag{A1}$$

where $\bar{y} \in \mathbb{R}^M$ and A is a $M \times N$ matrix. By definition, the number of measurements is much less than the size of the unknown signal, i.e., $M \ll N$. Suppose that the original signal x, is sparse. Accurate recovery can be achieved by solving the following convex optimization problem[1,2]

$$\min \|\bar{x}\|_1 \quad \text{subject to} \quad A\bar{x} = \bar{y}. \tag{A2}$$

Here we explain the method used in this paper to solve the convex optimization problem described by Eq. (A2). By inducing a new variable vector $u \in \mathbb{R}^N$, problem (A2) can be recast into a linear-constraint minimization problem

$$\min \sum_{i=1}^{N} u_i \quad \text{subject to} \quad \begin{cases} \bar{x}_i - u_i \le 0 \\ -\bar{x}_i - u_i \le 0 \\ A\bar{x} = \bar{y} \end{cases}. \tag{A3}$$

By defining $z = [\bar{x}^T, u^T]^T$, Eq. (A3) can be rewritten as

$$\langle c_0, z \rangle \quad \text{subject to} \quad \begin{cases} f_i(z) \le 0 \\ f'_i(z) \le 0 \\ A_0 z = \bar{y} \end{cases}, \tag{A4}$$

where $f_i(z) = \langle c_i, z \rangle, f'_i(z) = \langle c'_i, z \rangle$, $\langle . \rangle$ denotes inner product of two vectors, $c_0, c_i, c'_i \in \mathbb{R}^{2N}$, A_0 is a $M \times 2N$ matrix, $(c_0)_j = 0$ for $j \le N$, $(c_0)_j = 1$ for $j > N$, $(c_i)_j = 1$ for $j = i$, $(c_i)_j = -1$ for $j = N + i$, $(c'_i)_j = 1$ for $j = i$, $(c'_i)_j = -1$ for $j = N + i$, and $A_0 = [0^{M\times N}, A]$. To solve the linear constraint minimization problem (A4), one can use the Karush-Kuhn-Tucker conditions;[1,2] that is, at the optimal point z^*, there exists vectors $v^* \in \mathbb{R}^M$, $\lambda^* \in \mathbb{R}^N$, $\lambda'^* \in \mathbb{R}^N$, $\lambda^*, \lambda'^* \ge 0$ such that the following conditions are satisfied:

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$$
\begin{aligned}
c_0 + A_0^T v^* + \sum_i \lambda_i^* c_i + \sum_i \lambda'^*_i c'_i &= \mathbf{0}, \\
\lambda_i^* f_i(z^*) &= 0 \, (i = 1, ..., N), \\
\lambda_i'^* f_i'(z^*) &= 0 \, (i = 1, ..., N), \\
A_0 z^* &= \bar{y}, \\
f_i(z^*) &\le 0, \\
f_i'(z^*) &\le 0.
\end{aligned}
\tag{A5}
$$

The solution procedure of problem (A5) is the classical Newton method in the valid solution set determined by the inequality constraints in Eq. (A5)

$$\{\lambda \ge 0, \lambda' \ge 0, f_i(z) \le 0, f_i'(z) \le 0\}, \tag{A6}$$

where a point $(z, v, \lambda, \lambda')$ in this set is called as an interior point. We define a residual vector for all the equality conditions in Eq. (A5) as $r = [r_1^T, r_2^T, r_3^T, r_4^T]^T$, where

$$
\begin{aligned}
r_1 &= c_0 + A_0^T v + \sum_i \lambda_i c_i + \sum_i \lambda_i' c_i', \\
r_2 &= -\Lambda f, \\
r_3 &= -\Lambda' f', \\
r_4 &= A_0 z - \bar{y},
\end{aligned}
\tag{A7}
$$

$f = [f_1(z), f_2(z), ..., f_N(z)]^T$, $f' = [f_1'(z), f_2'(z), ..., f_N'(z)]^T$, and Λ, Λ' are diagonal matrices with $(\Lambda)_{ii} = \lambda_i$ and $(\Lambda')_{ii} = \lambda_i'$, respectively. To find the solution of (A5), we linearize the residual vector r using Taylor expansion about $(z, v, \lambda, \lambda')$, which gives

$$
\begin{aligned}
& r(z + \Delta z, v + \Delta v, \lambda + \Delta\lambda, \lambda' + \Delta\lambda') \\
&= r(z, v, \lambda, \lambda') + J(z, v, \lambda, \lambda') \begin{pmatrix} \Delta z \\ \Delta v \\ \Delta\lambda \\ \Delta\lambda' \end{pmatrix},
\end{aligned}
\tag{A8}
$$

where J is the Jacobian matrix of r

$$
J = \begin{pmatrix} \mathbf{0} & A_0^T & C^T & C'T \\ -\Lambda C & \mathbf{0} & -F & \mathbf{0} \\ -\Lambda' C' & \mathbf{0} & \mathbf{0} & -F' \\ A_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},
\tag{A9}
$$

the $N \times 2N$ matrices C and C' have c_i and c_i' as rows, respectively, and F and F' are diagonal with $(F)_{ii} = f_i(z)$ and $(F')_{ii} = f_i'(z)$. Thus, one can obtain the descent direction by setting the left-hand side of Eq. (A8) to be zero, which gives

$$
\begin{pmatrix} \Delta z \\ \Delta v \\ \Delta\lambda \\ \Delta\lambda' \end{pmatrix} = -J^{-1} r.
\tag{A10}
$$

With the descent direction, to solve (A5), one can update the solution by $z = z + s\Delta z, v = v + s\Delta v, \lambda = \lambda + s\Delta\lambda, \lambda' = \lambda' + s\Delta\lambda'$ with step length s. Here, s should guarantee that $(z + s\Delta z, v + s\Delta v, \lambda + s\Delta\lambda, \lambda' + s\Delta\lambda')$ is an interior point of valid solution set (A6). By iterating this procedure, the reconstructed sparse signal $\bar{x}$ can be obtained.

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