Transient chaos in optical metamaterials

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We investigate the dynamics of light rays in two classes of optical metamaterial systems: (1) time-dependent system with a volcano-shaped, inhomogeneous and isotropic refractive-index distribution, subject to external electromagnetic perturbations and (2) time-independent system consisting of three overlapping or non-overlapping refractive-index distributions. Utilizing a mechanical-optical analogy and coordinate transformation, the wave-propagation problem governed by the Maxwell's equations can be modeled by a set of ordinary differential equations for light rays. We find that transient chaotic dynamics, hyperbolic or nonhyperbolic, are common in optical metamaterial systems. Due to the analogy between light-ray dynamics in metamaterials and the motion of light in matter as described by general relativity, our results reinforce the recent idea that chaos in gravitational systems can be observed and studied in laboratory experiments. © *2011 American Institute of Physics*. [doi:10.1063/1.3623436]

Optical metamaterials, also referred to as negative refractive-index materials, are artificially designed materials with unconventional properties that natural materials do not typically possess. Although the concept of metamaterials was proposed theoretically in 1968, explosive growth in research occurred only about a decade ago, where now the area has become one of the most active, interdisciplinary fields. Significant applications include superlens overcoming the optical diffraction limit and electromagnetically invisible materials. Quite recently, a correspondence of light-ray dynamics in optical metamaterials to general gravitational systems was suggested and signatures of chaos were revealed, opening the avenue to explore fundamental phenomena in gravitational physics, which otherwise would not have been possible to be tested, in laboratory experiments. In this paper, we further probe chaos in metamaterial systems. Through systematic computations of light-ray trajectories in two classes of systems, one time-dependent and another time-independent, we establish the existence of transient chaotic dynamics, both hyperbolic and nonhyperbolic, in these systems. In light of the analogy between metamaterial optics and gravitational physics, our results suggest that transient chaos can be quite common in gravitational systems obeying Einstein's general relativity.

I. INTRODUCTION

Metamaterials are artificially designed, engineered, and fabricated structures possessing special (unconventional) properties that may not be readily available from natural materials. The last decade has witnessed an explosive growth of research on metamaterials in terms of both fundamental physics and potential applications. A primary research interest in metamaterials lies in their electromagnetic and optical properties. In this regard, negative refractive-index materials,[1–6] also referred to as left-handed media, are one of the most extensively investigated types of metamaterials. First conceived theoretically by Veselago[7] in 1968, this extraordinary material with both negative effective permittivity and permeability exhibited a remarkable potential for a variety of applications. For example, superlens[8] made of doubly negative metamaterials[9] can overcome the diffraction limit for conventional lenses and make subwavelength imaging possible. Another important application is invisible materials, where special cloak was realized in recent experiments for electromagnetic wave at optical frequencies,[10,11]

Quite recently, a link between metamaterial optics and celestial mechanics was proposed,[12] making it possible to investigate an array of gravitational phenomena predicted by Einstein's general relativity using optical analogies in the laboratory. For example, in general relativity, light can be trapped in some specific region in the space where a massive gravitational body exists, but such a trapping can be realized using metamaterials, generating an artificial "black hole" in the laboratory.[12] From this analogy, insights into the design of novel optical cavities and photon traps can be gained, with applications in areas such as micro-cavity lasers.

In this paper, we first study the light-ray dynamics in a class of inhomogeneous, isotropic optical metamaterials in the presence of a periodic, external electromagnetic perturbation. The driving, analogous to, e.g., a third-body perturbation in classical mechanics, provides a way to break the stable periodic orbits of light ray in the corresponding static material, making complex dynamics possible. Indeed, Ref. 12 predicted the appearance of chaotic dynamics in this class of systems. We then study a class of time-independent metamaterial systems with overlapping or non-overlapping refractive-index distributions. For both time-dependent and time-independent systems, we find that transient chaotic dynamics (or chaotic scattering dynamics)[13–17] of light rays are common. This means that, two incident light rays differing

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only slightly in initial conditions can exit the metamaterial system through drastically different states.[18] Besides providing direct evidence for transient chaos, we shall establish through computations the dynamical nature of the process, hyperbolic or nonhyperbolic (to be explained below). Due to the analogy between metamaterial optics and gravitational physics, our results suggest that transient chaos can be quite common in gravitational systems obeying Einstein's general relativity. In addition, since ray dynamics can be experimentally observed and investigated in optical metamaterials, our results reinforce the idea that chaotic dynamics in relativistic gravitational systems can be visualized and studied in laboratory experiments.[19]

In Sec. II, we describe the equations of motion governing the dynamical behavior of light rays in optical metamaterials. In Sec. III, we demonstrate transient chaos in metamaterial systems with time-dependent refractive index. In Sec. IV, we present evidence of transient chaos in time-independent systems. A brief conclusion is offered in Sec. V.

II. EQUATIONS OF MOTION

In general relativity, the geometry of the space is described by the four-dimensional space-time metric $g_{\mu\nu}(\mathbf{x}, t)$. The propagation of light rays in this empty but curved spacetime, which follows the natural geodesic lines, is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}[g_{00}(\mathbf{x}, t)\dot{t}^2 - g_{ij}(\mathbf{x}, t)\dot{x}^i\dot{x}^j]. \tag{1}$$

Here, the Einstein summation convention is used for spatial coordinate indices i and j, and the speed of light is normalized to $c = 1$. To relate the light propagation in curved space to that in the composite material, one needs to perform coordinate transformations[20] to Maxwell's equations. For isotropic media, an effective refraction index can be defined as $n = \sqrt{g/g_{00}}$, where $g = g_{ii}$ ($i = 1$, 2, and 3). Here we consider media with centro-symmetric effective refractive index n, and light-ray trajectories in the system can be further confined to the plane $\mathbf{r} = (x, y)$ due to the nature of planar motions of light rays in a centrally symmetric potential, as an orthogonal transformation always exists which brings the z-axis to being perpendicular to the plane of motion.

We now demonstrate that, after an appropriate coordinate transformation, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}[\dot{t}^2 - n^2(\rho, t)(\dot{\rho}^2 + \rho^2\dot{\phi}^2)], \tag{2}$$

where $\rho = |\mathbf{r}|$, ϕ denotes the azimuthal angle, and the derivatives are with respect to the proper time τ. In particular, the key to the optical-mechanical analogy is the invariance of the Maxwell's equations under coordinate transformations. It was demonstrated[20] that the general covariant form of the free-space Maxwell's equations

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0,$$
$$F^{\mu\nu}_{\ ,\nu} = \mu_0 J^\mu$$

is equivalent to the constitutive equations

$$\mathbf{D} = \varepsilon_0\varepsilon\mathbf{E} + c^{-1}\mathbf{w} \times \mathbf{H}$$
$$\mathbf{B} = \mu_0\mu\mathbf{H} - c^{-1}\mathbf{w} \times \mathbf{E}.$$

Here, the optical medium has the permittivity and permeability tensors $\varepsilon = \mu = \sqrt{-g}g^{ij}/g_{00}$, where $g = \det(g_{\mu\nu})$ and $w_i = g_{0i}/g_{00}$. Consider now light-ray motion in a special curved space-time metric $g_{\mu\nu}$ with the line element

$$ds^2 = g_{00}dt^2 - g_{ii}dx^i dx^i. \tag{3}$$

We can perform the coordinate transformation $x'^i = h_i x^i$ (no summation) that relates it to the Minkowski space, where $h_i = \sqrt{g_{ii}}$, denotes the Lamé coefficients of the transformation. To take into account the effect of the time-dilation factor $\sqrt{g_{00}}$, we use the normalized field quantities $E'^i = \sqrt{g_{00}}E^i$ and $H'^i = \sqrt{g_{00}}H^i$, which are observable in experiments. The material properties in this case can be written as

$$\varepsilon_{ij} = \mu_{ij} = h_1h_2h_3\delta_{ij}/(h_i\sqrt{g_{00}}). \tag{4}$$

For centrally symmetric space-time metric the line element can be expressed using the spherical coordinates as

$$ds^2 = g_{00}dt^2 - g_{\rho\rho}(d\rho^2 + \rho^2 d\Omega^2). \tag{5}$$

By defining an effective refractive index $n = \sqrt{g_{\rho\rho}/g_{00}}$, we arrive at the transformed Lagrangian (2). This particular refractive index could be realized experimentally using pure dielectric materials that are non-dissipative and non-dispersive.

By reversing such coordinate transformation, the light-ray motion in the material in flat space-time (as the case in a laboratory) is equivalent to that in empty but curved space-time, which can be studied by using the Euler-Lagrangian equations. We obtain

$$\begin{aligned} \rho' &= v, \\ v' &= [v^2 + \gamma(\rho, t)(\rho^2 + v^2)]/\rho, \\ t' &= n(\rho, t)(\rho^2 + v^2)^{1/2}, \end{aligned} \tag{6}$$

where $\gamma(\rho, t) = 1 + \rho\partial_\rho \ln n(\rho, t)$, $v \equiv d\rho/d\phi$, and all derivatives marked by prime are with respect to the azimuthal angle ϕ, e.g., $v' \equiv dv/d\phi$. For static media [i.e., $\partial_t n(\rho, t) = 0$], a recent stability analysis[12] provided a sufficient condition for having stable orbits inside the potential: $d\gamma(\rho)/d\rho \leq 0$, leading to a well-behaved refractive-index function

$$n(\rho) \sim \exp\left[\int d\rho\gamma(\rho)/\rho\right]/\rho. \tag{7}$$

A convenient case is given by the condition

$$n(\rho) = n_0(\rho/a)\exp(-2\rho/a),$$

where a is the radius of the circular orbit and n_0 is a constant characterizing the maximum refractive index $n_{max} = n_0/(2e)$. Figure 1 presents a schematic illustration of the static effective refraction index with an outer boundary ρ_{max}, outside which the refractive index $n_s(\rho)$ is truncated to unity. There are many periodic light-ray orbits [Figs. 1(b) and 1(c)] within

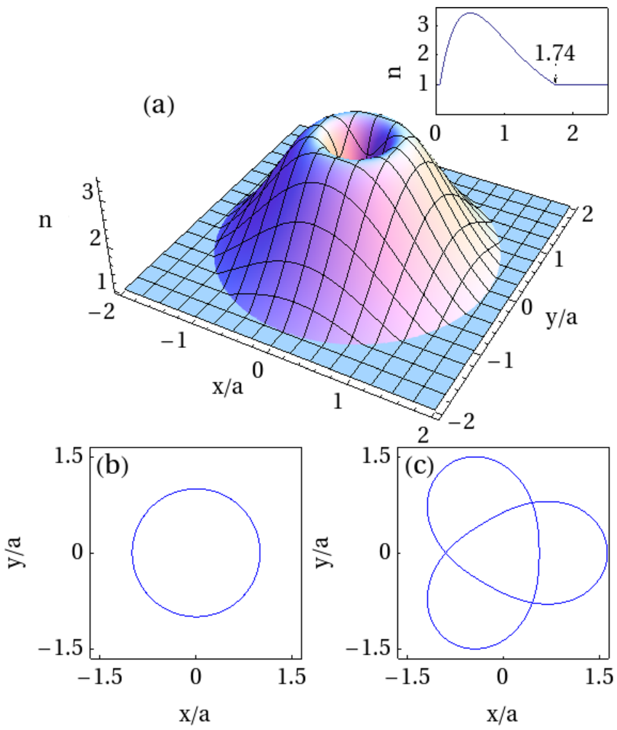

FIG. 1. (Color online) (a) Static effective refractive index and (b,c) stable periodic light ray orbits. In (a), the refractive index is truncated to unity for $\rho > \rho_{max} = 1.74a$, and the parameters are $n_0 = 18.65$ and $a = 15$ μm. In (b), the periodic orbit has the period 2π and radius a. In (c), the periodic orbit has the period 4π. The variables x and y are the Cartesian coordinates in two dimensions, which hold for subsequent figures in the paper.

the interaction region $\rho < \rho_{max}$ in this static system, making it a high quality optical cavity. However, in the presence of a time-dependent, external electromagnetic source, light-ray trajectories inside the cavity become more complicated. The time-dependent refractive index can be denoted as $n(\rho, t) = n(\rho) + \Delta n(t)$, where $\Delta n(t) = \xi n_{max} \sin(\omega t)$, ξ, and ω are the strength and the angular frequency of the perturbation, respectively.

A plausible scheme to realize the electromagnetic perturbation is as follows. In general, the refractive index of a material is controlled by its electric permittivity $\left(\varepsilon_{eff}\right)$ and magnetic permeability (μ_{eff}), which are basic quantities characterizing metamaterials and are functions of the frequency of propagating, polarized electromagnetic wave. The refractive index can thus be modulated by varying one of the parameters. For example, suppose we have a thin plate ($x-y$ plane) of optical metamaterial and a beam of polarized incident light, whose magnetic and electric components are in the z-direction and confined within the $x-y$ plane, respectively. The light beam can propagate within the $x-y$ plane and change its direction. According to the definitions of ε_{eff} and μ_{eff}: $\langle D\rangle \equiv \varepsilon_{eff}\langle E\rangle$ and $\langle B\rangle \equiv \mu_{eff}\langle H\rangle$, where $\langle\cdot\rangle$ denotes the average over one unit cell of the metamaterial structure, we cannot control ε_{eff} by adding a time-varying external electric field in x- or y-direction. This is due to the fact that the light beam keeps changing its direction during propagation. However, an external periodic magnetic field can be applied in the z-direction. If the field strength H_0 is small, its effect on the material property can be captured by first-order approximation, regardless of the functional form of $\mu_{eff}(\omega)$, where ω is the light frequency. In particular, we have $\mu = (\langle B\rangle + B_0)/\ \ (\langle H\rangle + H_0)\mu_{eff} + B_0/\langle H\rangle$. If we choose $B_0 \sim \sin\ (t)$, then the permeability and consequently the refractive index can be made to vary periodically with time.

III. TRANSIENT CHAOS IN SYSTEMS WITH TIME-DEPENDENT REFRACTIVE INDEX

Two parameters characterizing the initial conditions of the ray dynamics are the impact parameter b and the time t_{enter} at which the light ray enters the interaction region defined as $\rho < \rho_{max}$. The parameter t_{enter} is necessary to completely determine the trajectories of light rays because the refractive index is time-dependent. The initial time t_{enter} can affect the light-ray trajectories inside the interaction region even when two beams of light are launched toward the interaction region with the same impact parameter. The two initial conditions can be conveniently defined as follows. The center of the potential is set at the origin in the plane. The light rays are sent from far field ($\rho > \rho_{max}$) in the x direction toward the center of the interaction region. The impact parameter is then $b \equiv |y|$. The time t_{enter} then marks the instant when the light ray reaches the circular outer boundary. Since the external perturbation $\Delta n(t)$ has the period $T = 2\pi/\omega$, it is convenient to use t_{enter} [mod(T)] as the entering time. Figures 2(a) and 2(b) show two parallel but closely separated incoming light-ray trajectories entering the interaction region at the same time, where the resulting trajectories inside the interaction region are also close. However, for another pair of nearby incident beams entering simultaneously, the trajectories are completely different, as shown in Figs. 2(c) and

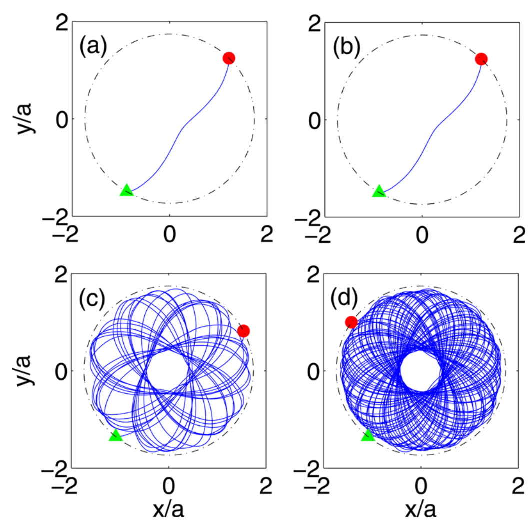

FIG. 2. (Color online) Scattering trajectories from two pairs of nearby initial conditions. The dash-dotted circles mark the outer boundary of the static refractive index potential, and the (green) triangles and (red) circles mark the incoming and outgoing positions of light ray at the boundary, respectively. All incident photons are sent from the $-x$ to $+x$ direction and the impact parameter is $b \equiv |y|$. The upper two panels (a: $b = 1.50002a$, $t_{enter} = 0.774T$) and (b: $b = 1.50003a$, $t_{enter} = 0.774T$) show two nearby incident positions with similar outgoing photon trajectories, while the bottom two panels (c: $b = 1.35995a$, $t_{enter} = 0.854T$) and (d: $b = 1.35996a$, $t_{enter} = 0.854T$) show two nearby incident rays with drastically different outgoing trajectories, indicating a sensitive dependence on initial conditions. The parameters are $a = 15$ μm, $\omega = 6c/a$, and $\xi = 0.2$.

2(d). The light-ray trajectories can thus be highly sensitive to the initial conditions, suggesting the emergence of transient chaos.

We now explore the dynamics of light rays in terms of the scattering functions, which are some quantities characterizing the rays after the interaction versus the impact parameter. In this regard, it is necessary to compensate the influence of the entering time t_{enter} on the light trajectory, which can be quite significant if incident light rays are launched, e.g., from a line segment perpendicular to the incident direction. To achieve this, we send the light rays from an arc of radius ρ_{max} such that the wave front of the incident light beam coincides with the outer boundary of the refractive index potential if the time-dependent term $\Delta n(t)$ were not present. In such an arrangement, the incident light rays with different impact parameters enter the interaction region at the same time, and so the scattering function can be obtained with respect to a single input variable, i.e., the impact parameter. The light rays stay in the region for a certain amount of time t_{delay} and then leave the interaction region. The output variables, the angle $\theta(b)[\mathrm{mod}(2\pi)]$ and delay time $t_{delay}(b)$ $[\mathrm{mod}(T)]$, are then plotted as functions of impact parameter b, as shown in Figure 3. These plots are characteristic of transient chaos in open Hamiltonian systems.[13]

There are two types of transient chaos in open Hamiltonian systems: hyperbolic[21] and nonhyperbolic.[22–26] In hyperbolic systems, all periodic orbits are unstable and the decay of particles from the interaction region is exponential.[21] In contrast, in nonhyperbolic dynamics, there are Kolmogorov-Arnold-Moser (KAM) tori and nonattracting chaotic sets coexisting in the phase space, and the particle decay is algebraic.[22,23] To determine the nature of transient chaos of optical rays in metamaterials, we compute and analyze the phase-space structure. In particular, without the time-dependent perturbation, there are two stable periodic orbits in the phase space, as shown in Figs. 1(b) and 1(c). In fact, if the refractive index $n(\rho) = n_0(\rho/a)e^{-2\rho/a}$ was not truncated for $\rho > \rho_{max}$, it can be less than unity. In that case, more periodic orbits of periods of multiples of 2π can exist, e.g., the third possible periodic orbit has period 6π. Physical reality requires, however, $n_s(\rho) > 1$ so that the truncation is necessary. Besides periodic orbits of periods 2π and 4π, there are an infinite number of quasiperiodic orbits in the interaction region. When the time-dependent perturbation is turned on, unstable periodic orbits are created, some of these quasiperiodic orbits survive, forming KAM tori, and nonattracting chaotic sets arise through the typical mechanism of homoclinic/heteroclinic intersections between the stable and unstable manifolds of the unstable periodic orbits. As the intensity ξ of the perturbation is increased, the regions containing the KAM tori shrink and the chaotic regions become more extensive in the phase space. A typical phase-space structure is shown in Fig. 4(a), where we observe both KAM tori and chaotic regions surrounding a central KAM island. Transient chaos is thus nonhyperbolic in this case. We note that, an analogous class of systems in classical mechanics exists, namely, soft-wall billiards with repulsive potentials, for which certain rigorous results on chaotic dynamics are available.[24–26]

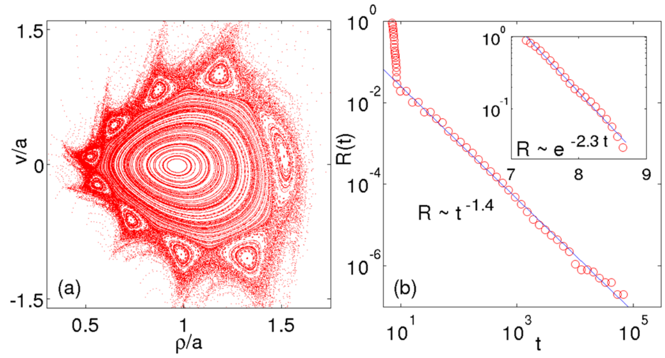

FIG. 4. (Color online) (a) Phase-space structure on a Poincaré surface of section for $\xi = 0.2$. There are both KAM islands and chaotic regions, indicating nonhyperbolic transient chaos. The data points are sampled at $t = mT$ for $m \in \mathbb{N}$. (b) Fraction of light rays remaining in the interaction region as a function of time. We see that the decay is mostly algebraic, except for the initial small-time interval where the decay is exponential, as demonstrated by the plot in the inset.

That the transient chaotic dynamics is nonhyperbolic can be further verified by examining the decay law of light rays. In particular, we define $R(t)$ to be the fraction of a large number of light rays (or photons in the short wavelength limit) still remaining in the interaction region $\rho < \rho_{max}$ at time t. Because of the time-dependent nature of the refractive index, we launch a large number of incident light rays successively and uniformly distributed in one period T of the external perturbation $\Delta n(t)$. The decay law of the light rays is shown in Fig. 4(b), where we see that $R(t)$ decreases exponentially for small t but algebraically for most of the time interval considered. We have, for $t < 8.7$, $R(t) \sim e^{-\alpha t}$ where $\alpha \approx 2.3$, and for $t \geq 8.7$, $R(t) \sim t^{-\beta}$ where $\beta \approx 1.4$.

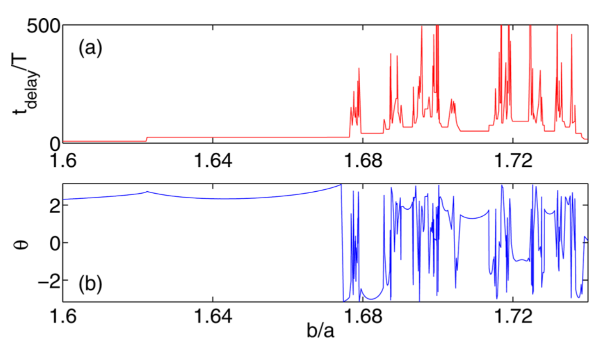

FIG. 3. (Color online) (a) Delay-time function $t_{delay}(b)[\mathrm{mod}(T)]$ and (b) angle function $\theta(b)[\mathrm{mod}(2\pi)]$ for $\xi = 0.2$.

IV. TRANSIENT CHAOS IN SYSTEMS WITH TIME-INDEPENDENT REFRACTIVE INDEX

To demonstrate the generality of transient chaos in optical-metamaterial systems, we now consider a class of systems in which the refractive index is time-independent. In contrast to the time-dependent case where chaos has been uncovered previously,[12] there has been no study of chaos in time-independent metamaterial systems. Our system consists of three equally spaced, static, and volcano-shaped refractive

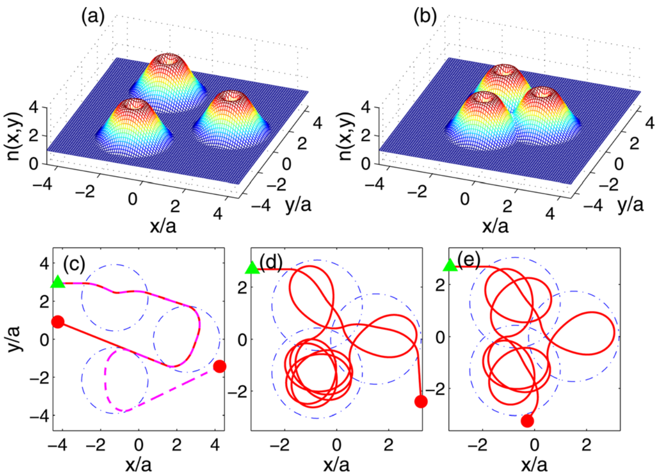

FIG. 5. (Color online) Time-independent effective-index distribution for (a) three separated potentials $d = (2.5/\sqrt{3})r_{max}$, and (b) three overlapping potentials $d = (1.5/\sqrt{3})r_{max}$. Sensitive dependence of light ray trajectories on initial conditions is shown in (c-e). The dash-dotted circles, (green) triangles and (red) circles have the same meaning as in Fig. 2. For the non-overlapping potential case (a), we show two trajectories in (c). The trajectory marked by the solid lines is for $b = 2.92724527$ and the one marked by dashed line is for a slightly different b value (increased by 10^8). In (d) $b = 2.69012$, and (e) $b = 2.69013$, two distinct trajectories from two close impact parameters for the overlapping potential case (b) are shown, respectively.

index potentials, as shown in Figs. 5(a) and 5(b). The Lagrangian of the system is of the same form as Eq. (2) except that now the refractive index is constant. Moreover, due to the loss of the central symmetry in the potential, an additional dynamical variable θ_v, the velocity angle with respect to the $+x$ direction, is needed to describe the dynamics. We obtain

$$\dot{x} = \cos\theta_v / n(x, y), \tag{8}$$
$$\dot{y} = \sin\theta_v / n(x, y),$$
$$\dot{\theta}_v = (d\hat{\mathbf{v}}/d\theta_v) \cdot \nabla n / n^2,$$

where $\hat{\mathbf{v}}$ is the unit vector in the velocity direction, and the derivatives are with respect to any affine parameter. In this system, the characteristics of transient chaos can be quite different in terms of whether the three refractive index potentials overlap. Figures 5(a) and 5(b) show two different configurations of the potentials. In Fig. 5(a), where the potentials are spatially separated, there are stable periodic orbits in each potential region and unstable orbits circling the three potentials. In the overlapping case [Fig. 5(b)], some of the stable orbits are destroyed, giving rise to complicated trajectories. In this case, light ray trajectories can bounce back and forth between the original stable orbits within a single potential region and the unstable orbits that connect the three potentials, forming new unstable periodic orbits.

For the time-independent case, two convenient dynamical variables characterizing the transient dynamics are the impact parameter and the angle of incident light ray. To be concrete, we focus on scattering functions and the decay law with respect to variation in the impact parameter. An example of sensitive dependence of the trajectories on initial conditions is shown in the bottom panels of Fig. 5. In the non-overlapping potential case, the stable orbits inside each potential region cannot be reached by the trajectories starting from outside, i.e., the stable and the unstable orbits are well separated. Figure 5(c) shows two distinct trajectories from two extremely closed initial impact parameters. For the overlapping case, Figs. 5(d) and 5(e) show two different trajectories from two nearby impact parameters. One can still see that the light ray encircles around the original stable orbits within the single potential region but finally leaves, due to the fact that the overlapping regions break the original stable orbits and connect them to the regions outside.

Typical scattering functions, for which the incident angle of the light rays is fixed to be along the $+x$ direction, and the associated light-ray decay law are shown in Fig. 6. We observe typical features of transient chaos. As shown in Fig. 6(c), the decay law is exponential in this case, indicating the hyperbolic nature of the transient chaotic dynamics. The physical reason is that, since the potentials are non-overlapping, the stable and unstable periodic orbits are well separated in the phase space. Since decay law is meaningfully defined by light rays from outside the interaction region in all directions with random impact parameters, the KAM islands surrounding the stable periodic orbits are isolated from the regions outside and so are inaccessible to these rays, as shown in Figs. 7(a) and 7(b), respectively. For the overlapping-potential case, the dynamics is nonhyperbolic, as demonstrated by the phase-space structures shown in Figs. 7(c) and 7(d). In this case, three potentials penetrate into each other so that the originally inaccessible KAM islands are now accessible to light rays initiated afar from the interaction region, leading to an algebraic decay law. There is then a crossover from exponential to algebraic decay as a system parameter changes so that the refractive-index potentials begin to overlap with each other, a known phenomenon in chaotic scattering in potential systems.[27,28]

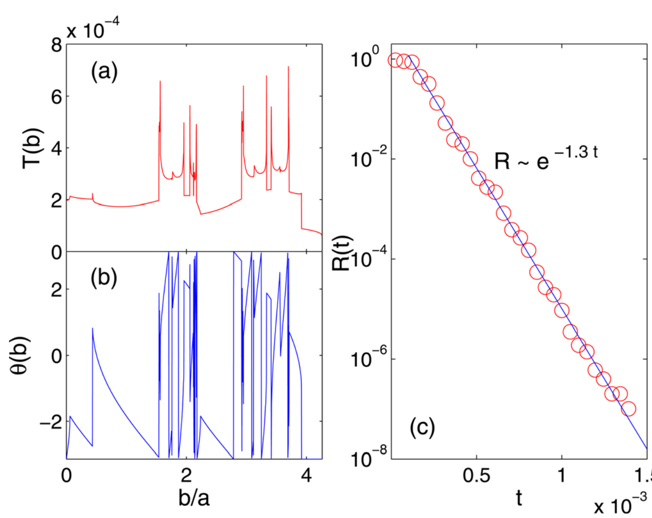

FIG. 6. (Color online) For the non-overlapping refractive-index potential case $\left(d = (2.5/\sqrt{3})r_{max}\right)$, (a,b) scattering functions and (c) exponential light-ray decay law.

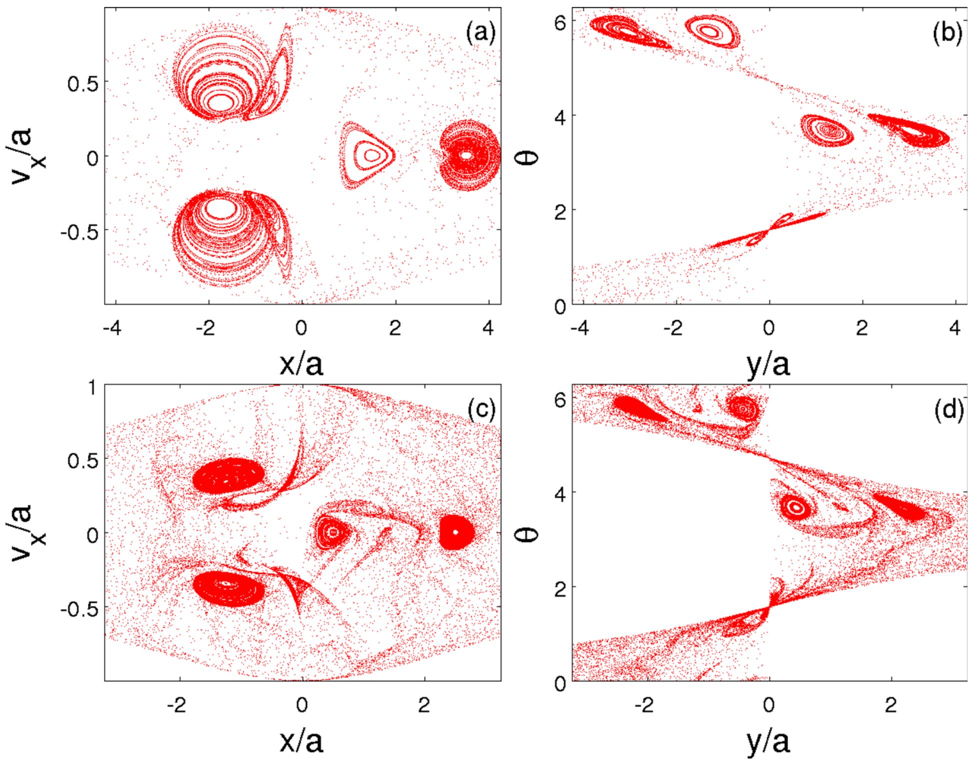

FIG. 7. (Color online) Phase-space structure for the non-overlapping (a,b) and overlapping (c,d) potential cases. The points are sampled when the direction of velocity (represented by θ_v) forms $\pi/2$ angle with respect to the polar angle φ. This angle can be adjusted so that the resulting phase-space diagram looks similar but with each KAM island rotated.

V. CONCLUSION

We have demonstrated transient chaotic dynamics of light rays in optical metamaterials under time-dependent perturbations, which can be realized by an external electromagnetic field. The transient dynamics is typically nonhyperbolic in this case. We have also demonstrated that, even without the external time-dependent perturbations, transient chaos can arise from a class of static refractive-index potential configurations. To our knowledge, chaos in optical metamaterial systems with static refractive index has not been observed previously. A rigorous mathematical understanding of the chaotic dynamics in optical metamaterials is not available at the present, but insights can be obtained from previous mathematical works on chaos in soft-wall billiards.[24–26] Based on the recently established connection between optical metamaterial and relativistic gravitational systems,[12] our results reinforce the idea that complex chaotic dynamics in the latter can potentially be observed and tested in laboratory experiments using optical metamaterials.

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[18]In chaotic scattering, a scattering function contains an uncountably infinite number of singularities. Near any of the singularities, an arbitrarily small change in the input variable can cause a large change in the output variable. This is a sensitive dependence on initial conditions that signifies the appearance of chaos. Dynamically, chaotic scattering is due to the existence of non-attracting chaotic sets in the phase space (Refs. 13, 17). It is generally understood now that chaotic scattering is the physical manifestation of transient chaos (Ref. 29) in open Hamiltonian systems.
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