Onset of synchronization in complex gradient networks

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Recently, it has been found that the synchronizability of a scale-free network can be enhanced by introducing some proper gradient in the coupling. This result has been obtained by using eigenvalue-spectrum analysis under the assumption of identical node dynamics. Here we obtain an analytic formula for the onset of synchronization by incorporating the Kuramoto model on gradient scale-free networks. Our result provides quantitative support for the enhancement of synchronization in such networks, further justifying their ubiquity in natural and in technological systems.
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One important issue in the study of network synchronization is about how to optimize the network configuration, by adjusting either the network topology or the network couplings, in order of a higher synchronization propensity. For a network of fixed topology and total coupling cost, the optimization of network synchronization relies on only the redistribution of the weight and direction of the network couplings. Previous studies on this had been mainly carried out for networks of identical node dynamics by the standard approach of eigenratio analysis, where a number of configuration criteria had been proposed to improve the network synchronizability. As node dynamics in realistic networks is typically nonidentical, a natural question, therefore, is whether the configuration criteria established in identical networks are still workable for nonidentical networks. Meanwhile, realistic networks generally consist of a large amount of nodes, which makes the eigenratio approach infeasible in practice. So the other question is how to evaluate analytically the improved network synchronization brought by the configuration criteria. Here, by investigating the onset synchronization of a generalized Kuramoto model, we show analytically how the adoption of gradient coupling, an efficient coupling scheme discovered in identical networks, is helpful in enhancing the synchronization of nonidentical networks.

I. INTRODUCTION

There has been much recent interest in synchronization in complex networks,[1–9] motivated by the observation that synchronization is fundamental to various processes in nature that involve the interactions of many interconnected components.[10] Currently there are two approaches to the network-synchronization problem. The first is based on analyzing the eigenvalue spectrum of the coupling matrix of the network under consideration. By assuming the idealized setting where the dynamical processes on individual nodes are identical across the entire network, a synchronization state can be conveniently defined and its stability with respect to external, desynchronous perturbations can then be determined. An elegant result coming out of this approach is that, given specific node dynamics, a network is more likely to achieve synchronization if the spread of the eigenvalues in the underlying coupling matrix is smaller.[2] In the past several years, this approach has been applied to analyzing the synchronizability of small-world networks,[1–3] scale-free networks,[4] weighted complex networks,[5,6] complex clustered networks,[8] and complex gradient networks.[9] While there is in principle no restriction on the complexity of the individual node dynamics in this approach, analysis becomes possible due to the assumption of identical node dynamics. One issue is thus what might happen when this assumption is relaxed to a certain extent. If there is heterogeneity in the node dynamics, a simple synchronization state cannot be defined. In this case, to make the problem tractable and the analysis feasible, it is necessary to reduce the complexity of the local dynamics. This leads to the second approach: to adopt the classical Kuramoto model[11,12] of globally connected networks to complex networks.[13] The individual node dynamics in the Kuramoto model is extremely simple, as it is given by that of a uniform rotation: $\dot{\theta} = \omega$, where θ is a phase variable and ω is the frequency. Heterogeneity in the local dynamics can be realized by choosing different frequencies for different nodes. Usually the frequencies are chosen randomly from a prescribed distribution.[11,12] This approach has yielded explicit formulas describing the transition to synchronization in scale-free networks.[13] In particular, as a coupling parameter ε is increased through a critical value ε_c, partial synchronization occurs, where nodes begin to form distinct, synchronous clusters. The value of ε_c depends explicitly on statistical properties of both the degree distribution of the underlying network and the frequency distribution of the oscillators.[13] As ε is increased further through another criti-

cal value ε_c', full or global synchronization occurs where all nodes in the network are synchronized. Recently, the Kuramoto-model approach has been extended to predicting ε_c' (Ref. 14) for complex clustered networks.[15]

A hallmark of modern network science is the discovery of the ubiquity of scale-free networks in nature and in man-made systems,[16,17] in addition to the identification of small-world property in complex networks.[18] However, initial comparison study of network synchronizability indicates that a scale-free network, while having smaller network distances than a small-world network of the same size, can actually be more difficult to synchronize.[4] While this somewhat counterintuitive result can be understood as the consequence of a blockade of communication on the network by a small set of hub nodes in the network, the ubiquity of scale-free networks and the importance of synchronization in network functions seems to have generated a paradox. Since the setting under which the result is obtained is unweighted and undirected scale-free networks,[4] a possible resolution to the paradox is to consider weighted and directional interactions. Indeed, several recent works have shown that coupling schemes incorporating weights and directionality can be articulated so that scale-free networks so designed have a stronger synchronizability than homogeneous networks of comparable parameters.[5–7,9] For example, we have constructed a general class of gradient networks to account for both the directionality and asymmetry.[9] The idea is that an asymmetrical and weighted network can be regarded effectively as the "superposition" of a symmetrically coupled weighed network and a directed weighed network, but a weighted, directed network is in fact a gradient network,[19,20] a network for which the interactions among nodes are determined by some gradient field. A few basic considerations of dynamics on realistic networks lead to the construction of some appropriate gradient field, which in turn give rise to scale-free networks that are significantly more synchronizable. Note that most of the recent works[5–7,9] along these lines assume identical node dynamics and are based on eigenvalue analysis to infer the network synchronizability. For the few studies where networks of nonidentical node dynamics are considered, the results are based on direct simulations and are mainly used to verify the findings obtained in identical networks.

In this paper, we investigate the onset of synchronization in complex gradient networks in the Kuramoto framework. There are two aims: (i) to provide analytic support for the recent result[9] that gradient field can enhance the network synchronizability, and (ii) to infer the generality of the result by incorporating heterogeneity in node dynamics. The tractability of the Kuramoto paradigm allows us to obtain an analytic formula for the critical coupling required for the onset of synchronization as a function of the strength of the gradient field, which is verified by numerical results. This result provides a more solid footing that scale-free networks can be quite synchronizable, giving further justification for their ubiquitous appearance in large networked systems in nature and technology.

In Sec. II, we describe our gradient network model and derive a formula for the critical coupling parameter required

for the onset of synchronization. In Sec. III, we provide numerical verifications. A brief discussion is presented in Sec. IV.

II. GRADIENT SCALE-FREE NETWORKS AND FORMULA FOR THE ONSET OF SYNCHRONIZATION

A. Gradient scale-free networks

A theoretical framework to encompass synchronization in gradient networks can be quite general. In fact, most scenarios of network synchronization considered so far[2–7] can be understood from the gradient-network point of view.

In the existing random gradient-field model,[19,20] once a field is established, the link between a pair of nodes is purely directed. For example, if the gradient field stipulates a link from node i to node j, it is directed in the sense that only i can influence j, but not the other way around. However, communication and/or interactions between two nodes in the network can in general be bidirectional, although the coupling strength can depend on the direction of interaction. It is thus necessary to consider the situation in which, for instance, the link from node n to node m carries more weight than the link from m to n. The links can therefore be highly asymmetrical, and so is the coupling matrix. Such a network can be regarded as the superposition of two subgraphs: one undirected, symmetric network and another unidirected gradient network. In particular, let C_{nm} be the coupling from node m to node n and we have $C_{nm} \neq C_{mn}$. Defining $\Delta C_{nm} \equiv C_{nm} - C_{mn}$, we can write $C_{nm} = (C_{nm} + C_{mn})/2 + \Delta C_{nm}/2$, where the first term is a symmetrical coupling, and the second term represents a directed coupling. Since $\Delta C_{nm} = -\Delta C_{mn}$, the direction of the coupling is defined to be from node m to n if $\Delta C_{nm} > 0$ and vice versa. The original network can thus be regarded as being composed of a symmetrical network characterized by the symmetrical coupling term, and a gradient network represented by ΔC_{nm}. Both networks are weighted since the coupling value depends on the indices n and m.

The class of gradient networks to be treated in this paper is constructed as follows. We start from the adjacent matrix $A = \{a_{nm}\}$, where $a_{nm} = 1$ if there is a link between nodes n and m and 0 otherwise, and $a_{nn} = 0$. The degree of node n is thus $k_n = \Sigma_m a_{nm}$. From the adjacent matrix A, we set $s_{nm} = a_{nm}(1 + g)$ if $k_n < k_m$, and $s_{nm} = a_{nm}(1 - g)$ if $k_n > k_m$. We then obtain the gradient matrix $S = \{s_{nm}\}$, where g characterizes the strength of the gradient. The coupling matrix $C = \{c_{nm}\}$ is defined by $c_{nm} = k_n s_{nm} / \Sigma_j s_{nj}$. The coupling gradient from node m to node n, therefore, is $\Delta c_{nm} = c_{nm} - c_{mn}$. For positive g, the gradient points from nodes with large degrees to nodes with small degrees.

B. Kuramoto dynamics and order parameter

The Kuramoto model on a gradient network can be written as

$$\dot{\theta}_n = \omega_n + \varepsilon \sum_{m=1}^{N} c_{nm} \sin(\theta_m - \theta_n), \tag{1}$$

where θ_n and ω_n are the phase and natural frequency of oscillator n, respectively, and ε represents the overall coupling strength. In general, the frequency ω_n follows some probability distribution $\rho(\omega)$. For theoretical tractability, we assume that the network is densely connected and has a large size. A global order parameter characterizing the degree of coherence in the network is[13]

$$r \equiv \frac{\sum_{n=1}^{N} r_n}{\sum_{n=1}^{N} d_n^{in}}, \tag{2}$$

where the local order parameter r_n is given by

$$r_n e^{i\psi_n} \equiv \sum_{m=1}^{N} c_{nm} \langle e^{i\theta_m} \rangle_t \tag{3}$$

and

$$d_n^{in} \equiv \sum_{m=1}^{N} c_{nm} \tag{4}$$

is the total *incoming* coupling strength of node n. The critical coupling value ε_c for the onset of synchronization is defined to be the point where r starts to increase from 0. In Ref. [13], a general formula is provided for the order parameter for the coupling regime $\varepsilon \gtrsim \varepsilon_c$. It is

$$r^2 = \frac{1}{\alpha_1 \alpha_2^2} \frac{\langle d^{in} d^{out} \rangle^3}{\langle (d^{in})^3 d^{out} \rangle \langle d^{in} \rangle^2} \left(\frac{\varepsilon}{\varepsilon_c} - 1 \right) \left(\frac{\varepsilon}{\varepsilon_c} \right)^{-3}, \tag{5}$$

where $d_n^{out} = \sum_{m=1}^{N} c_{mn}$ denotes the total outgoing coupling strength of node n, and the two constants α_1 and α_2 are given by

$$\alpha_1 = 2/[\pi\rho(0)] \quad \text{and} \quad \alpha_2 = -\pi\rho''(0)\alpha_1/16, \tag{6}$$

which are determined by the first-order $[\rho(0)]$ and the second-order $[\rho''(0)]$ approximations of the frequency distribution, respectively. The critical coupling ε_c is given by

$$\varepsilon_c = \alpha_1 \frac{\langle d^{in} \rangle}{\langle d^{in} d^{out} \rangle}, \tag{7}$$

where $\langle \cdot \rangle$ denotes some ensemble average.

For our gradient-field network model, the total incoming coupling strength d^{in} and the total outgoing coupling strength d^{out} at each node assume real values and they are in general unequal. The main task of this paper is to investigate, analytically, how the balance between d^{in} and d^{out} will affect the onset of network synchronization.

C. Formula for onset of synchronization

Since d^{in} and d^{out} determine the onset of synchronization, we shall evaluate these quantities analytically. First, let us examine d^{in}. For node n, $d_n^{in} = \sum_m c_{nm} = \sum_m k_n s_{nm} / \sum_j s_{nj} = k_n$. Therefore, the total incoming coupling d^{in} is just the degree k. By definition, the total outgoing coupling of node n is $d_n^{out} = \sum_j c_{jn} = \sum_j k_j s_{jn} / \sum_l s_{jl}$. To work out d^{out}, rearrange the nodes by their degrees in descending order, i.e., $k_1 \geq k_2$

$\geq \cdots \geq k_N$. Thus the gradient matrix S becomes regular in the sense that the elements s_{nm} in the upper triangular region of the matrix ($m > n$) are $(1-g)$ or 0, depending on whether the two nodes are connected or not, and the elements in the lower triangular region are $(1+g)$ or 0. Denoting

$$\Omega_i = \sum_j s_{ij}/k_i,$$

we have

$$\begin{aligned}
\Omega_i &= \frac{1}{k_i} \sum_j s_{ij} \\
&= \frac{1}{k_i} \left[\sum_{j<i} a_{ij}(1+g) + \sum_{j>i} a_{ij}(1-g) \right] \\
&= \frac{1}{k_i} \cdot [k_i Q(k > k_i)(1+g) + k_i Q(k < k_i)(1-g)] \\
&= 1 + g[Q(k > k_i) - Q(k < k_i)],
\end{aligned}$$

where

$$Q(k) = kP(k)/\langle k \rangle$$

is the probability that the node has degree k if it is reached by following a randomly selected link, or the degree distribution of neighboring nodes. Assuming $P(k) = C_P k^{-\gamma}$, $k \geq k_{min}$, $k_{max} \gg k_{min}$, and $\gamma > 2$, then for a large network, we have $C_P \approx (\gamma-1)k_{min}^{\gamma-1}$. And $Q(k)$ can be written as $Q(k) = C_Q k^{1-\gamma}, k \geq k_{min}$, where $C_Q \approx (\gamma-2)k_{min}^{\gamma-2}$. Therefore, we get

$$Q(k > k_i) = \int_{k_i}^{k_{max}} Q(k)dk = \frac{C_Q}{\gamma-2}(k_i^{2-\gamma} - k_{max}^{2-\gamma}),$$

$$Q(k < k_i) = \int_{k_{min}}^{k_i} Q(k)dk = \frac{C_Q}{\gamma-2}(k_{min}^{2-\gamma} - k_i^{2-\gamma}),$$

where k_{min} and k_{max} are the minimum and the maximum node degree of the network. This leads to

$$\Omega_i = 1 + g\frac{C_Q}{\gamma-2}[2k_i^{2-\gamma} - (k_{min}^{2-\gamma} + k_{max}^{2-\gamma})].$$

Letting $C_2 = k_{min}^{2-\gamma} + k_{max}^{2-\gamma}$, we can further simplify the above expression as

$$\Omega_i = 1 - \frac{gC_Q C_2}{\gamma-2} + \frac{2gC_Q}{\gamma-2}k_i^{2-\gamma} = a + bk_i^{2-\gamma}, \tag{8}$$

where $a = 1 - gC_Q C_2/(\gamma-2)$ and $b = 2gC_Q/(\gamma-2)$.

Going back to the coupling matrix, we have $c_{in} = s_{in}/\Omega_i$ and, hence,

$$\begin{aligned}
d_n^{out} &= \sum_i c_{in} \\
&= \sum_i \frac{s_{in}}{\Omega_i} \\
&= \sum_{i<n} \frac{a_{in}(1-g)}{\Omega_i} + \sum_{i>n} \frac{a_{in}(1+g)}{\Omega_i} \\
&= k_n \left[\int_{k_n}^{k_{max}} \frac{Q(k)(1-g)}{a+bk^{2-\gamma}}dk + \int_{k_{min}}^{k_n} \frac{Q(k)(1+g)}{a+bk^{2-\gamma}}dk \right]
\end{aligned}$$

$$= k_n \left[\int_{k_{min}}^{k_{max}} \frac{Q(k)}{a + bk^{2-\gamma}} dk \right.$$

$$\left. + g \left(\int_{k_{min}}^{k_n} \frac{Q(k)}{a + bk^{2-\gamma}} dk - \int_{k_n}^{k_{max}} \frac{Q(k)}{a + bk^{2-\gamma}} dk \right) \right].$$

Noting that

$$\int \frac{Q(k)}{a + bk^{2-\gamma}} dk = \int \frac{C_Q k^{1-\gamma}}{a + bk^{2-\gamma}} dk$$

$$= \frac{C_Q}{(2 - \gamma)b} \ln(a + bk^{2-\gamma}),$$

we have

$$d_n^{out} = k_n \left[\frac{C_Q}{(2 - \gamma)b} \ln \frac{a + bk_{max}^{2-\gamma}}{a + bk_{min}^{2-\gamma}} \right.$$

$$\left. + g \frac{C_Q}{(2 - \gamma)b} \ln \frac{(a + bk_n^{2-\gamma})^2}{(a + bk_{min}^{2-\gamma})(a + bk_{max}^{2-\gamma})} \right]. \tag{9}$$

Since $b = 2gC_Q/(\gamma - 2)$, we have $C_Q/[(2-\gamma)b] = -1/(2g)$. Since it is assumed that the degree exponent $\gamma > 2$ and $k_{max} \gg 1$, we have $k_{max}^{2-\gamma} \ll 1$. Then we can neglect the term $k_{max}^{2-\gamma}$ and further simplify Eq. (9). Recalling that $C_Q = (\gamma - 2)k_{min}^{\gamma-2}$ and $C_2 = k_{min}^{2-\gamma} + k_{max}^{2-\gamma} \approx k_{min}^{2-\gamma}$, we obtain

$$a = 1 - \frac{gC_Q C_2}{\gamma - 2} = 1 - \frac{gC_Q}{\gamma - 2} k_{min}^{2-\gamma} = 1 - g$$

and

$$b = \frac{2gC_Q}{\gamma - 2} = \frac{2g}{k_{min}^{2-\gamma}}.$$

Inserting all these expressions into Eq. (9), we obtain

$$d_n^{out} = k_n \left[\frac{1}{2g} \left(\ln \frac{1+g}{1-g} + g \ln(1+g)(1-g) \right) \right.$$

$$\left. - \ln \left(1 - g + 2g \left(\frac{k_n}{k_{min}} \right)^{2-\gamma} \right) \right]. \tag{10}$$

Letting

$$F = \frac{1}{2g} \left(\ln \frac{1+g}{1-g} + g \ln(1+g)(1-g) \right)$$

$$= \frac{1}{2g} [(1+g)\ln(1+g) - (1-g)\ln(1-g)] \tag{11}$$

and

$$G_n = -\ln \left(1 - g + 2g \left(\frac{k_n}{k_{min}} \right)^{2-\gamma} \right), \tag{12}$$

we can rewrite Eq. (10) as

$$d_n^{out} = k_n(F + G_n). \tag{13}$$

Since F does not depend on the node's degree, it can be regarded as the symmetrical part of the couplings of each link, which depends only on the gradient parameter g and is decreased as the absolute value of g increases. In contrast, the term G is a joint function of g and k_n. While G depends

on g monotonically, its exact value, however, is strongly affected by the node's degree: nodes of larger degrees have larger G [Eq. (12)]. For small g, we have

$$F \simeq 1 - g^2/6 \quad \text{and} \quad G \simeq g - 2g(k/k_{min})^{2-\gamma}.$$

Therefore, the leading term of G is g, and when g approaches 0, d_n^{out} returns to k_n.

Finally, we have

$$\langle d^{in} d^{out} \rangle = \int_{k_{min}}^{k_{max}} [F + G(k)]k^2 P(k) dk$$

$$= F \cdot \langle k^2 \rangle + \langle G(k)k^2 \rangle. \tag{14}$$

Equation (14) is our key result, which gives, implicitly, the dependence of ε_c on the coupling gradient parameter g and the network-topology parameter (γ).

From Eq. (13), we see that the introduction of coupling gradient changes only the weights $H = F + G$ of the outgoing couplings at each node, while the total coupling cost of the network is kept unchanged. That is to say, gradient changes the distribution of H from an even form ($H = 1$ in an unweighted network) to an uneven form [$H = H(g, k)$ in a weighted network]. According to the value of H, we are able to divide the nodes into two groups: nodes with degrees larger than a critical value k_c have $H > 1$, while those with degrees smaller than k_c have $H < 1$ (for positive g). The critical degree k_c can be calculated by requiring $H = 1$,

$$F - \ln \left(1 - g + 2g \left(\frac{k_c}{k_{min}} \right)^{2-\gamma} \right) = 1,$$

which yields

$$k_c = k_{min} \cdot \ln \left[\frac{1}{2} - \frac{1}{2g} (1 - e^{F-1}) \right]^{1/2-\gamma}. \tag{15}$$

The heterogeneous distribution of H becomes more apparent when considering the extreme cases of $k \approx k_{max}$ and $k \approx k_{min}$. From Eqs. (11) and (12), we have

$$H(k_{max}) \simeq F - \ln(1 - g) = \frac{1}{2g} (1 + g) \ln \frac{1+g}{1-g} \tag{16}$$

and

$$H(k_{min}) \simeq F - \ln(1 + g) = \frac{1}{2g} (1 - g) \ln \frac{1+g}{1-g}. \tag{17}$$

Clearly, $H(k_{max}) > H(k_{min})$. Since the total outgoing coupling is a constant for the network, i.e., $\Sigma_i d_i^{out} = \Sigma_i H_i k_i = \Sigma_{i,j} c_{ij} = \Sigma_i k_i^{in} = \Sigma_i k_i$, the gradient effect can thus be understood as a shifting of partial of the outgoing coupling from small-degree nodes to large-degree nodes.

Now we can write down H explicitly,

$$H = F - \ln \left[1 - g + 2g \left(\frac{k}{k_{min}} \right)^{2-\gamma} \right]. \tag{18}$$

For a fixed gradient strength g, an increase of the network homogeneity, i.e., the degree exponent γ, leads to a suppression of the term $2g(k/k_{min})^{2-\gamma}$, which will make the distribution of H more homogeneous. As a result, it will hinder synchronization in the sense that the value of critical cou-

pling ε_c increases with the increase of the degree exponent γ.

Our analysis provides a base for understanding the interplay between the coupling gradient and the network topology in shaping the network synchronization, as follows. A change in the gradient strength g or in the degree exponent γ does not change the total coupling cost of the network; it only redistributes the weight of the outgoing couplings at each node according to its degree information. When gradient $g > 0$ is introduced, the outgoing couplings of small-degree nodes with $k < k_c$ are reduced by an amount that is added to large-degree nodes having degree $k > k_c$. As a result, a heterogeneous distribution in H arises which, in turn, decreases the value of ε_c [see Eq. (14)]. This enhancement of network synchronization, however, is modulated by the network topology. By increasing the degree exponent γ, the distribution of H tends to be homogeneous (i.e., $H \sim 1$) and, consequently, network synchronization is suppressed. These are the mechanisms that govern the roles of coupling gradient and network topology in synchronization. Our analysis suggests that (i) in the presence of coupling gradient, synchronization in a network of even nonidentical node dynamics can generally be enhanced, and (ii) in comparison with homogeneous networks, coupling gradient appears to be more advantageous for heterogeneous networks in the sense that, under the same gradient strength, they are more synchronizable.

III. NUMERICAL VERIFICATIONS

To provide numerical support for our theory, we use generalized scale-free networks generated by using the algorithm in Ref. 21, where the degree exponent γ can be adjusted. The frequency distribution of the phase oscillators is chosen to be

$$\rho(\omega) = \begin{cases} (3/4)(1-\omega^2), & -1 \le \omega \le 1 \\ 0, & \omega < -1 \text{ or } \omega > 1. \end{cases} \quad (19)$$

The initial phases of the oscillators are randomly distributed in $[0, 2\pi]$. In the computations, data for a transient period of time ($T_0 = 100$) are disregarded and the global order parameter is calculated for $T = 100$ for different values of the gradient parameter g. Figure 1 shows, for $g = 0.5$, 0, and -0.5, r^2 versus the coupling parameter ε. For each case, r starts to increase from zero when ε increases through some critical value ε_c, indicating the onset of synchronization among the phase oscillators in the network. The remarkable observation is that the value of ε_c for $g = 0.5$ is smaller than that for $g = 0$. Since a positive value of g corresponds to the situation in which the coupling gradient points from larger-degree nodes to small-degree nodes, we see that such a positive gradient field enhances the network synchronization. In contrast, when the gradient field points in the opposite direction, i.e., from small-degree nodes to large-degree nodes, the network becomes more difficult to synchronize, as characterized by a larger value of ε_c comparing with that in the nongradient ($g = 0$) case. The solid curves in Fig. 1 are from Eq. (5), which agree reasonably well with the numerics. In particular, the agreement between the theoretical and numerical values of ε_c is good.
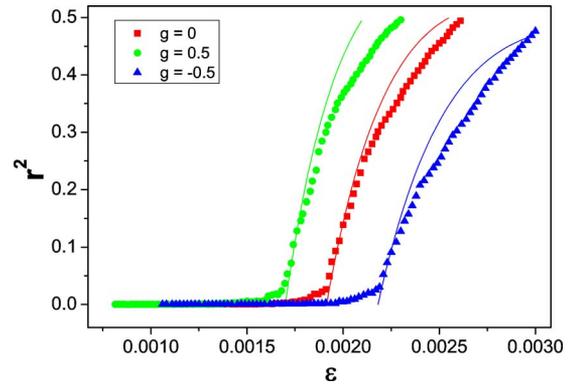

FIG. 1. (Color online) For an ensemble of ten scale-free networks of $N = 1500$ nodes and of degree exponent $\gamma = 3$, the average value of the square of the global order parameter r^2 vs the coupling parameter ε for three values of the gradient parameter g. The average degree of the networks is chosen to be rather large: $\langle k \rangle = 400$. [The reason is that the theoretical formulas Eqs. (5) and (7), originally derived in Ref. 13, hold only for reasonably densely connected networks.] We see that a positive gradient field, which points from large-degree to small-degree nodes, can enhance synchronization in the sense that the onset of synchronization occurs for a smaller value of ε as compared with the nongradient case ($g = 0$). A negative gradient field tends to hinder synchronization.

Numerical support for our main analytic result Eq. (14), which enables the dependence of ε_c on the gradient parameter g to be implicitly calculated, is shown in Fig. 2, where the data points are from direct numerical simulation, and the solid curve is our analytic prediction. In simulation, ϵ_c is defined as the point from where $r^2 > 2 \times 10^{-2}$ in Fig. 1. The agreement is reasonable. We observe a monotonic reduction of ε_c as g is increased from a negative value. Figure 2 thus represents a *quantitative* verification, on a paradigmatic class of solvable network-dynamics model, for the qualitative result concerning the synchronizability of gradient networks obtained previously.[9]

Simulations have also been conducted to uncover the dependence of ε_c on γ. In particular, the degree exponent γ is varied systematically from 3 to 25, during which the size and average degree of the network are maintained at constant. Our result, Eq. (18), suggests that ε_c increase mono-
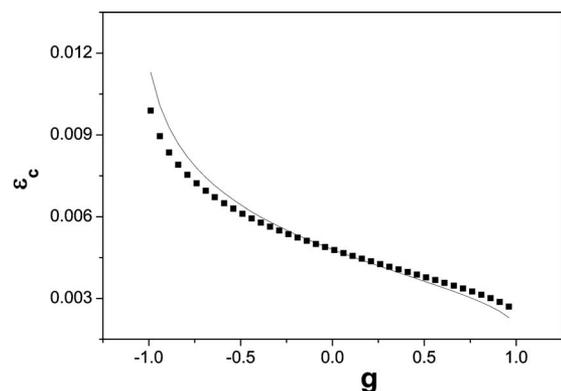

FIG. 2. For an ensemble of scale-free networks, the critical coupling parameter ε_c vs the gradient parameter g. The solid curve is from the main analytic result Eq. (14). The network parameters are $N = 5000$, $\gamma = 3$, and $\langle k \rangle = 100$. We observe a monotonic decrease in ε_c as g is increased from a negative value and a reasonable agreement between theory and numerics.
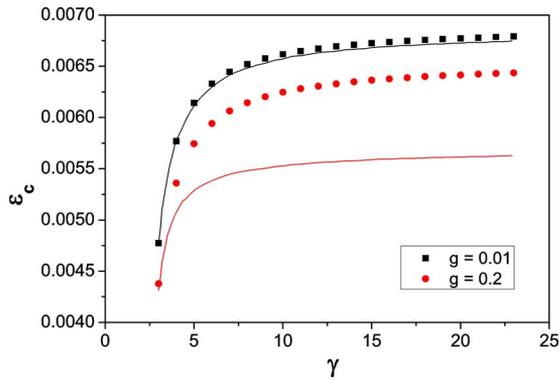
FIG. 3. (Color online) For scale-free networks of 5000 nodes, average degree 100, ε_c vs the degree exponent γ for $g = 0.01$ (the upper curve) and $g = 0.2$ (the lower curve). There is a monotonic increase of ε_c with γ, as predicted by our theory. The agreement between theory and numerics is better for the small-g case.

tonically with γ, and direct numerical computations indeed indicate so, as shown in Fig. 3 for the cases of $g = 0.01$ and 0.2. For $g = 0.01$, the analytic prediction (the solid curve that matches the corresponding data) is very good. However, for $g = 0.2$, the theoretical and numerical results do not agree with each other well, except for the cases in which γ is small. This is somewhat expected, as the starting point of our theoretical treatment, Eqs. (5) and (7), is valid under the assumption of large and dense networks. For large value of g, as γ is increased, the network size and linkage density need to be increased to maintain the applicability of Eqs. (5) and (7) while, for comparison purpose, the simulations in Fig. 3 are carried out under constant network size and average degree.

Remarks. By a descending order of the predicating precision (or, by an ascending order of the computing efficiency), in Ref. 13 the authors proposed four approximating approaches: the time average theory (TAT), the frequency distribution approximation (FDA), the perturbation theory (PT), and the mean-field theory (MF). Among them, the MF approach requires only knowledge of the frequency distribution and the degree distribution of the network, and thus is the only feasible approach for analyzing realistic networks when detailed information about the network structure (the coupling matrix) is not available. We have thus chosen the MF approach. As discussed in Ref. 13, the MF approximation is based on the assumption that the eigenvector u associated with the largest eigenvalue λ satisfies $u_n o \propto k_n$, with k_n the degree of node n. We have verified that this assumption is usually satisfied for relatively small gradient strength, e.g., $g < 0.1$. Thus predictions based on the MF theory are good for the small g regime. As g becomes large, the discrepancies between the MF predictions and numerics grow. To improve the theoretical prediction in the large g regime, one can use the more exact formula obtained, for example, by using the perturbation theory. In this case, there is no explicit formula for the largest eigenvalue of the coupling matrix; it needs to be calculated numerically.

IV. DISCUSSIONS

It has been demonstrated that gradient field on a complex network, when properly designed, can enhance the network's ability to achieve synchronization.[9] For example, for a scale-free network, when the gradient field is such that the couplings from hub nodes to smaller-degree nodes in the network are stronger than the respective couplings in the opposite direction, network synchronizability can be enhanced significantly. This result provides insight into a paradox in network science: scale-free networks are ubiquitous in natural and technological systems but they appear to be more difficult to synchronize than random networks.[4] The key is weight and asymmetrical interactions: their effects on network can be understood by the dynamics of some equivalent gradient networks. This study of the synchronizability of gradient networks is *qualitative* in the sense that it is based on the standard approach of master-stability function[22] and eigenratio analysis,[2] which requires that all node dynamics be identical. The purpose of the present paper is to provide analytic results to place the phenomenon of enhanced synchronization in complex gradient networks on a more *quantitative* and therefore firmer ground.

We have shown here that analytic treatment of the gradient-network synchronization problem is indeed possible when the complexity of the local node dynamics is reduced. Our choice is the classical Kuramoto model where the local dynamics is that of a simple phase oscillator. Since the local oscillator dynamics are initially different with different frequencies, the Kuramoto model provides a paradigm for addressing synchronization in a network with heterogeneous local dynamics, making the results more general as compared with those from the synchronizability analysis. A key to our work is the recent analysis of the Kuramoto model on scale-free networks where analytic formulas of the order parameter and of the critical coupling parameter for the onset of synchronization have been derived.[13] Based on these results, we are able to obtain, for complex gradient networks, an implicit formula relating the critical coupling parameter to the gradient field strength and a parameter characterizing the network topology. The formula has been verified by numerical results on a class of scale-free networks. The formula can also be used to understand recent numerical findings[23] that the onset of synchronization in a scale-free network requires less coupling than that for more homogeneous networks. While our theory is derived for large and densely connected networks, the general finding that synchronization can be enhanced by gradient couplings applies to any network. Synchronization in complex networks has been an active field of research, but most existing results concern the network synchronizability and they are qualitative. Our result represents one of the few more quantitative results in this area.

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[1] L. F. Lago-Fernandez, R. Huerta, F. Corbacho, and J. A. Siguenza, Phys. Rev. Lett. **84**, 2758 (2000).
[2] M. B. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[3] X. F. Wang and G. Chen, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 187 (2002); IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **49**, 54 (2002).
[4] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[5] A. E. Motter, C. Zhou, and J. Kurths, Europhys. Lett. **69**, 334 (2005); Phys. Rev. E **71**, 016116 (2005); AIP Conf. Proc. **778**, 201 (2005); C. Zhou, A. E. Motter, and J. Kurths, Phys. Rev. Lett. **96**, 034101 (2006).C. Zhou and J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006).
[6] M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005).
[7] T. Nishikawa and A. E. Motter, Phys. Rev. E **73**, 065106 (2006).
[8] L. Huang, K. Park, Y.-C. Lai, L. Yang, and K. Yang, Phys. Rev. Lett. **97**, 164101 (2006).
[9] X. G. Wang, Y.-C. Lai, and C.-H. Lai, Phys. Rev. E **75**, 056205 (2007); X. G. Wang, L. Huang, Y.-C. Lai, and C.-H. Lai, ibid. **76**, 056113 (2007).
[10] S. Strogatz, Sync: The Emerging Science of Spontaneous Order (Hyperion, New York, 2003).
[11] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence (Springer-Verlag, Berlin, 1984).
[12] S. H. Strogatz, Physica D **143**, 1 (2000).
[13] J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. E **71**, 036151 (2005); Chaos **16**, 015107 (2005).
[14] S.-G. Guan, X.-G. Wang, Y.-C. Lai, and C. H. Lai, Phys. Rev. E **77**, 046211 (2008).
[15] D. J. Watts, P. S. Dodds, and M. E. J. Newman, Science **296**, 1302 (2002); R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon, ibid. **298**, 824 (2002); K. A. Eriksen, I. Simonsen, S. Maslov, and K. Sneppen, Phys. Rev. Lett. **90**, 148701 (2003); A. E. Motter, T. Nishikawa, and Y.-C. Lai, Phys. Rev. E **68**, 036105 (2003); E. Oh, K. Rho, H. Hong, and B. Kahng, ibid. **72**, 047101 (2005).
[16] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[17] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[18] D. J. Watts and S. H. Strogatz, Nature **393**, 440 (1998).
[19] Z. Toroczkai and K. E. Bassler, Nature **428**, 716 (2004); Z. Toroczkai, B. Kozma, K. E. Bassler, N. W. Hengartner, and G. Korniss, arXiv:cond-mat/0408262.
[20] K. Park, L. Zhao, Y.-C. Lai, and N. Ye, Phys. Rev. E **71**, 065105 (2005).
[21] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).
[22] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[23] J. Gómez-Gardeñes, Y. Moreno, and A. Arenas, Phys. Rev. Lett. **98**, 034101 (2007).