Effect of noise on the neutral direction of chaotic attractor

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A chaotic attractor from a deterministic flow must necessarily possess a neutral direction, as characterized by a null Lyapunov exponent. We show that for a wide class of chaotic attractors, particularly those having multiple scrolls in the phase space, the existence of the neutral direction can be extremely fragile in the sense that it is typically destroyed by noise of arbitrarily small amplitude. A universal scaling law quantifying the increase of the Lyapunov exponent with noise is obtained. A way to observe the scaling law in experiments is suggested. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1637735]

The various effects of noise on deterministic dynamical systems have been a topic of continuous interest and of fundamental importance in nonlinear and statistical physics. Here we focus on how noise affects the Lyapunov spectrum of a chaotic attractor. Under small noise, the originally null Lyapunov exponent, which reflects the existence of a neutral direction in the deterministic case, is more likely to change as compared with the other nonzero exponents. The neutral direction may thus be "fragile" in the sense that it can be destroyed by small noise. We find, however, that there exist two natural classes of chaotic oscillators, one for which the neutral direction is robust under noise while another for which the direction is fragile. The former corresponds to chaotic attractors with well-defined rotational structure such as the Rössler attractor, and the latter to attractors with multiple scrolls in the phase space such as those from the Lorenz system. We give a physical theory to explain why Rössler-like chaotic attractors can have a well-defined neutral direction but it is typically destroyed by arbitrarily small noise for Lorenz-like attractors. We also find that for the latter, the increase of the Lyapunov exponent from zero with noise obeys an algebraic scaling law. For double-scroll chaotic attractors in the three-dimensional phase space, the scaling exponent assumes the universal value of two. Our result, besides its importance from a basic standpoint, can also be useful for practical applications such as estimating the internal noise level of chaotic systems. Our work is also directly relevant to the study of chaotic phase synchronization.

I. INTRODUCTION

The behavior of a deterministic chaotic system under noise is a fundamental problem in nonlinear and statistical physics, and in applied mathematics as well. This is so because physical systems are often under noise, and it is important to assess how properties of deterministic chaotic dynamics are affected by noise. Mathematically, this is related to the structural stability of a dynamical system. Indeed, there has been a continuous interest in this area of research.[1–11]

An elementary fact in nonlinear dynamics is that a chaotic attractor arising from a deterministic flow must have a neutral direction in the phase space along which small distances are preserved. This neutral direction is, of course, the direction of the flow itself in the phase space, which is characterized by a null Lyapunov exponent. The purpose of this paper is to show that for a wide class of chaotic attractors, particularly those having multiple scrolls in the phase space (such as the Lorenz attractor), the existence of the neutral direction can be extremely fragile in the sense that it is typically destroyed by noise, no matter how weak. Let $\lambda^{(0)}(0)=0$ be the null Lyapunov exponent in the absence of noise. We find that λ_0 becomes positive as the noise amplitude D is increased from zero. We will show that $\lambda^{(0)}(D)$ obeys the following algebraic scaling law:

$$\lambda^{(0)}(D)\sim D^{\alpha} \quad \text{for } D\gtrsim 0, \tag{1}$$

where $\alpha>0$ is the scaling exponent. For double-scroll chaotic attractors from three-dimensional flows, the scaling exponent assumes the value of two. We will argue that the disturbance caused by noise to the consistency of the motions of chaotic trajectories on distinct phase-space scrolls, which exists in the determinstic flow, is the fundamental mechanism leading to the destruction of the neutral direction and to the scaling law (1). We will present numerical support using the classical Lorenz[12] attractor. We note that in dimension three, the noise-free attractor possesses only one positive Lyapunov exponent. But as soon as there is small noise, the system exhibits two positive Lyapunov exponents. Thus the phenomenon we report here represents an interesting case where arbitrarily weak noise can induce high-dimensional chaos with more than one positive Lyapunov exponent even for well-known low-dimensional chaotic systems.

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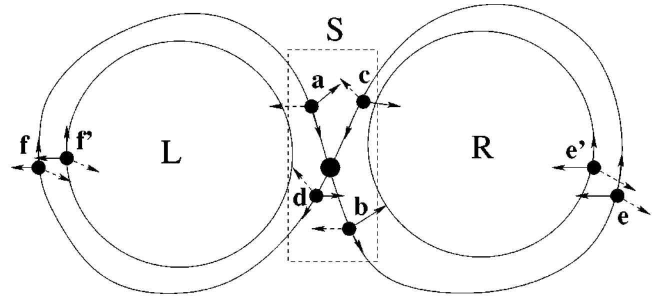

FIG. 1. Schematic illustration of a double-scroll chaotic attractor and the associated dynamical consistency, e.g., point a (c) can only go to b (d).

Note that the prefactor in Eq. (1) depends on system details, whose magnitude determines the value of the positive Lyapunov exponent induced by noise. The prefactor thus determines how drastic this effect can be, in terms of the relative magnitude of the noise-induced exponent to the largest Lyapunov exponent of the system. For example, for the classical Lorenz system, we find numerically that the prefactor is about 0.04, versus the largest exponent which is on the order of unity.

In Sec. II, we give a physical theory to explain the behavior of the null Lyapunov exponent under noise for Rössler- and Lorenz-like chaotic attractors, and derive the scaling law (1). Numerical support is provided in Sec. III, and a brief discussion is presented in Sec. IV.

II. THEORY

For convenience, we consider a double-scroll chaotic attractor in the three-dimensional phase space, which arises from a dynamical system described by $d\mathbf{x}/dt=\mathbf{F}(\mathbf{x})$, as shown schematically in Fig. 1. The classical Lorenz attractor, perhaps the best known chaotic attractor, belongs to this type. The left- and right-hand side scrolls are denoted by "L" and "R," respectively. A typical trajectory visits both scrolls in time, and it tends to stay in one scroll executing chaotic motion for a time, switch to the other scroll, wander chaotically for some time there, switch back, and so on. Switchings occur in the region denoted by "S" in which there is an unstable steady state.[13] A key feature to notice is that in the deterministic case, the way that switchings occur must be *consistent* with the natural dynamics. For instance, a trajectory moving to point a near the switching region must go to point b after the switching. It cannot go to point c. Similarly, under the dynamics point c can only move to point d. Another aspect of the dynamical consistency is that the relative frequencies with which a trajectory visits "L" and "R" appear to be constant, as can be easily verified numerically.

Our idea is that noise can disturb the dynamical consistency and consequently destroy the neutral direction. To see how, we note that, depending on the location of a trajectory, the effect of noise can be quite different. When the trajectory is not in the switching region, noise can perturb its position, say from point e (f) to point e' (f') or vice versa. As shown in Fig. 1, perturbations at such locations will have little effect on the local eigenspace. Taking a pair of original and perturbed points (e,e') as an example, we see that the original eigenvector in the neutral direction (direction of the flow) at e remains to be a neutral direction at the perturbed point e'. There can, of course, be small deviations from the neutral direction, but they will be averaged out by noise as the trajectory moves in region "R." This is in fact the reason why the neutral direction associated with a single-scroll chaotic attractor, such as the Rössler attractor,[14] can persist under noise. When a trajectory is in the switching region "S," noise of arbitrarily small amplitude can alter the local eigenspace *in a significant way*. For instance, when the trajectory is at point a, noise can kick it to point c. Such a perturbation has two effects. First, since the local eigenspaces at the two points are distinct, the neutral eigenvector at a, when carried over by the trajectory perturbed to c, will not be in the neutral direction at c. The vector typically will have a component in the unstable direction at c and its length will consequently be stretched exponentially. Thus the length of the neutral vector on the attractor, when it is perturbed in the switching region as described, will generally increase exponentially, causing the originally null Lyapunov exponent to become positive. Second, the noisy perturbation that moves the trajectory from a to c, is in fact *inconsistent* with the deterministic dynamics because, in the absence of noise, the trajectory would move passing point b. This effect can in fact be observed in numerical experiments by computing the frequency of switching as a function of the noise amplitude. As we will show, this effect should be relatively easy to be observed and quantified in laboratory experiments.

Let $f^{\mathrm{L}}(D)$ and $f^{\mathrm{R}}(D)$ be the frequencies of visits of a typical trajectory to the L and R scrolls, respectively, under noise of amplitude D, and let $f^{\mathrm{S}}(D)$ be the probability that the trajectory experiences *inconsistent* perturbations in the switching region S, where $f^{\mathrm{L}}(D)+f^{\mathrm{R}}(D)+f^{\mathrm{S}}(D)=1$. (If the trajectory simply passes through the switching region in a way consistent with the deterministic flow, we regard it as either in "L" or in "R.") In the noise-free case, we have $f^{\mathrm{S}}(0)=0$ so the Lyapunov exponents of the chaotic attractor can be written as $\lambda_i=f^{\mathrm{L}}(0)\lambda_i^{\mathrm{L}}+f^{\mathrm{R}}(0)\lambda_i^{\mathrm{R}}$ $(i=1,2,3)$, where λ_i^{L} and λ_i^{R} are the average rates of change of infinitesimal vectors along the corresponding eigendirections when the trajectory is in the left and right scroll, respectively. In particular, the null exponent can be trivially written as $\lambda^{(0)}(0)\equiv\lambda_2(0)=f^{\mathrm{L}}(0)\lambda_2^{\mathrm{L}}+f^{\mathrm{R}}(0)\lambda_2^{\mathrm{R}}=0$. Under noise, when the trajectory is perturbed inconsistently in the switching region, the neutral vector is stretched exponentially there. In the typical case where there is a dominant unstable steady state in the switching region, the rate is mainly determined by the largest eigenvalue of the steady state. Let $\overline{\lambda}>0$ be the Lyapunov exponent associated with this eigenvalue. We have

$$\lambda_2(D)\approx f^{\mathrm{L}}(D)\lambda_2^{\mathrm{L}}+f^{\mathrm{R}}(D)\lambda_2^{\mathrm{R}}+f^{\mathrm{S}}(D)\overline{\lambda}\approx f^{\mathrm{S}}(D)\overline{\lambda}. \qquad (2)$$

Note that $f^{\mathrm{S}}(D)$ is proportional to the probability that a trajectory falls in the switching region, which is proportional to the probability that a trajectory crosses the stable manifold of the dominant unstable steady state. In the three-dimensional phase space, a noisy trajectory near the unstable steady state can be found in a sphere of radius D centered at the steady

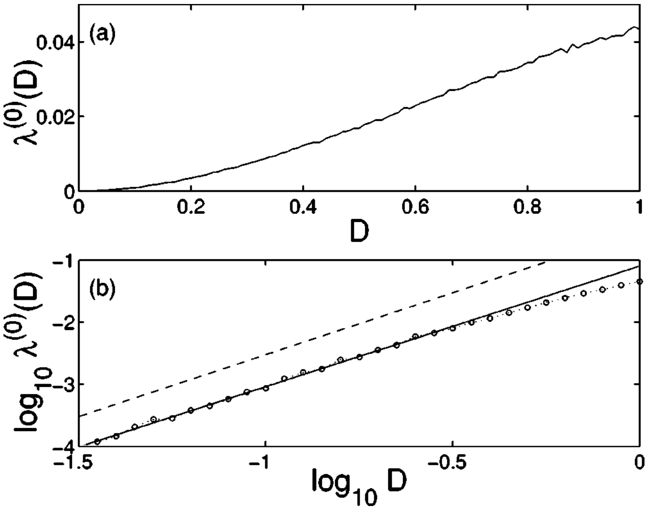

FIG. 2. For the classical Lorenz chaotic attractor under additive noise, (a) the originally null Lyapunov exponent vs the noise amplitude; (b) scaling of this exponent with noise, where the dashed line indicates the theoretically predicted slope of 2.

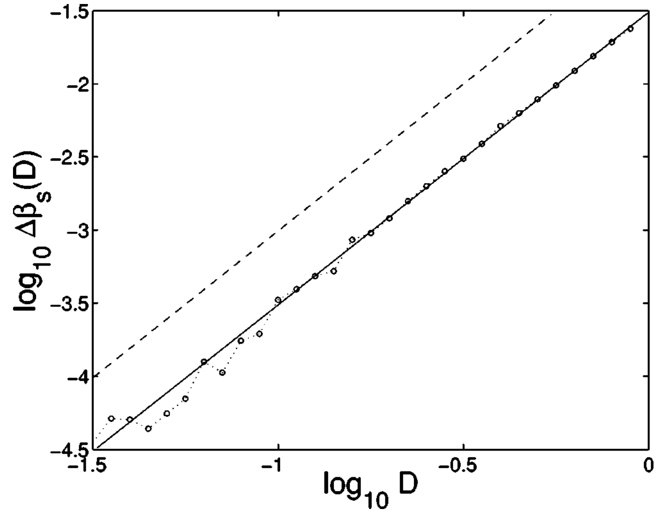

FIG. 3. For the classical Lorenz attractor under additive noise, scaling of the switching frequency $\Delta\beta_S$ with the noise. This scaling law is experimentally more accessible.

state. Since the dimension of the stable manifold of the unstable steady state is 2, we have $f^S(D) \sim D^2$, which gives the scaling law (1).

Since $f^S(D)$ is the probability of inconsistent perturbations, which cause changes in the frequencies of visit to "L" and "R" relative to those in the deterministic case, we expect to see a corresponding change in the frequency that a trajectory switches from "L" to "R" and vice versa. In computations, the switching frequency $\beta^S(D)$ can be obtained as follows. Construct a Poincaré surface of section and use symbols **l** and **r** to denote, on the section, the location of the trajectory point in the left and right scrolls, respectively. A symbolic string of length N can then be generated associated with a long trajectory: $\sigma_1\sigma_2,\ldots,\sigma_N$, where σ_i is either **l** or **r**. Let N_S be the number of times in the symbolic string that pairs of two different symbols (**lr** or **rl**) are observed. We then have $\beta_S(D) = \lim_{N\to\infty} N_S/N$ and the scaling relation

$$\Delta\beta_S(D) \equiv \beta_S(D) - \beta_S(0) \sim f^S(D) \sim D^\alpha, \tag{3}$$

where $\alpha = 2$ for three-dimensional flows. In experiments where the system equations are not available, the scaling law (3) can be obtained relatively easily. In contrast, it may be difficult to observe the scaling law (1) in experiments, as estimating a Lyapunov exponent close to zero from time series to a required precision is difficult. Because both the noise-induced exponent and the switching frequency are proportional to the probability $f^S(D)$, we see that relation (3) can be considered as an indirect way to verify the scaling law (1) in laboratory experiments.

In deriving Eq. (3), we implicitly assumed that the probability of inconsistent perturbation does not depend on the position of the trajectory (e.g., in the symbolic string). While in actuality the probability of switching between the scrolls depends on the location of the trajectory in the switching region, this has little effect on Eq. (3) because the quantity of interest here is the *average* switching frequency in the long-time limit.

III. NUMERICAL SUPPORT

We now present numerical support for the scalings laws (1) and (3). We consider the classical Lorenz system with additive noise: $dx/dt = 10(y-x) + D\xi_1(t)$, $dy/dt = 28x - y - xz + D\xi_2(t)$, and $dz/dt = -(8/3)z + xy + D\xi_3(t)$, where $\xi_i(t)$ $(i = 1,2,3)$ are Gaussian random variables of zero mean and unit standard deviation. Figure 2(a) shows the originally null Lyapunov exponent versus the noise amplitude, where it can be seen that the null exponent starts to increase as soon as the noise is turned on, indicating the destruction of the neutral direction of the noisy chaotic flow. The algebraic scaling law (1) is shown in Fig. 2(b), where the dashed line indicates the slope of 2. The noise scaling of the experimentally accessible quantity $\Delta\beta_S(D)$ is shown in Fig. 3, where the scaling exponent is also 2. Numerical computations reveal the same scaling laws with the Lorenz system under multiplicative noise, as shown in Figs. 4(a) and 4(b).

Our heuristic argument suggests that $\Delta\beta_S(D)$ and $\lambda^0(D)$ depend mainly on $f^S(D)$, the switching frequency. The deviation of the numerically computed noise-induced exponent $\lambda^0(D)$ from the predicted behavior (1) occurs for relatively large value of D. This is so because our heuristic analysis leading to Eq. (1) is valid only in the small noise regime. We observe, however, that the numerical results of $\Delta\beta_S(D)$ argee with the predicted scaling law (3) even for large values of D. This is expected because the quantity $\Delta\beta_S(D)$ itself is in fact the switching frequency. The relatively large fluctuations of $\Delta\beta_S(D)$ in the small noise regime in Figs. 3 and 4(b) come from the finite numerics as, when the noise amplitude is small, it is more difficult to calculate accurately the deviation of the switching frequency from that in the deterministic case.

IV. DISCUSSION

In summary, we have discovered that noise can have a metamorphic effect on one of the fundamental properties of chaotic attractors with multiple scrolls in the phase space in

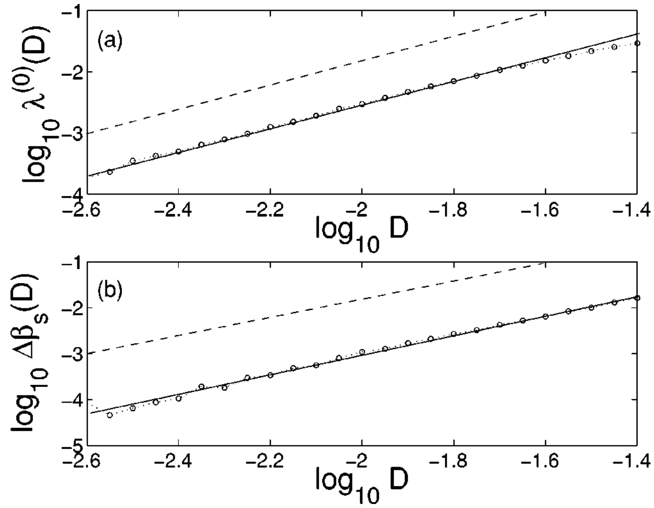

FIG. 4. For the classical Lorenz chaotic attractor under multiplicative noise: $dx/dt=10(y-x)+Dx\xi_1(t)$, $dy/dt=28x-y-xz+Dy\xi_2(t)$, and $dz/dt=-(8/3)z+xy+Dz\xi_3(t)$, where $\xi_i(t)$ $(i=1,2,3)$ are Gaussian random variables of zero mean and unit standard deviation, (a) scaling of the originally null Lyapunov exponent with noise, and (b) noisy scaling of the switching frequency. The dashed lines indicate the theoretical slope of 2.

the sense that the neutral direction of the flow is fragile as it can be destroyed by arbitrarily weak noise. We have obtained universal scaling laws for the Lyapunov exponent and the switching frequency; the latter is experimentally more accessible. We have utilized the setting of three-dimensional flows to describe our findings, both for convenience and for the consideration that these attractors are well studied.[15] It is surprising that noise can cause such a fundamental modification to the flow, which to our knowledge, has not been noticed previously.

Besides its theoretical value, our scaling law[16] can possibly be useful for applications such as assessing the strength of internal noise in a chaotic system through the measurement of a near-zero Lyapunov exponent. For instance, one can measure the exponent in the absence of external noise and perform the same measurements for a set of systematically varying levels of the external noise. From the behavior of the exponent versus the external noise level, the strength of the internal noise can be estimated. Our result is also relevant to chaotic phase synchronization. In particular, the neutral direction in a chaotic flow is of considerable recent interest because it characterizes the *phase* of the flow. When chaotic oscillators are coupled, phase synchronization[17] can occur, as have been identified in many physical, chemical, and biological systems.[18] The observation that the neutral direction can be preserved under noise for single-scroll chaotic attractors such as those from the Rössler oscillator suggests that the concept of phase and consequently, phase synchronization, are robust for such systems, which has indeed been observed and studied extensively.[18] Our result that the neutral directions in the Lorenz-type of double-scroll chaotic attractors are fragile under noise implies a possible difficulty to define the phase and to study phase synchronization (if it exists) in such systems under noise. To our knowledge this remains an open problem that warrants future attention, as the Lorenz-type of chaotic attractors is common in nonlinear dynamical systems.

ACKNOWLEDGMENT

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[1] Some pioneering works in this direction are the following. The effect of noise on period-doubling transition to chaos was studied by Crutchfield *et al.* (Refs. 2 and 3), where a renormalization-group approach was used to analyze the scaling behavior of the Lyapunov exponent near the transition (Ref. 3). The effect of noise on type-I intermittency was investigated by Hirsch *et al.* (Ref. 4). The influence of noise on periodic attractors for the Lorenz system was studied by Fedchenia *et al.* (Ref. 5). Noise-induced chaos in a system with homoclinic points was discussed by Anishchenko and Herzel (Ref. 6) and the opposite phenomenon of noise stabilization of chaotic dynamics was studied by Herzel (Ref. 7). The problem of noise-induced chaos also has similarities with the problem of noise activation of excitable systems (Ref. 8). Transition to noisy chaos for dynamical systems in periodic windows has recently been investigated (Ref. 9), which is relevant to problems in, for instance, laser physics (Ref. 10) and biology (Ref. 11).
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