Controlled test for predictive power of Lyapunov exponents: Their inability to predict epileptic seizures

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Lyapunov exponents are a set of fundamental dynamical invariants characterizing a system's sensitive dependence on initial conditions. For more than a decade, it has been claimed that the exponents computed from electroencephalogram (EEG) or electrocorticogram (ECoG) signals can be used for prediction of epileptic seizures minutes or even tens of minutes in advance. The purpose of this paper is to examine the predictive power of Lyapunov exponents. Three approaches are employed. (1) We present qualitative arguments suggesting that the Lyapunov exponents generally are not useful for seizure prediction. (2) We construct a two-dimensional, nonstationary chaotic map with a parameter slowly varying in a range containing a crisis, and test whether this critical event can be predicted by monitoring the evolution of finite-time Lyapunov exponents. This can thus be regarded as a "control test" for the claimed predictive power of the exponents for seizure. We find that two major obstacles arise in this application: statistical fluctuations of the Lyapunov exponents due to finite time computation and noise from the time series. We show that increasing the amount of data in a moving window will not improve the exponents' detective power for characteristic system changes, and that the presence of small noise can ruin completely the predictive power of the exponents. (3) We report negative results obtained from ECoG signals recorded from patients with epilepsy. All these indicate firmly that, the use of Lyapunov exponents for seizure prediction is practically impossible as the brain dynamical system generating the ECoG signals is more complicated than low-dimensional chaotic systems, and is noisy. [DOI: 10.1063/1.1777831]

The necessity of designing and carrying out controlled tests to validate a new finding or a new methodology is an elementary notion in scientific research. For example, to claim that a new phenomenon has been discovered in a new material under certain conditions, controls must be run on some different materials under similar conditions to show that the phenomenon does not occur. Similarly, if a new methodology is claimed to be able to predict critical events based on measured data, the method must be tested to work for a control system for which some critical events are designed to occur at known times through some well understood mechanism. This paper is focused on seizure prediction with a set of fundamental dynamical invariant quantities, the Lyapunov exponents, computed from measured time series. While there has been a gradual recognition in the applied nonlinear dynamics community that these exponents may not be useful for prediction, efforts have continued that are devoted to utilizing them for prediction in problems of significant interest. In particular, in the area of epilepsy, it has been claimed for more than a decade that by monitoring the evolution of the largest Lyapunov exponent calculated from electroencephalogram (EEG) or electrocorticogram (ECoG) data, epileptic seizures can be predicted minutes or even tens of minutes in advance of their clinical onset (the most recent claim being that this time can be longer than 80 min).[1–4] This claim, if true, would clearly have significant implications. Our concern is that so far, there appears to be no systematic effort to assess the predictive power of Lyapunov exponents. The purpose of this paper is to report our results of such a study through three approaches: (1) qualitative arguments, (2) controlled tests, and (3) tests using real data. All these indicate strongly that the Lyapunov exponents are generally not useful for predicting or detecting epileptic seizures. For the control test, we construct a two-dimensional chaotic map with a time-varying parameter $p(t)$. The system is thus nonstationary and we assume that the range of the parameter variation contains a crisis at which the chaotic attractor suddenly increases its size. Assuming that starting from an initial value p_0 the parameter varies slowly with time and passes through the critical point p_c at a later time and eventually comes back to p_0. A typical time series thus consists of segments of small-amplitude

chaotic oscillations for $p<p_c$ and a segment of large-amplitude oscillations for $p \geqslant p_c$, mimicking EEG or ECoG signals containing a seizure. Lyapunov exponents are then computed from a moving window across the time series to test whether the system change at $p=p_c$ reflects itself in the computed time-varying exponents and whether this gradual parameter drift can be detected in advance of the crisis in order to predict the impending crisis event. This class of control systems is the simplest that we can imagine to test the predictive power of Lyapunov exponents. Our analysis and computations indicate that there are two major factors that can prevent the exponents from being effective tools to predict characteristic system changes: statistical fluctuations and noise. As can be expected intuitively, even for low-dimensional, deterministic chaotic systems, the power of the Lyapunov exponents to detect parameter drift into crisis holds only in noiseless or extremely low-noise situations. In a realistic situation, especially in a system as high-dimensional and noisy as the brain, it appears unlikely that the Lyapunov exponents, or related quantities from nonlinear dynamics such as the correlation dimension, can be useful for predicting or detecting epileptic seizures.

I. INTRODUCTION

An outstanding problem in biomedical sciences is to devise techniques to detect and to predict epileptic seizures that affect about 1% of the population. Seizures are accompanied by abnormal electrical activities in different regions of the brain, and can be monitored by electroencephalogram (EEG) recorded via electrodes attached to the scalp, or by electrocorticogram (ECoG) from electrodes in direct contact with the cortex. These recordings provide a window, perhaps the only practically accessible window at present, through which the origin and dynamics of epilepsy can be investigated. Analysis of EEG or ECoG has thus become a topic of paramount importance.

An approach that is gaining increasing attention in dealing with these signals is to use techniques from time series analysis developed in nonlinear dynamics and chaos.[1–18] Early evidence suggested that the brain activity generating the EEG or ECoG signals can be described by low-dimensional dynamical systems,[5–7] which implies that detection and even dynamical control of epilepsy are possible, since prediction[19–22] and control[23,24] of low-dimensional chaotic systems are indeed achievable. However, re-examination of these early claims indicated a lack of low-dimensional dynamical structure in brain signals.[25,26] Whether a low-dimensional, deterministic interpretation of the EEG or ECoG signals is appropriate remains to be a debated issue, with no consensus in sight. Intuitively, the EEG or ECoG signals represent the collective behavior of a large number [approximately 10^5–10^8 (Ref. 27)] of neurons with complicated interconnections, and it is quite unlikely that the resulting behavior would be generally low-dimensional. Despite this uncertainty, measures useful for characterizing low-dimensional chaotic systems, such as the correlation dimension and the Lyapunov exponents, have been utilized to study the EEG or ECoG signals[1–4,8,10,11,15] with various claims that epileptic seizures can be predicted in advance.[1–4,10,11]

The focus of this paper is on Lyapunov exponents, which are fundamental invariant quantities characterizing the expansion or contraction of infinitesimal vectors in the phase space of nonlinear dynamical systems. The existing reports of prediction of seizures through measures that rely on Lyapunov exponents[1–4] prompt us to systematically investigate their seizure prediction power on ECoG time series. Unfortunately, our results suggest that this power is quite sensitive to factors such as random noise.

There have been reports in the literature of a pre-seizure state characterized a change in the Lyapunov exponent,[28] but no consensus on this topic seemed to have been reached.[29] Assuming the existence of this preseizure state, we investigate whether the Lyapunov exponents are sufficiently robust to noise and sensitive for detecting subtle changes in the system state that may precede seizures. The goal of this paper is to investigate the predictive power of Lyapunov exponents for nonstationary dynamical systems in the presence of noise with direct application to epilepsy. (A brief account of this work has been published recently.[30]) We use the general setting of a time series evaluated through data in a moving window in order to simulate the situation of on-line analysis. From each window's data, a phase space is reconstructed by using the delay-coordinate embedding method,[31,32] and the spectrum of all Lyapunov exponents is computed. Variations in the values of these exponents are examined, with particular attention to statistically significant changes which may indicate a precursor of a critical event such as the seizure onset.

To run the controls, we construct a low-dimensional, nonstationary chaotic system, with temporal parameter variation to produce nonstationarity, which allows the predictive power of the Lyapunov exponents to be addressed in a controllable way. We design the parameter variation to be a gradual transition to a critical event with significant changes in the system characteristics. Before the onset of the critical event, there are no noticeable, characteristic changes in the system state. The critical event is analogous to the occurrence of a seizure, and the slow parameter change to the possible preictal precursor to the seizure. Successful detection of the transition to a preictal state through the Lyapunov exponents would indicate their predictive power for the seizure.

We have identified two major factors that critically influence the detective and hence the predictive powers of the Lyapunov exponents computed from time series: (1) their statistical fluctuations in finite times and (2) the presence of noise. Counterintuitively, increasing the size of the moving window (and thus the amount of data contained therein) generally will not improve the detective power of the Lyapunov exponents in the noiseless case. We also find that the predictive power of the exponents is destroyed completely by noise of amplitude as small as 1% of the typical variation of the time series. We also tested ECoG data segments recorded from patients with epilepsy, systematically varying the key parameters in the algorithm, but found no indication that the evolving behaviors are capable of predicting or detecting seizure. Our basic conclusion is that if the brain's dynamical system generating the ECoG signals is more complicated than a low-dimensional, deterministic system, and if there is an appreciable but reasonable amount of noise present, it is extremely unlikely that the Lyapunov exponents will be more valuable for seizure prediction than simple, linear techniques.

In Sec. II, we provide some intuitive reasonings for the inability of Lyapunov exponents to predict epileptic seizures. In Sec. III we describe the implementation of our algorithm for computing all Lyapunov exponents from time series and justify its validity by using time series from low-dimensional chaotic systems for which the Lyapunov exponents are known. In Sec. IV, we report results of control tests from our low-dimensional chaotic model and compare them with those from a simple autocorrelation method. In Sec. V, we present results of systematic computations of Lyapunov exponents from test ECoG data sets containing seizures and address the effects of varying the size of the moving window. A discussion is presented in Sec. VI.

II. DIFFICULTIES ASSOCIATED WITH LYAPUNOV EXPONENTS AS A PREDICTIVE TOOL FOR EPILEPTIC SEIZURES

There are several technical difficulties associated with the predictive power of the Lyapunov exponents in the context of epilepsy.

(1) Difficulty in estimating Lyapunov exponents from time series. The spectrum of Lyapunov exponents is among the most difficult to compute from dynamical systems, particularly from time series when the system equations are not available. Despite the existence of numerical algorithms,[33–37] important issues such as the distribution of spurious exponents[38] and uncertainties in the estimates of the exponents[39] have begun to be addressed only recently. In general, the uncertainties can be severe,[39] which can have some effect on the predictive power of the exponents in realistic situations.

If one intends to compute only the largest Lyapunov exponent, the situation is ''better'' in the sense that the difficulty with spurious exponents does not exist. Indeed, there are methods for this task even in the presence of measurement noise of a few percent in magnitude (e.g., by Rosenstein *et al.*[40] and by Kantz[41]). Although the methods appear to be robust for data from maps, difficulties can arise for data from flows such as the Lorenz system.[41]

(2) Complexity of the brain dynamical system. The Lyapunov exponents may be useful for prediction if they can be computed reliably and accurately from short time series. This can hopefully be accomplished if the underlying dynamical system is primarily deterministic and relatively low-dimensional.[33–39] These requirements typically are not met by the brain dynamical system that generates the EEG or ECoG signals.

(3) Fundamental relationship between fractal dimension and Lyapunov spectrum. Our recent analysis of the correlation dimension for ECoG (Ref. 42) suggests strongly that this measure, while having the capability of tracking seizures, generally is not useful for seizure prediction. We have shown that it is not superior to simple methods based on quantities from linear stochastic analysis such as the autocorrelation. Since the fractal-dimension spectrum and the Lyapunov exponents of a dynamical system are fundamentally related, it is reasonable to suspect that the exponents will perform no better than the correlation dimension for the prediction of seizures.

The last point can be seen more explicitly, as follows. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d$ be the spectrum of Lyapunov exponents of a d-dimensional dynamical system. It is an elementary notion in nonlinear dynamics that the information dimension is an upper bound for the correlation dimension. In typical situations, the values of these two dimensions are close. The information dimension D_1, on the other hand, is conjectured by Kaplan and Yorke[43] to be the same as the Lyapunov dimension D_L, which is defined in terms of the Lyapunov spectrum, as follows:

$$D_L = J + \frac{\sum_{i=1}^{J} \lambda_i}{|\lambda_{J+1}|}, \tag{1}$$

where $1 \leqslant J \leqslant d$ is an integer that satisfies

$$\sum_{i=1}^{J} \lambda_i > 0 > \sum_{i=1}^{J+1} \lambda_i .$$

The Kaplan–Yorke conjecture was shown to be exact for random dynamical systems (e.g., deterministic system under noise) by Ledrappier[44] and Young.[45] Since $D_2 \lesssim D_1$ and since there is evidence[42] that D_2 is ineffective for early detection of seizures, Eq. (1) suggests that the Lyapunov exponents would not be useful for predicting seizures.

III. LYAPUNOV EXPONENTS FROM TIME SERIES

The Lyapunov exponents characterize how a set of orthonormal, infinitesimal distances evolve under the dynamics. For a d-dimensional dynamical system, there are d Lyapunov exponents. Here we briefly review the issues associated with computing them from time series.

Consider a dynamical system described by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \tag{2}$$

where $\mathbf{x} \in \mathbf{R}^d$ is a d-dimensional vector. Taking the variation of both sides of Eq. (2) yields an equation governing the evolution of infinitesimal vector $\delta\mathbf{x}$ in the tangent space at $\mathbf{x}(t)$,

$$\frac{d\,\delta\mathbf{x}}{dt} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \cdot \delta\mathbf{x}. \tag{3}$$

Solving for Eq. (3) gives

$$\delta\mathbf{x}(t) = \mathbf{A}^t \delta\mathbf{x}(0), \tag{4}$$

where $\mathbf{A}^t$ is a linear operator that evolves an infinitesimal vector from time 0 to time t. The mean exponential rate of divergence of the tangent vector is then given by

$$\lambda[\mathbf{x}(0),\delta\mathbf{x}(0)]=\lim_{t\to\infty}\frac{1}{t}\ln\frac{\|\delta\mathbf{x}(t)\|}{\|\delta\mathbf{x}(0)\|}, \tag{5}$$

where $\|\cdot\|$ denotes the length of the vector inside with respect to a Riemannian metric. In typical situations there exists a set of d-dimensional basis vectors $\{\mathbf{e}_i\}$ $(i=1,\ldots,d)$, in the following sense:

$$\lambda_i\equiv\lambda[\mathbf{x}(0),\mathbf{e}_i]. \tag{6}$$

These λ_i's define the Lyapunov spectrum, which can be ordered, as follows:

$$\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_d. \tag{7}$$

Values of λ_i do not depend on the choice of the initial condition $\mathbf{x}(0)$, insofar as it is chosen randomly in a proper phase-space region.

Computationally, there exist several methods for computing the Lyapunov spectrum from time series.[33–37] While details of these methods are different, they share the same basic principle. We have implemented the one developed by Eckmann *et al.*[36] The key is to find approximate Jacobian matrices along the trajectory in the reconstructed, m-dimensional phase space by using the delay-coordinate embedding technique, where $m>2d$.[31,32] The matrices govern the evolution of infinitesimal vectors in the tangent space. Given a trajectory point $\mathbf{x}_i$, we locate a point in a small ϵ-neighborhood of $\mathbf{x}_i$ and monitor how it evolves under dynamics, in order to figure out how a small vector $\delta\mathbf{x}_i$ at $\mathbf{x}_i$ evolves. Suppose after one time unit the small vector becomes $\delta\mathbf{x}_{i+1}$. We then have

$$\delta\mathbf{x}_{i+1}\approx\mathbf{T}(\mathbf{x}_i)\cdot\delta\mathbf{x}_i,$$

where $\mathbf{T}(\mathbf{x}_i)$ is the $m\times m$ Jacobian matrix at $\mathbf{x}_i$. In order to uniquely determine the matrix, m independent, orthonormal vectors in the neighborhood of $\mathbf{x}_i$ are required. Thus it is necessary to collect a number of points around $\mathbf{x}_i$. After the Jacobian matrices are computed, a QR-decomposition procedure can be used to yield the spectrum of Lyapunov exponents.[35,36]

Because the Jacobian matrices are meaningful only in the linear neighborhoods of trajectory points, the sizes of the neighborhoods must be small enough to ensure that the dynamics within are approximately linear. While smaller sizes in general can yield more accurate matrices, the length of the time series required will be greater. Roughly, in order to have a fixed number of points in a small region, as its size ϵ is decreased, the required length of the time series increases. Experience with time series from low-dimensional chaotic systems suggests that ϵ should be about a few percent of the size of the attractor. That is, if the time series is normalized to the unit interval, the choice of ϵ should be less than 5% (usually between 1% and 5%), and greater than the error due to the digitization precision of the data.

Another issue in the computation of Lyapunov exponents from time series is the inevitable occurrence of the *spurious* exponents. If the invariant set is d-dimensional, and an m-dimensional embedding space is used $(m>2d)$, there will be $m-d$ spurious Lyapunov exponents. For convenience, we call λ_i^e $(i=1,\ldots,m)$, all m exponents computed from time series, the *pseudo-Lyapunov spectrum*. For low-dimensional dynamical systems, *for specific choice of* m, and in a noise-free case there are criteria for distinguishing the spurious exponents from the true ones.[38] For instance, for a one-dimensional chaotic map where there is a positive Lyapunov exponent $\lambda>0$, the $(m-1)$ spurious exponents are $2\lambda,\ldots,m\lambda$. For a two-dimensional map (or equivalently, a three-dimensional flow) where there is a positive and a negative exponent, $\lambda_1>0>\lambda_2$ and for $m=5$, the pseudo-Lyapunov spectrum is: $\lambda_1^e\approx2\lambda_1$, $\lambda_2^e\approx\lambda_1$, $\lambda_3^e\approx\lambda_1+\lambda_2$, $\lambda_4^e\approx\lambda_2$, and $\lambda_5^e\approx2\lambda_2$. Unfortunately, at present there are no general criteria for determining the distribution of spurious Lyapunov exponents for an arbitrary system with arbitrary embedding dimension. The existing specific criteria can, however, be used to test whether the algorithm is coded correctly.

Based on the procedure described, we have developed a code for computing all Lyapunov exponents from a time series. To demonstrate that the code is reliable, we use two numerical examples: (1) the logistic map:[46] $x_{n+1}=4x_n(1-x_n)$ for which the only positive Lyapunov exponent is $\lambda=\ln 2$, and (2) the two-dimensional IHJM (Ikeda–Hammel–Jones–Moloney) map[47] that models the dynamics of a nonlinear optical cavity:

$$z_{n+1}=A+Bz_n\exp\left[ik-\frac{ip}{1+|z_n|^2}\right], \tag{8}$$

where $A=0.85$, $B=0.9$, $k=0.4$, $z=x+iy$ is a complex number, and p is a parameter. For $p=7.25$, the two Lyapunov exponents are $\lambda_1\approx0.36$ and $\lambda_2\approx-0.57$.

A time series from the logistic map was computed using an observed variable $y_n=\sin^3(5x_n)$. Three exponents were found using $m=3$, delay time $\tau=1$, and moving window length $N=10000$: $\lambda_1^e\approx2.3$, $\lambda_2^e\approx1.5$, and $\lambda_3^e\approx0.74$, of which the third is approximately the true exponent, while the first two are spurious and follow the distribution of spurious exponents for one-dimensional maps.[38] For the IHJM map, we set the system parameter $p=7.25$, embedding dimension $m=5$, delay time $\tau=1$, and moving window length $N=10000$. The pseudoexponents are approximately $\lambda_1^e\approx0.71\approx2\lambda_1$, $\lambda_2^e\approx0.33\approx\lambda_1$, $\lambda_3^e\approx-0.18\approx\lambda_1+\lambda_2$, $\lambda_4^e\approx-0.59\approx\lambda_2$, and $\lambda_5^e\approx-1.11\approx2\lambda_2$, which are the correct ones for two-dimensional maps.[38] These results indicate that the algorithm can correctly compute the Lyapunov exponents using finite time series from low-dimensional chaotic systems.

While our algorithm has been tested for maps, difficulties can arise if the data are from flow with unknown sampling rate. In this case, the Jacobian matrices can be such that the stretching rate per sampling interval is small, causing different eigendirections in the tangent space to have almost degenerate expansion/contraction factors. This can lead to inaccurate estimates of the Lyapunov exponents or spurious exponents. However, the relevant issue here is how any changes in the exponent can possibly be used to predict or detect critical events in the system. In this sense, whether the estimated exponents are true ones is less important. As we will show, even for data from map, the Lyapunov exponents are in general not useful for prediction. EEG or ECoG sig-

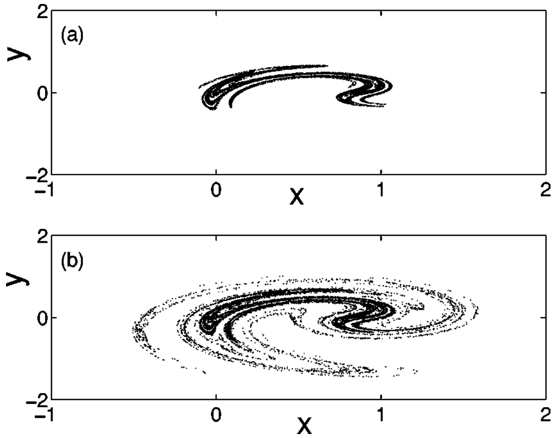

FIG. 1. For the IHJM map in Eq. (8), (a) a relatively small chaotic attractor before the interior crisis for $p=7.25$, (b) the larger attractor after the crisis for $p=7.4$.

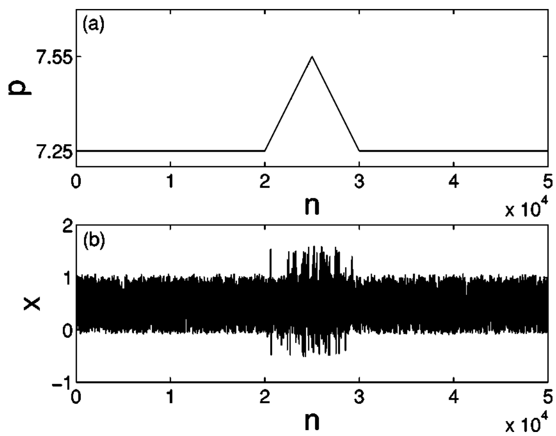

FIG. 2. For the IHJM map in Eq. (8), (a) parameter variation as described by Eq. (9), and (b) a typical time series which mimics a segment of ECoG data with a seizure.

nals are considered flow data. Thus it is reasonable to expect that the exponents will have no predictive power for seizures.

IV. PREDICTIVE POWER OF LYAPUNOV EXPONENTS IN LOW-DIMENSIONAL, DETERMINISTIC SYSTEMS

A. Discrete-time map model

To assess the predictive power of Lyapunov exponents computed from time series in a controllable way, we seek a model of deterministic chaotic system with parameter variations to simulate the nonstationary nature of ECoG data with seizure. We choose the IHJM map in Eq. (8) and allow temporal variation in the parameter p. To mimic ECoG data with seizure,[48] we choose p from an interval about the nominal value $p_c \approx 7.27$, at which there is an interior crisis.[49] Specifically, for $p \lesssim p_c$, there is a chaotic attractor of relatively small size in the phase space, as shown in Fig. 1(a) for $p=7.25$. At $p=p_c$, the small attractor collides with a preexisting, nonattracting chaotic set (chaotic saddle)[50] to form a larger attractor, as shown in Fig. 1(b) for $p=7.4$. For $p \gtrsim p_c$, a trajectory spends most its time in the phase-space region where the original small attractor resides, with occasional visits to the region in which the original chaotic saddle lies. A typical time series then consists of chaotic behavior of smaller amplitude most of the time, with occasional, randomly occurring bursts of relatively larger amplitude. Assume $p_0 \lesssim p_c$ so that the system is in a precrisis state but it is about to undergo a crisis. Then, the parameter changes through the critical value p_c, after sometime it comes back to the original, precrisis value p_0. During the time interval in which the parameter changes, we expect to observe characteristically different behavior (e.g., random motion of larger amplitude, as in the ictal phase in ECoG data). To be concrete, for the IHJM map we choose a time interval of 50000 iterations, and assume the following variation of the parameter p:

$$p_n = \begin{cases} p_0, & n < t_i = 20000 \\ p_0 + n(p_1 - p_0)/5000, & t_i \leqslant n < t_m = 25000 \\ p_1 - n(p_1 - p_0)/5000, & t_m \leqslant n < t_f = 30000 \\ p_0, & n > t_f, \end{cases} \tag{9}$$

where $p_0 = 7.25$ and $p_1 = 7.55$, are shown in Fig. 2(a). A typical time series $\{x_n\}$ is shown in Fig. 2(b), where we see a different behavior for $20000 < n \leqslant 30000$ during which the parameter variation occurs. The time series in Fig. 2(b) mimics a segment of ECoG data with a seizure. The average values of the two Lyapunov exponents in this ''ictal'' phase are $\lambda_1 \approx 0.42$ and $\lambda_2 \approx -0.63$.

We then choose a moving window and explore the predictability of the parameter change based on the pseudo-Lyapunov spectrum computed from the finite data set in the window.

B. Size of moving window and detectability

If the number N of data points in the moving window is small, the computed pseudo-Lyapunov exponents will have large fluctuations, as shown in Figs. 3(a)–3(f) for $m=5$ and $N=630$, where Fig. 3(a) shows the nonstationary time series and Figs. 3(b)–3(f) are the evolutions of λ_i^e $(i=1,\ldots,m)$. We

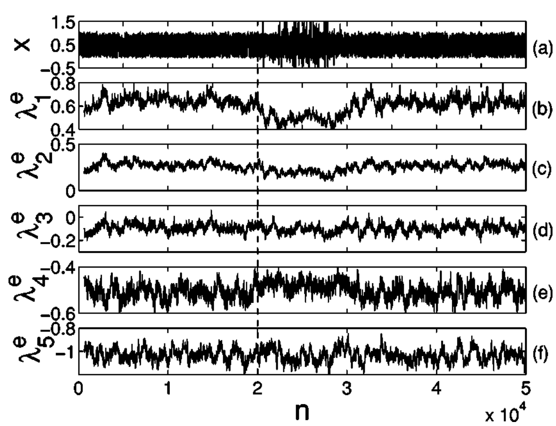

FIG. 3. For the IHJM map in Eq. (8), $m=5$ and $N=630$, (a) nonstationary time series, (b–f) temporal evolution of λ_i^e for $i=1,\ldots,5$, respectively.

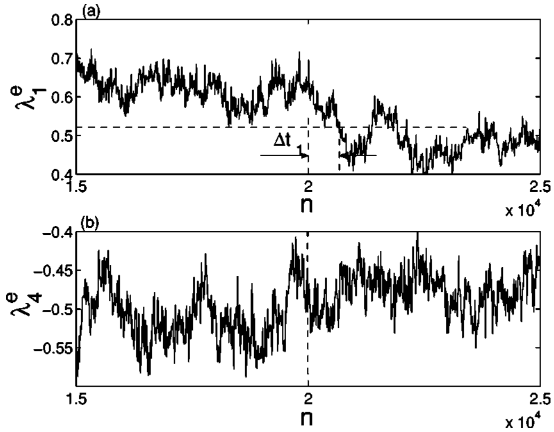

FIG. 4. (a,b) Blow-ups of Figs. 3(b) and 3(e), respectively.

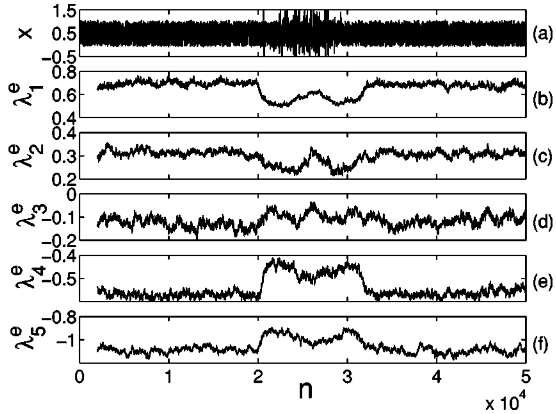

FIG. 5. For the IHJM map in Eq. (8), $m=5$ and $N=1995$, (a) nonstationary time series, (b–f) temporal evolution of λ_i^e for $i=1,\ldots,5$, respectively.

see that, comparing with the asymptotic values of the two Lyapunov exponents ($\lambda_1 \approx 0.42$ and $\lambda_2 \approx -0.63$), the second and the fourth (λ_2^e and λ_4^e) are approximately the true exponents, while the remaining are spurious ones. The vertical dashed line indicates t_i, the time when the control parameter p starts to change. The parameter change is somewhat reflected in λ_1^e. For the average change in λ_1^e to be statistically significant, where the change should be greater than the average amount of fluctuation, the time required is about $\Delta t_1 \approx 700$ after t_i for λ_1^e, as can be seen in Fig. 4(a), a blowup of part of Fig. 3(b) around t_i. Other exponents show no statistically discernible changes after t_i, as represented by the behavior of λ_4^e in Fig. 4(b), which is a blowup of part of Fig. 3(e). One question is whether increasing N would help reduce Δt.

Consider the situation where N is large. Given this finite time N, we imagine choosing a large number of initial conditions and compute the Lyapunov spectra for all the resulting trajectories of length N. The exponents computed in finite time are effectively random variables whose histograms can be constructed. For trajectories on a chaotic attractor, the typical distribution of a finite-time Lyapunov exponent λ_N is[51]

$$P(\lambda_N,N) \approx \left[\frac{NG''(\overline{\lambda})}{2\pi}\right]^{1/2} \exp\left[-\frac{N}{2}G''(\overline{\lambda})(\lambda_N-\overline{\lambda})^2\right], \tag{10}$$

where $\overline{\lambda}$ is the asymptotic value of λ_N in the limit $N \to \infty$, and $G(x)$ is a function satisfying $G(\overline{\lambda})=0$, $G'(\overline{\lambda})=0$, and $G''(\overline{\lambda})>0$. For large N, the standard deviation of λ_N is

$$\sigma_{\lambda_N} \sim \frac{1}{\sqrt{N}}. \tag{11}$$

If the moving time window is located completely in $t<t_i$, the average Lyapunov exponent is

$$\lambda_N = \frac{1}{N}\sum_{i=1}^{N} \lambda^{(1)}(i), \tag{12}$$

where $\lambda^{(1)}(i)$ is the time-one Lyapunov exponent for $t<t_i$. Now consider a moving time window across the critical time t_i, where N_1 points are before t_i, N_2 points are after, and $N_1+N_2=N$. The computed exponent is

$$\lambda_N' = \frac{1}{N}\left[\sum_{i=1}^{N_1} \lambda^{(1)}(i) + \sum_{i=1}^{N_2} \lambda^{(2)}(i)\right], \tag{13}$$

where $\lambda^{(2)}(i)$ is the time-one Lyapunov exponent for $t>t_i$. Let $\overline{\lambda^{(1)}}$ and $\overline{\lambda^{(2)}}$ be the asymptotic values of the Lyapunov exponent for $t<t_i$ and $t>t_i$, respectively. If $N_1 \gg 1$, $N_2 \gg 1$, $N_1 \sim N$, and $N_2 \sim N$, we can write

$$\sum_{i=1}^{N} \lambda^{(1)}(i) = N\overline{\lambda^{(1)}} + \mathcal{O}(1/\sqrt{N}), \tag{14}$$

$$\sum_{i=1}^{N_1} \lambda^{(1)}(i) = N_1\overline{\lambda^{(1)}} + \mathcal{O}(1/\sqrt{N_1}) \approx N_1\overline{\lambda^{(1)}} + \mathcal{O}(1/\sqrt{N}),$$

$$\sum_{i=1}^{N_2} \lambda^{(2)}(i) = N_2\overline{\lambda^{(2)}} + \mathcal{O}(1/\sqrt{N_2}) \approx N_2\overline{\lambda^{(2)}} + \mathcal{O}(1/\sqrt{N}),$$

where $\mathcal{O}(1/\sqrt{N})$ is a number on the order of $1/\sqrt{N}$. The change in the computed time-N exponent is thus

$$\begin{aligned}\Delta\lambda_N &= \lambda_N - \lambda_N' \\ &\approx \frac{1}{N}[N\lambda^{(1)} - N_1\lambda^{(1)} - N_2\lambda^{(2)}] + \mathcal{O}(1/\sqrt{N}) \\ &= \frac{N_2}{N}(\lambda^{(1)} - \lambda^{(2)}) + \mathcal{O}(1/\sqrt{N}) \sim \frac{N_2}{N}.\end{aligned} \tag{15}$$

For the change in the Lyapunov exponent to be statistically significant and thus detectable, we require $\Delta\lambda_N \gtrsim \sigma_{\lambda_N}$, which gives the time required to detect the change,

$$\Delta t = N_2 \gtrsim \sqrt{N}. \tag{16}$$

We see that increasing the size of the moving window in fact causes an increase in the time required to detect a change in the Lyapunov exponent. The increase is, however, incremental as compared to the increase in N and therefore may not be easily observed. In numerical experiments, we will not see an apparent decrease in Δt when N is increased.

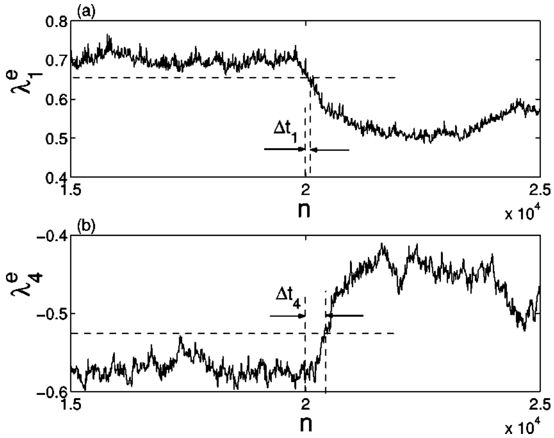

FIG. 6. (a,b) Enlargements of Figs. 5(b) and 5(e), respectively. The respective detection times are $\Delta t_1 \approx 100$ and $\Delta t_4 \approx 700$.

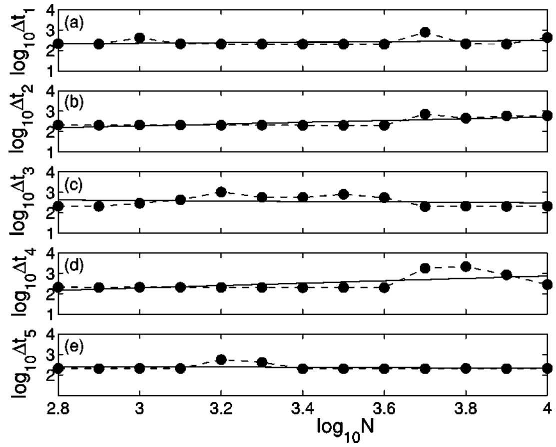

FIG. 7. For the IHJM map in Eq. (8) and $m=5$, (a–e) detection times Δt_i for λ_i^e ($i=1,\ldots,5$, respectively) versus N.

Figures 5(b)–5(f) show, for $m=5$ and $N=1995$, the temporal evolution of λ_i^e ($i=1,\ldots,5$), respectively. A discernible change in the exponents can be seen for $t \gtrsim t_i$, particularly in λ_1^e and λ_4^e. Figures 6(a) and 6(b) show enlargements of the evolution of λ_1^e and λ_4^e around t_i, respectively, where we see that the time required for detection of the system change is $\Delta t_1 \approx 100$ for λ_1^e and $\Delta t_4 \approx 700$ for λ_4^e. As N is increased, the level of fluctuations in the Lyapunov exponents is reduced but the detection time is not reduced, as shown in Figs. 7(a)–7(e), numerically determined detection times versus N for λ_i^e ($i=1,\ldots,5$), respectively. In all cases, the detection time shows a slight increase as N is increased, which is consistent with our analysis.

C. Predictive power of Lyapunov exponents

Our results in Sec. IV B indicate that while the critical change of the system state can be detected through the pseudo-Lyapunov spectrum from time series, it is not clear whether the change can be *predicted* in advance. To goal of our control test is to assess, for the model system, whether Lyapunov exponents possess any predictive power for critical change of the system state.

We conceive that the onset of seizure corresponds to the transition of the system through a critical state. In order to be able to predict the seizure in advance, it may be assumed that the state of system undergoes slow changes before seizure onset. The question is whether any state change before the critical point (onset of the seizure) can be detected through the pseudo-Lyapunov exponents. Motivated by this, we consider the following relatively simple situation: Suppose a critical event occurs in which the system bifurcates to a characteristically different state. However, before the event, the parameter changes smoothly toward the critical bifurcation, although perhaps not at the same rate as that at which it passes through the critical point. For the IHJM map, we thus consider the scheme of parameter change, as shown in Fig. 8(a), where initially the parameter p is fixed at a constant value ($p=7.1$) below the critical point p_c. As p passes through p_c at about $n \gtrsim 20000$, a critical event (interior crisis) occurs. Before this, p is assumed to change at a slower rate for $10000 < n < 20000$. The entire time interval of interest is taken to be 40000 iterations. If we measure the time series before the critical point there is no apparent characteristic change, despite the slow change in parameter,[52] as shown in Fig. 8(b). This setting thus represents an appropriate test bed for the predictive power of pseudo-Lyapunov exponents for critical events.

We proceed by choosing a moving window containing N data points and examining any possible changes in the pseudo-Lyapunov spectrum. When N is small, the large fluctuations in the exponents render undetectable the slow parameter changes preceding the onset of crisis. This indicates that the crisis cannot be predicted when N is small. As N is increased, the fluctuations are reduced so that the system change preceding the crisis can be detected, as shown in the behaviors of λ_i^e in Figs. 9(b)–9(f), respectively, for $m=5$ and $N=3981$. The change indeed can be detected at time $n \gtrsim 10000$, which far precedes the crisis. *While this seems to indicate that the exponents have the predictive power for crisis, we find that the presence of small noise can wipe out this power completely*.

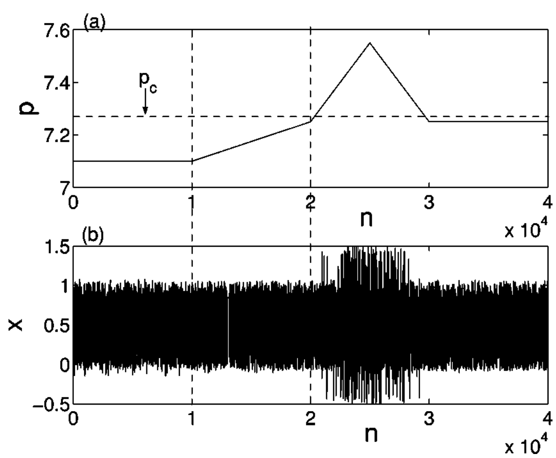

FIG. 8. Our scheme for testing the predictive power for critical event of Lyapunov exponents from time series, utilizing the IHJM map. (a) Parameter variation with time, where it changes slowly before the interior-crisis point, and (b) a typical time series that shows no characteristic change before the crisis, despite the parameter change.

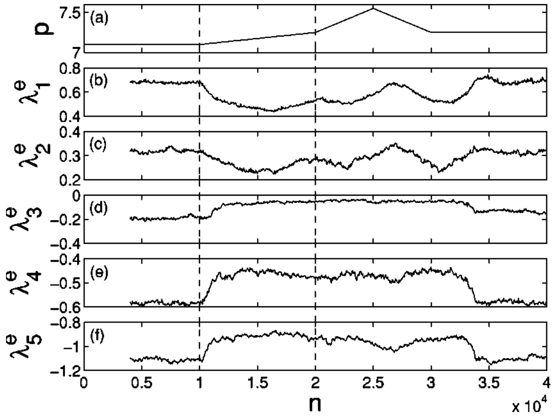

FIG. 9. (a) Scheme of parameter variation with time. (b–f) Temporal evolutions of λ_i^e ($i=1,\ldots,5$) for $m=5$ and $N=3981$, in the absence of noise. In this case, the parameter change preceding the crisis can be detected through the pseudo-Lyapunov exponents.

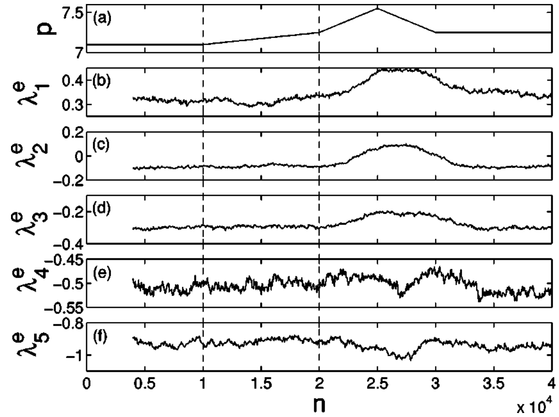

FIG. 11. (a) Scheme of parameter variation with time, (b–f) temporal evolutions of λ_i^e ($i=1,\ldots,5$) for $m=5$, $N=3981$, and noise amplitude $D=10^{-2.0}$ (corresponding to about 0.5% of the amplitude of the measured data). At this noise level the crisis cannot be predicted in advance because the parameter change preceding the crisis cannot be detected.

To simulate noise, we add two terms $D\xi_n^x$ and $D\xi_n^y$ to the x- and y-equations of the IHJM map, where D is the noise amplitude, and ξ_n^x and ξ_n^y are independent random variables uniformly distributed in $[-1,1]$. Figures 10(b)–10(f), 11(b)–11(f), and 12(b)–12(f) show, for $m=5$ and $N=3981$, temporal evolutions of the five pseudo-Lyapunov exponents for noise levels $D=10^{-2.6}$, $D=10^{-2.0}$, and $D=10^{-1.0}$, respectively. Note that the range of the time series from the IHJM map is about 2.0, so these noise levels roughly correspond to 0.1%, 0.5%, and 5% of the variation of the dynamical variable, which can be considered as small. We observe that there is a progressive deterioration of the predictive power of the exponents, as the parameter change preceding the crisis can no longer be detected at the noise level of about $D=10^{-2.0}$. For relatively larger noise [$D=10^{-1.0}$ in Figs. 12(b)–12(f)], even the critical event (crisis) itself cannot be detected through the variation of these exponents. These results suggest that in practical situations where small noise is inevitable, one should not expect the Lyapunov exponents computed from time series to have any predictive power, as it appears unlikely that their variations are statistically significant enough to allow for detection of system change preceding a critical event.

D. Comparison with autocorrelation

To provide a means for comparison of the Lyapunov exponents' predictive abilities, we compute from time series $x(t)$ an approximation of the decay of the autocorrelation envelope,[53]

$$\alpha=\frac{1}{M}\sum_{k=1}^{M}|R_k|^{1/k}, \tag{17}$$

where

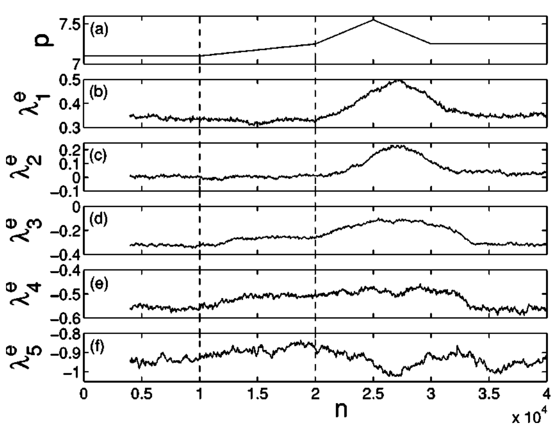

FIG. 10. (a) Scheme of parameter variation with time, (b–f) temporal evolutions of λ_i^e ($i=1,\ldots,5$) for $m=5$, $N=3981$, and noise amplitude $D=10^{-2.6}$ (corresponding to about 0.1% of the variation of the time series). At this noise level the crisis arguably can be predicted in advance (through, for example, λ_3^e, λ_4^e, and λ_5^e).

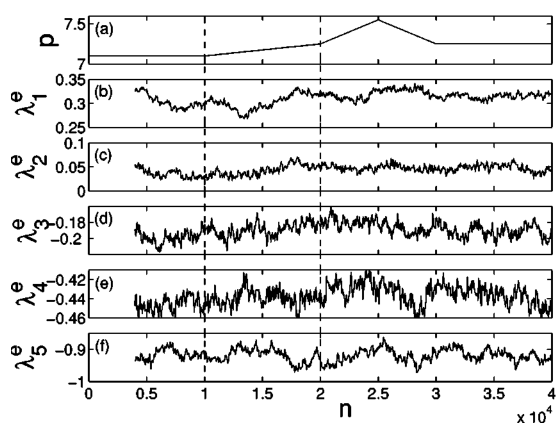

FIG. 12. (a) Scheme of parameter variation with time, (b–f) temporal evolutions of λ_i^e ($i=1,\ldots,5$) for $m=5$, $N=3981$, and noise amplitude $D=10^{-1.0}$. At this noise level, which is about 5% of the amplitude of the measured time series, even the crisis itself cannot be detected through the exponents.

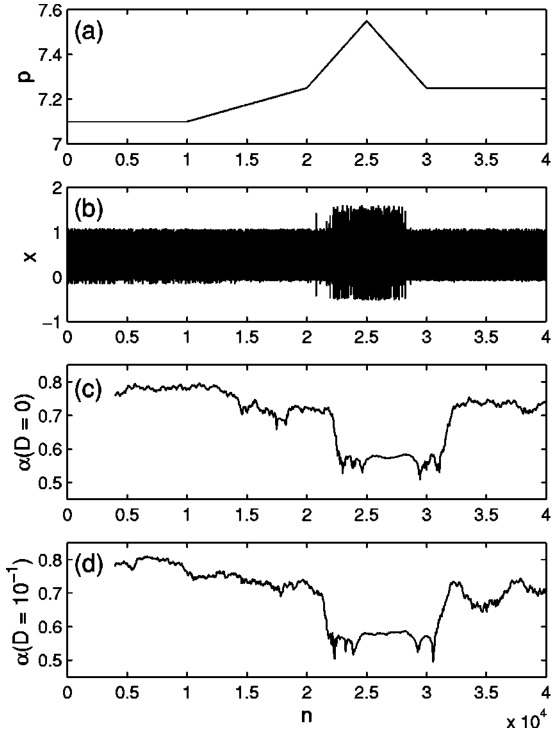

FIG. 13. (a) Parameter variation in time. (b) Time series of **x** in the absence of noise $(D=0)$. (c) Plot of α computed in 4000 point windows, overlapped by 3960 points computed on the time series in (b). (d) The same as in (c) except that there is a noise of amplitude $D=10^{-1}$.

$$R_\tau=\frac{\Sigma x(t)x(t+\tau)}{\sqrt{\Sigma x^2(t)\Sigma x^2(t+\tau)}}, \tag{18}$$

and $M=6$. Using the same model as in Fig. 8, we compute α on sliding windows of length 4000 points with an overlap of 3960 points. The noise-free case is shown in Fig. 13(c). Both the parameter drift preceding the crisis and the crisis itself are clearly visible in the α time series. In contrast to the Lyapunov exponents, the autocorrelation's ability to detect this drift appears robust even under moderate noise ($D=10^{-1}$, or 5%), as shown in Fig. 13(d). For this noise value, the Lyapunov spectrum could not even detect the crisis as shown in Fig. 12.

We recently performed a comparison of the autocorrelation and the correlation dimension,[42] and found that the two measures tend to track each other in seizure, though neither demonstrated any predictive ability. Before and after the seizure, the value of the correlation dimension is approximately constant, but it fluctuates significantly during the seizure, which we showed indicates a dramatic loss and gain of the autocorrelation, alternating in time. The implication, as in the present case, is that traditional analyses of stochastic processes or linear time-frequency analyses may be as effective (if not more effective) for analysis of ECoG signals, including seizure prediction.

E. Continuous-time model

While we used a discrete-time map model to illustrate the predictive and detective powers of the Lyapunov exponents, a question is what happens to continuous-time systems. A related issue concerns the nature of the bifurcation. In particular, in our discrete-time map model, the critical event that we used to model seizure is interior crisis, which is a global bifurcation. One might argue that epileptic seizures may be a local bifurcation. Indeed, for an interior crisis, the properties of the attractor before the bifurcation are generally not affected by the fact that the crisis will happen. In particular, the sensitivity of the dynamical invariants of the attractor such as the dimensions, Lyapunov exponents, and entropies to the control parameter is about the same in the parameter regime before the crisis. For a local bifurcation, for instance, a Hopf bifurcation at which a stable steady state becomes unstable and a stable limit cycle is born, the Lyapunov exponent may be a stronger indicator for the bifurcation. To address these issues we now consider a continuous-time model with a Hopf bifurcation and investigate the sensitivity of Lyapunov exponents to parameter changes in the presence of noise.

We use the following two-dimensional canonical model for Hopf bifurcation, under white noise of amplitude D:

$$\frac{dx}{dt}=-y+x[a(t)-x^2-y^2]+D\xi_1(t),$$

$$\frac{dy}{dt}=x+y[a(t)-x^2-y^2]+D\xi_2(t), \tag{19}$$

where $a(t)$ is a control parameter that can vary with time, and $\xi_1(t)$ and $\xi_2(t)$ are independent Gaussian random variables of zero mean and unit variance with the following properties: $\langle\xi_1(t)\xi_1(t')\rangle=\delta(t-t')$, $\langle\xi_2(t)\xi_2(t')\rangle=\delta(t-t')$, and $\langle\xi_1(t)\xi_2(t')\rangle=0$. When $a(t)$ is constant, the stochastic processes $x(t)$ and $y(t)$ are stationary; otherwise they are nonstationary. For the deterministic system $(D=0)$, if $a<0$, the attractor of the sytem is a steady state defined by $x=0$ and $y=0$. The Hopf bifurcation occurs at $a_c=0$ where for $a>0$, the steady state becomes unstable and a limit-cycle attractor, given by $x(t)=a\cos t$ and $y=a\sin t$, becomes stable. The period of the oscillation $(T_0\equiv 2\pi)$ thus defines the natural time scale of the system. To mimic a seizure, we examine a time interval of 9000 cycles of oscillation, which corresponds to actual time of $T=9000T_0$, and divide this time into three intervals: $(0,T_1)$, (T_1,T_2), and (T_2,T). The parameter variations in these intervals are chosen such that in the first and third intervals the attractor of the system is the steady state ($x=0$ and $y=0$) but in the middle interval the attractor is the limit-cycle oscillator. In particular, we assume

$$a(t)=\begin{cases} a_0+(a_1-a_0)t/T_1 & \text{for } 0<t<T_1, \\ a_1+2(a_2-a_1)(t-T_1)/(T_2-T_1) & \text{for } T_1<t<T_1+(T_2-T_1)/2, \\ a_2-2(a_2-a_1)[t-T_1-(T_2-T_1)/2]/(T_2-T_1) & \text{for } T_1+(T_2-T_1)/2<t<T_2, \\ a_1-(a_1-a_0)(t-T_2)/(T-T_2) & \text{for } T_2<t<T, \end{cases} \tag{20}$$

as shown in Fig. 14(a) for $a_0=-0.25$, $a_1=0.0$, $a_2=1.0$, $T_1=4000T_0$, and $T_2=5000T_0$. A typical time series from this nonstationary system is shown in Fig. 14(b), where the noise amplitude is $D=10^{-2}$. Analogous to the terms of epilepsy, the three intervals of time can be conveniently called preictal, ictal, and postictal phases, respectively. For the stationary system $[a(t)=a=\text{constant}]$, the theoretical values of the two Lyapunov exponents for $a<0$ are $\lambda_1=\lambda_2=a<0$. After the limit-cycle attractor is born via the Hopf bifurcation at a_c, the exponents are $\lambda_1=0$ and $\lambda_2=-2a<0$. Thus, for the nonstationary system as in Figs. 14(a) and 14(b), the theoretical value of the largest Lyapunov exponent is negative for the preictal and postictal phases, while it is zero for the ictal phase.

To obtain time series from the model (19), we use the standard second-order, Heun's method for solving stochastic differential equations.[54] In particular, for the system in Figs. 14(a) and 14(b), we use the step size $h=0.01$ in numerical integration and generate time series $x(t)$ [or $y(t)$] using the sampling interval of $t_s=40h$, corresponding to approximately 16 points per oscillating period. Lyapunov exponents are then computed from the time series using moving time window of width $\Delta t\approx 636T_0$, spaced at $t_w\approx 12.7T_0$. The delay time used is $\tau\approx 0.95T_0$ (approximately one cycle of the natural oscillation) and the embedding dimension is chosen to be $m=3$ (considering that the steady-state and limit-cycle attractors are only zero- and one-dimensional, respectively). Due to noise, for the preictal and postictal phases, the first two Lyapunov exponents from the moving windows are positive, which are spurious, as shown in Figs. 14(c) and 14(d). However, as indicated in these plots, in the ictal phase where theoretically the largest exponent is zero, the algorithm seems to be able to capture the correct value. There is thus a relatively sharp change in the estimated value of the exponents shortly after the onset of the ictal phase, indicating that the exponents are capable of detecting the local, Hopf bifurcation in spite of the presence of noise. Note that, however, the noise level for Figs. 14(c) and 14(d) are relatively small: about 1% of the amplitude of the oscillation in the ictal phase. As the time series becomes more noisy, as shown in Fig. 15(b) for $D=0.1$ (about 10% of the oscillation in the ictal phase), the ability for the Lyapunov exponents to detect even this local bifurcation deteriorate, as shown in Figs. 15(c) and 15(d).

Our results thus demonstrate that Lyapunov exponents from time series are capable of detecting simple, local bifurcations in the presence of noise. However, as we described, this task of detection can also be accomplished by using measures from traditional stochastic analysis such as the autocorrelation. Taking into account the computational complexity, Lyapunov exponents are arguably disadvantageous for detection. On the other hand, ECoG signals typically come from a large number of neurons [approximately 10^5-10^8 (Ref. 27)]. It may not be suitable to regard epileptic seizures as being caused by some local bifurcations. This again suggests that Lyapunov exponents are not useful for predicting or detecting seizures.

V. TESTS USING ECoG DATA

The data used here were collected from patients with pharmaco-resistant seizures who underwent evaluation for

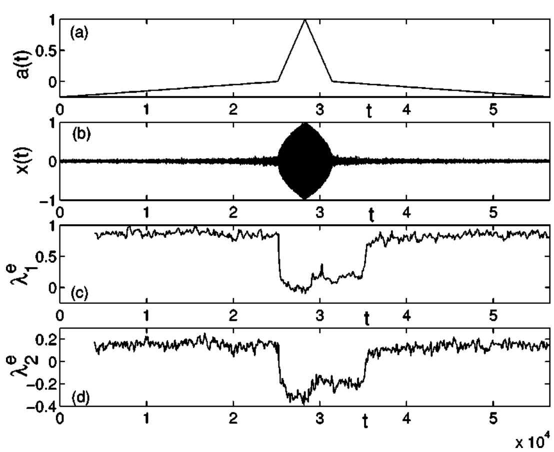

FIG. 14. For the nonstationary, continuous-time model (19) with a local Hopf bifurcation under noise of amplitude $D=0.01$, (a) the parameter variation, (b) the noisy time series $x(t)$, and (c,d) the first two Lyapunov exponents computed from moving window over the time series, which appear to be able to detect the bifurcation. See text for simulation parameters.

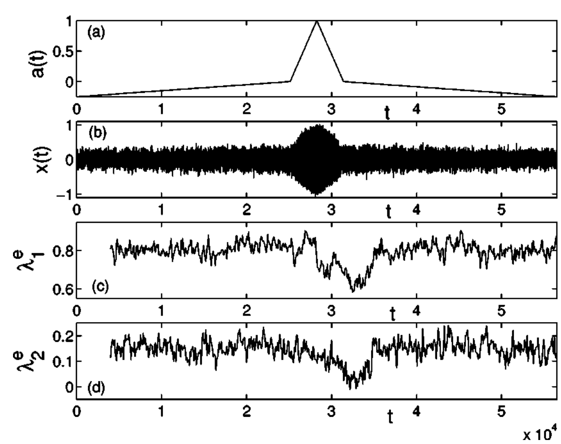

FIG. 15. (a–d) The same as in Figs. 14(a)–14(d), respectively, except that the noise level is now $D=0.1$. The ability for the Lyapunov exponents to detect even this local bifurcation apparently deteriorate, as compared with the case of lower noise in Fig. 14.

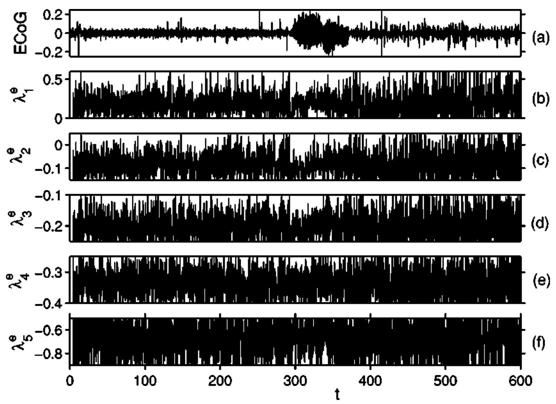

FIG. 16. (a) A segment of ECoG time series containing a seizure which starts at approximately $t=300$ s and lasts for about 80 s. (b–f) For $m=5$ and $\Delta t \approx 4.17$ s (corresponding to $N=1000$ data points), the five computed Lyapunov exponents versus time, where time is counted as the end of the moving window.

epilepsy surgery at the University of Kansas Comprehensive Epilepsy Center. The data were recorded via depth electrodes (Ad-Tech), implanted stereotaxically into the amygdalo-hippocampal region. Correctness of the placement is assessed with MRI. The signal is sampled at a rate of 240 Hz, amplified to a dynamic range of ± 300 μV, and digitized to 10 bits precision with 0.59 μV/bit using commercially available devices (Nicolet, Madison, WI). The recording was deemed of good technical quality and suitable for analysis. For convenience, the data set is linearly normalized to the unit interval. We have tested 11 seizures from two patients, all indicating a lack of predictive power of the Lyapunov exponents. In the following we present results with one seizure.

When computing the pseudo-Lyapunov exponents from ECoG time series, there are several computational parameters that can affect the results. These are: the length N of the moving window, the embedding dimension m, the delay time τ, and the size ϵ of the linear neighborhood. We find that the computed exponents are relatively robust against variations in τ and ϵ, insofar as they are chosen properly. The choice of the delay time τ is quite straightforward. The empirical criterion is that adjacent time-delayed components should serve as independent variables. If τ is too small, the adjacent components will be too correlated for them to serve as independent coordinates. If τ is too large, then neighboring components are too uncorrelated. Empirically, given an ECoG signal $x(t)$, one chooses τ such that $R_\tau = 1/e$.[53] We fix $\tau = 1/12$ s. For ϵ, we find that computational results vary little when it is chosen to be around 0.02 (2% of the amplitude of the ECoG signal). We thus fix $\epsilon = 0.02$. In what follows, we will systematically examine the effects of varying the two key parameters: N and m.

Intuitively, shorter time series result in larger fluctuations in the computed Lyapunov exponents. As we increase the length of the moving window, we expect to see an apparent decrease in the level of fluctuations. To demonstrate this effect, we fix the embedding dimension at $m=5$. Figure 16(a) shows the segment of ECoG time series of 600 s containing a seizure, which occurs at $t \approx 300$ s. Figures 16(b)–16(f) show, for a moving window of length $\Delta t \approx 4.17$ s (corresponding to $N=1000$ data points), the five computed Lyapunov exponents versus time, where the time is recorded at the right edge of the window. All exponents exhibit significant fluctuations, which are reduced as N is increased, as shown in Figs. 17(b)–17(f) for $\Delta t \approx 13.18$ s (corresponding to $N=10^{3.5}=3162$), in Figs. 18(b)–18(f) for $\Delta t \approx 52.5$ s (corresponding to $N=10^{4.1}=12589$), and in Figs. 19(b)–19(f) for $\Delta t \approx 131.8$ s (corresponding to $N=10^{4.5}=31623$). Despite the reduction in the fluctuations of the pseudo-Lyapunov exponents, there is no indication that any statistically significant change in these exponents occur before, during, and after the seizure, suggesting that the computed exponents are not capable of distinguishing among preseizure, seizure, and postseizure phases, let alone being able to *predict* the occurrence of the seizure in advance.

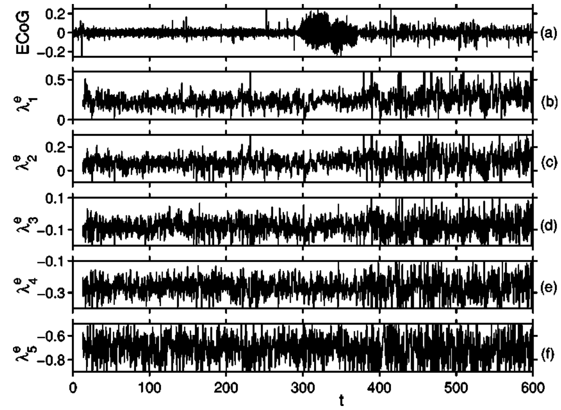

FIG. 17. (b–f) For $m=5$ and $\Delta t \approx 13.18$ s (corresponding to $N=10^{3.5}=3162$), the five computed Lyapunov exponents versus time.

There is thus no indication that the temporal behavior of the pseudo-Lyapunov exponents (Figs. 16–19) predicts the seizure.

VI. DISCUSSIONS

Successful and robust prediction of epileptic seizures is challenging. Our experience suggests that a systematic and generally applicable methodology for seizure prediction is

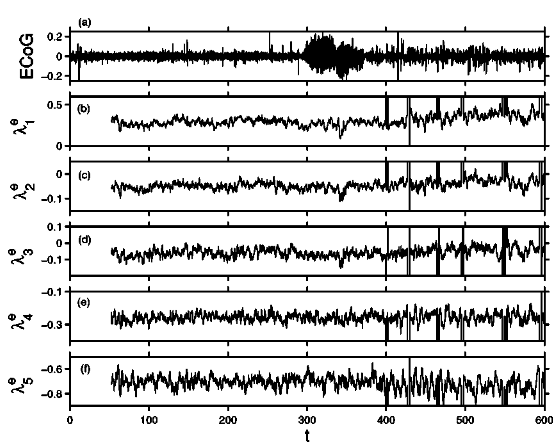

FIG. 18. (b–f) For $m=5$ and $\Delta t \approx 52.5$ s (corresponding to $N=10^{4.1}=12589$), the five computed Lyapunov exponents versus time.

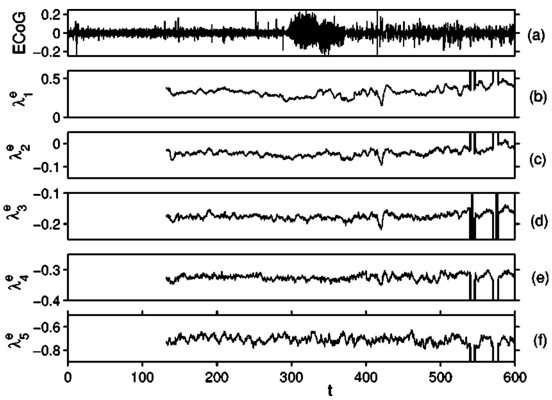

FIG. 19. (b–f) For $m=5$ and $\Delta t \approx 131.8$ s (corresponding to $N=10^{4.5}=31623$) the five computed Lyapunov exponents versus time.

still lacking, despite existing claims.[1–4,10,11] This is especially true when techniques designed for low-dimensional nonlinear dynamical systems are used. There are two fundamental reasons for this: (1) EEG or ECoG signals are complicated, nonlinear, nonstationary, high-dimensional and noisy; (2) the techniques may not be sufficiently sensitive to discriminate random behaviors with subtle differences, though they are highly effective in distinguishing between regular and chaotic behaviors. From this viewpoint, it is uncertain whether nonlinear-dynamics based techniques would perform better than the techniques from random signal processing or linear time-frequency-energy techniques. Prediction of seizure based on EEG or ECoG signals thus remains largely an open problem.

Lyapunov exponents are fundamental invariant quantities characterizing a dynamical system. They measure the exponential growth rates of orthonormal, infinitesimal vectors in the phase space. To determine them from time series, when the underlying mathematical model is unknown, is one of the most challenging tasks in nonlinear dynamics. While algorithms based on phase-space reconstruction by delay-coordinate embedding have existed for about two decades, issues such as the distribution of spurious exponents in low-dimensional chaotic systems[38] and the accuracy of the estimated exponents[39] have been addressed only recently. In deterministic chaotic systems, the Lyapunov exponents depend on parameter values. This is perhaps one of the main facts that motivate researchers to explore the possibility of utilizing the Lyapunov exponents for significant applications in biomedical sciences and engineering, despite the computational difficulty.

In realistic situations Lyapunov exponents can be computed only in finite time windows. This is particularly relevant to applications concerning prediction, where a moving time window containing a finite number of data points is used and the exponents are computed from this finite data set. This paper addresses the predictive power of the Lyapunov exponents in a systematic way. Our results indicate that there are two major factors that can prevent the Lyapunov exponents from being effective to predict characteristic system changes: *statistical fluctuations and noise*.

It is known[51] in nonlinear dynamics that finite-time Lyapunov exponents can exhibit random fluctuations [Eq. (10)], which present a serious obstacle to their predictive power because any changes in the exponents must be larger than the fluctuations in order for them to be indicative of system changes. Increasing the size of a moving window will not decrease the detection time for system changes, as we have shown in this paper. This implies that any characteristic change of the system must be significant enough for it to be detected through the Lyapunov exponents, regardless of the size of the finite data set contained in the moving window, insofar as it is statistically meaningful.

We have also demonstrated that relatively small parameter changes in the system, which precede a critical event, can indeed be detected through the changes in the Lyapunov exponents. Thus, if the small system changes are regarded as "precursors" of the critical event, its occurrence can indeed be predicted in advance. However, this predictive power of the Lyapunov exponents can be ruined completely by noise with magnitude as small as less than 1% of the variation of the system variable. As the noise level is increased to about 5% of the variation, even the detective power of the exponents is lost.

We have obtained these results through a deterministic chaotic systems modeled by the two-dimensional, IHJM map. Since the map has been a paradigm to address many fundamental issues in chaotic dynamics, we believe our results are fairly general, at least for low-dimensional chaotic systems. The basic message is that even for such low-dimensional, relatively controllable systems, the predictive power of the Lyapunov exponents holds only in noiseless or extremely low-noise situations. In realistic situations where an appreciable but reasonable amount of noise is present, the exponents are useless for predictions even for low-dimensional, deterministic dynamical systems.

The brain dynamical systems responsible for the epileptic seizures are much more complicated than low-dimensional chaotic systems or even idealized high-dimensional systems such as coupled map lattices. In epilepsy, all information is from a few dozen probes, each sensing approximately 10^5–10^8 neurons[27] into the corresponding neuron ensemble in the brain about which relatively little is known. The signals so obtained (ECoG) are inevitably noisy. These considerations suggest that the Lyapunov exponents do not appear to have any predictive or detective powers for epileptic seizures.

We are certainly hopeful that nonlinear dynamics can offer useful methodology for understanding[55,56] and possibly predicting seizures, but the tools would perhaps be based on spatiotemporal information as can be offered by multichannel ECoG recordings. Possible candidates include synchronization-based techniques[57] that has been successful in detecting very subtle correlations between biomedical signals.[58]

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