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Perspectives on relativistic quantum chaos

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Abstract

Quantum Chaos has been investigated for about a half century. It is an old yet vigorous interdisciplinary field with new concepts and interesting topics emerging constantly. Recent years have witnessed a growing interest in quantum chaos in relativistic quantum systems, leading to the still developing field of *relativistic quantum chaos*. The purpose of this paper is not to provide a thorough review of this area, but rather to outline the basics and introduce the key concepts and methods in a concise way. A few representative topics are discussed, which may help the readers to quickly grasp the essentials of relativistic quantum chaos. A brief overview of the general topics in quantum chaos has also been provided with rich references.

Keywords: quantum chaos, relativistic quantum systems, Dirac billiards

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum chaos is a branch of fundamental physics investigating the intercapillary field of quantum mechanics, statistical physics, and nonlinear dynamics [1–8]. Even before the establishment of quantum mechanics, in 1913, Bohr proposed quantization rule and used it to successfully predict the energy spectrum of hydrogen atom, which explained the Balmer formula obtained from experimental observations well. Later in 1917, Einstein extended Bohr's quantization rule to integrable systems with global torus structure in phase space [9]. Then he noticed that these quantization rules are only applicable to integrable systems, and would fail for more general, non-integrable systems [9, 10]. About a half century later, in 1970s, inspired by extensive investigations of nonlinear dynamics and chaos, the issue of how to extend the semiclassical quantization rule to non-integrable systems was perceived again by the community, leading to the development of Gutzwiller's trace formula that although being measure zero, the unstable periodic orbits play a crucial rule in shaping the quantum spectral fluctuation behaviors [5, 11–23]. There are quantum systems, e.g. quantum

billiards, whose classical counterpart can be chaotic. It is thus mystical that since the Schrödinger equation is linear and thus there is no real chaos in the quantum system, how does it emerge in the semiclassical limit? Note that here 'quantum system' is specifically for the single particle system described by the Schrödinger equation, where many-body effects are excluded. Alternatively, how does the nonlinear and chaotic dynamics that are ubiquitous in the classical world affect the behavior of the corresponding quantum systems? Are there any indicators in the quantum system that can be used to tell whether its semiclassical limit is integrable or chaotic? The efforts to understand these questions and the results constitute the field of quantum chaos, which has attracted extensive attention during the past half century, and has led to profound understandings to the principles of the classical-quantum correspondence [3].

An intriguing phenomenon regarding classical orbits in quantum systems is quantum scar [15, 24–33], where chaotic systems leave behind scars of paths that seem to be retraced in the quantum world [34–40]. Classically, a chaotic system has periodic orbits, but the chaotic nature renders all the orbits being unstable, i.e. an arbitrarily small perturbation to the

particle moving along such an orbit could push its motion out of the orbit completely. A paradigmatic model is the two dimensional billiard system, where a particle moves freely inside the billiard, and reflects specularly at the boundary. Thus the shape of the boundary determines the dynamics of the billiard system [8, 41, 42]. A quantum billiard can be constructed similarly, i.e. a two dimensional infinite potential well whose boundary has the same shape of the corresponding classical billiard. Thus in the short wavelength limit, wave dynamics become ray dynamics, and the quantum billiard degenerates to the classical billiard. Classically, the probability to find a particle moving exactly on an unstable periodic orbit is zero, as the measure of these orbits in the phase space vanishes. This leads to the ergodicity of the dynamics. Semiclassically, the averaged Wigner function can be assumed to take the ‘microcanonical’ form [43], resulting in a Gaussian random function in the coordinate space. Surprisingly, for the quantum counterpart, some of the eigenwavefunctions would concentrate especially around these orbits, forming the quantum scars. The density of a scar along the orbit is a constant and does not depend on \hbar sensitively. But the width of the scars is typically of the order of the wavelength, as \hbar or the wavelength goes to zero, the scar will finally disappear in the random background. ‘In this way, the scar ‘heals’ as $\hbar \rightarrow 0$ ’ [24]. Scars have been searched and analyzed in mesoscopic systems [44–54]. Due to the similarity of the equations for different types of waves, scars have been observed in microwave [55–61], optical fiber and microcavity [62–66], acoustic and liquid surface wave systems [67–70]. Quantum scars in phase space could reveal more information about the classical orbit [24] and have been discussed in [35, 71–77], with numerical evidence of antiscars provided in [76]. The statistical properties of scars has been discussed in [78]. Many-body effects in billiard models have been investigated using the Kohn and Sham (KS) equations in the mean-field approximation, i.e. noninteracting particles moving in some fictitious effective field, and scarring could take place when the disorder is weak and the electron density is sufficiently high [30]. For a comprehensive analysis of quantum scars, please see [79, 80]. Besides the conventional situations, scars on quantum networks are found to be insensitive to the Lyapunov exponents [81]. Quantum scars have also been identified by accumulation of atomic density for certain energies in spin–orbit-coupled atomic gases [82], and observed in the two-dimensional harmonic oscillators due to local impurities [83–85]. Quantum many-body scars become a hot topic recently due to the weak ergodicity breaking caused by these states [86–95], which provides a new route to the departure from the eigenstate thermalization hypothesis (ETH) scenario other than many-body localization (MBL). Their analogy in a driven fracton system, namely, dynamical scar, has also been observed [96]. Note that in these investigations, scar states do not relate to classical periodic orbits as in its original setup, but rather a small number of localized states in an otherwise thermalizing spectrum, while in contrast to both ETH where most of the states behave like thermal states, and MBL in which essentially all eigenstates are athermal.

Another cornerstone of quantum chaos is the random matrix theory (RMT), which was mostly developed in 1950s by Wigner [97–99] when dealing with the energy spectrum of complex quantum systems such as the complex nuclei and later in 1960s by Dyson [100–104]. These works form the basis for random matrix theory (see [105–111] for an overview and recent developments). The idea is such that since the interactions are so complex, it could be efficient to approximate the Hamiltonian by a random matrix with elements following certain statistical properties imposed by the symmetry of the system [105–112]. Density distributions of the energy levels are given, and the distributions of the spacings between nearest neighboring levels are investigated extensively due to the findings that they follow different universal functional forms, namely, Poisson [113] or Wigner–Dyson statistics [114], if the corresponding classical dynamics are integrable or chaotic, respectively. In particular, for systems corresponding to classically chaotic dynamics with no additional geometric symmetry, if time reversal symmetry is preserved, the level spacing statistics would follow RMT with Gaussian orthogonal ensembles (GOE). If time reversal symmetry is broken, then Gaussian unitary ensembles (GUE) would apply. Gaussian symplectic ensembles (GSE) would also appear if the system possesses symplectic symmetry. Long range correlations, i.e. the number variance Σ_2 and spectral rigidity Δ_3 , and higher order correlations in spectra are also found to have distinct behavior for quantum systems with integrable or chaotic classical dynamics [106, 115, 116]. Various numerical and experimental evidences are provided [117–121]. Through a series of works [115, 122–125], a connection between classical periodic orbits in chaotic systems and the spectral correlation of the corresponding quantum system represented by the form factor was established, laid the foundation of the universality of the spectral statistics for classically chaotic systems in RMT, which has also been extended into spin 1/2 [126] and many-body situations [127]. These findings are quite prominent as they could serve as the quantum indications of their classical dynamics and symmetry properties. Note that these statements are for generic systems. Non-generic systems, however, may violate such rules [128]. Level spacing statistics of quantum quasidegeneracy has been investigated and Shnirelman peak was identified [129]. Model for experimental level spacing distributions with missing and spurious levels has also been considered [130, 131]. Generalization of the nearest level spacing statistics to open chaotic wave systems with non-Hermitian Hamiltonian was demonstrated in [132]. Between integrable and chaotic systems, there are pseudo-integrable systems [133], and also mixed dynamical systems with both Kolmogorov–Arnold–Moser (KAM) islands and chaotic sea in the phase space [134]. Non-universal behaviors have been noticed [133–138]. In particular, for singular quantum billiards with a point-like scatterer inside an integrable billiard [137], the level spacing statistics may exhibit intermediate statistics, namely, semi-Poisson [139], that exhibits both level repulsion (in the chaotic case) so the probability to find closeby levels are small, and exponential decay for large spacings as in the Poisson distribution (the

integrable case) [140–146]. This feature also appears in quantum systems with parameters close to the metal-insulator transition [147]. While for another class of pseudo-integrable systems, namely, polygonal (particularly triangular) quantum billiards that introduce dividing scattering only at the corners [133, 135], although there are conjectures and numerical discussions [139], the spectral statistic is rather complex and is still an open issue for general cases.

There are many other important topics involving quantum billiards, such as nodal line structures and wavefunction statistics [148–157], quantum chaotic scattering [158–166] with experimental demonstrations on two dimensional electron gas (2DEG) [50, 167–170] where the electrons are described by the two dimensional Schrödinger equation, quantum pointer (preferred) states and decoherence [52, 171–175], universal conductance fluctuations [176–180], chaos-assisted quantum tunneling [181–195], effects of electron–electron interaction [30, 195–199], Loschmidt echo and fidelity [200–236], etc. Being described by the same Helmholtz equation for the spatial wavefunction, e.g. $(\nabla^2 + k^2)\psi = 0$ or its extensions, where ∇^2 is the Laplacian operator and k is the wave number, the quantum billiard can be simulated with other wave systems, such as microwave [55, 59–61, 182, 237–254], light in optical fibers [62, 66] and optical microcavities [255–264], acoustic waves and plate vibrations [68, 265–270], liquid surface waves [67, 69, 70, 271–273], etc.

Other prototypical models that have been investigated extensively in the development of the field of quantum chaos include kicked rotors in terms of diffusion [274–288], entanglement as signatures of referring classical chaos [289–291], experimental realizations [292–295], and other related topics [285, 296–299], and the Dicke model [300–306] and the Lipkin–Meshkov–Glick model [307–310] to account for the many-body effects. In particular, in the phenomena of dynamical localization of kicked rotors with parameters in the classically chaotic region, the momentum localization length has an integer scaling property versus the reduced Planck constant \hbar ; while in the vicinity of the golden cantori, a fractional \hbar scaling is observed, which was argued as the quantum signature of the golden cantori [311–316]. However, in a following work with a random-pair-kicked particle model, it is found that the fractional \hbar scaling can emerge in systems even without the golden cantori structure at all [296], thus it is not a quantum signature of the classical cantori, but has an origin of inherent quantum nature. Abnormal diffusion in one-dimensional tight-binding lattices [317–322] is another interesting subject which is related to the kicked rotor model, as if the diagonal potential is periodic, it can be mapped into a periodically driven time-dependent quantum problem [276]. Ionization of Rydberg atoms [323–344] and dynamics of Bose–Einstein condensation [197, 345–353] have also been investigated extensively in quantum chaos. Due to the interdisciplinary nature, phenomena and effects in quantum chaos have broad applications in nuclei physics [110, 111, 354, 355], cold atom physics [356–361], controlled laser emission [64, 256, 257, 259, 362–364], quantum information [365–370], etc.

Although being an old field, there are still hot topics and astonishing findings emerging recently due to deeper understandings of the theory, the advances of the computation power and the experimental techniques, such as quantum graphs and their microwave network simulations [131, 371–389], universal quantum manifestations for different classical dynamics [125, 284, 380, 390–396], many-body localization [397–404], quantum thermalization [309, 405–417], quantum thermodynamics [304, 418–426], out-of-time-ordered correlator (OTOC) [427–436], and other related topics [437, 438]. In particular, there are a number of active groups in China publishing recent works in Communications in Theoretical Physics [166, 207, 236, 285, 340, 359, 439, 440], Chinese Physics B [254, 320, 341, 343, 369, 398, 403, 441–443], Chinese Physics Letters [175, 344, 416, 444, 445], and Science China Physics, Mechanics & Astronomy [288, 446–449].

Among the new developments, one interesting field is to expand the broadly investigated quantum chaos into relativistic quantum systems, and see what happens when the relativistic effect cannot be neglected, e.g. what classical chaos can bring to the relativistic quantum systems. This resulted in the still developing field of relativistic quantum chaos. The first study of relativistic quantum chaos was carried out in 1987, by Berry and Mondragon [450], who invented a two dimensional neutrino billiard by imposing infinite mass confinement on the boundary, i.e. the MIT bag model [451], where the neutrino, with spin 1/2, at that time was believed to be massless, and was described by the massless Dirac equation. The reason for not choosing the conventional electric potential for confinement is that the Klein tunneling for relativistic particles will invalidate such a confinement. We shall denote such system as *massless Dirac billiards*. The intriguing point is that, with the infinite mass potential, the Hamiltonian breaks the time reversal symmetry, leading to GUE spectral statistics. In a following work by Antoine *et al* [452], a 2D fermionic billiard in a curved space coupled with a magnetic field is considered. Results were obtained under a generalized boundary condition, which confirmed the results by Berry and Mondragon when the boundary condition reduces to the same one. The generalization of the neutrino billiard in a three dimensional cavity has been investigated in [453], with the finding that the orbital lengths seem to be the same as in the scalar spinless case. After Berry and Mondragon’s seminal investigation of neutrino (or massless Dirac) billiards [450], there are only a few works in this topic [452, 453]. Only when graphene and other 2D Dirac materials emerged in 2000s [454–460] rendering experimental observation of the effects possible, the field became prosperous and many different aspects have been investigated extensively, e.g. graphene/Dirac billiards were proposed [461, 462] and has started the enthusiasm in this field with either graphene, or microwave artificial graphene (microwave photonic crystal with honeycomb lattice), or by directly solving the massless Dirac equation in a confined region [463, 464].

Since there are different approaches for *relativistic quantum chaos*, before we proceed further, we shall define the boundary of the our discussions clearly to avoid confusion.

First, it should be distinguished from *relativistic chaos*, where the motion of the particle is in relativistic regime, i.e. its speed is comparable to the speed of light, but is described by the classical dynamics, not quantum mechanics. There are many interesting results in this topic [465–479], but it is not considered as relativistic *quantum* chaos. Secondly, quantum chaos has a great motivation regarding classical-quantum correspondence. While for relativistic quantum systems, there are quantities such as spin that does not have a classical correspondence. However, for a relativistic quantum systems and its ‘corresponding’ classical counterpart by just considering the ‘trajectory’ of the particle, the properties of the former can be affected significantly by the classical dynamics, i.e. whether chaotic or integrable, of the latter. Therefore, studies of this field are to reveal how classical dynamics may have influence to the ‘corresponding’ relativistic quantum systems, not to demonstrate the one-to-one correspondence of the two limiting cases. In this sense, there are studies of the relation between entanglement and classical dynamics [289–291, 353, 443, 445, 480–483], spin transport versus classical dynamics [484–489], that although there are no direct one-to-one correspondence, there are significant influence to the behavior of entanglement and spin transport from classical dynamics. Thirdly, the term relativistic quantum chaos is actually not new, but was proposed explicitly about three decades ago in a paper by Tomaschitz entitled ‘Relativistic quantum chaos in Robertson–Walker cosmologies’ [490]. In this work, Tomaschitz found localized wave fields, which are solutions of the Klein–Gordon equation, quantized on the bounded trajectories in the classical geodesic motion. Actually there is a series of works on this line in quantum cosmology [490–498] concerning chaotic quantum billiards in the vicinity of a cosmological singularity in quantum cosmology, where the local behavior of a part of the metric functions can be described by a billiard on a space of constant negative curvature, leading to the formation of spatial chaos. These results could be helpful to understand the early stages of the Universe. Another line is to examine the spectral properties of the quantum chromodynamics (QCD) lattice Dirac operator [499–507] and Dirac operator on quantum graphs [508, 509], where agreement with chiral random matrix theory has been confirmed.

While our focus has been on billiard systems, kicked rotor was an important model for this field and is still an active research topic [442, 510–512]. As a side note, looking for semiclassical treatment of quantum spinor particles has been a persistent effort from 1930s by Pauli [513] to early of this century [514–531]. These semiclassical results provide insights in understanding the spectral fluctuations in graphene nano-structures [532, 533].

2. Spectral statistics

The main results in level spacing statistics are as follows. For a system with energy levels $\{E_n, n = 1, 2, \dots\}$, let $\tilde{N}(E)$ be the number of levels below E . Generally, the density of the spectra

is not uniform, therefore, to make comparison of level spacings meaningful, the spectra needs to be unfolded: $x_n \equiv \langle \tilde{N}(E_n) \rangle$, where $\langle \tilde{N}(E) \rangle$ is the smooth part of $\tilde{N}(E)$. Then in general the statistics of x_n follow universal rules depending only on the symmetry of the original quantum system and the corresponding classical dynamics [113, 114], not on the details of the systems. One important quantity is the level spacing distribution $P(S)$, which is the distribution function of the nearest-neighbor spacing, e.g. $S_n = x_{n+1} - x_n$, of the unfolded spectrum $\{x_n\}$. Another quantity is the spectral rigidity $\Delta_3(L)$, for detailed calculations please refer to page 5 of [534] and references therein.

2.1. Berry and Mondragon's result revisited

Berry and Mondragon investigated two dimensional billiard with the African shape (guaranteeing chaotic dynamics with no geometric symmetry) of confined massless spin-1/2 particles, and found GUE statistics [450]. This result is quite surprising as there is no magnetic field or magnetic flux in the system which are typically required to break the T-symmetry. Here the time reversal symmetry is broken by the confinement boundary. The Hamiltonian is given by

$$\hat{H} = -i\hbar v \hat{\sigma} \cdot \nabla + V(\mathbf{r}) \hat{\sigma}_z, \quad (1)$$

where v is the Fermi velocity for quasiparticles or the speed of light for a true massless relativistic particle, $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y)$ and $\hat{\sigma}_z$ are Pauli matrices, and $V(\mathbf{r})$ is the infinite-mass confinement potential, i.e. $V(\mathbf{r}) = 0$ for \mathbf{r} inside the billiard region D , while $V(\mathbf{r}) = \infty$ otherwise. The time-reversal operator is given by $\hat{T} = i\sigma_y \hat{K}$, where \hat{K} denotes complex conjugate. It can be readily verified that $\hat{T} \hat{H} \hat{T}^{-1} = -i\hbar v \hat{\sigma} \cdot \nabla - V(\mathbf{r}) \hat{\sigma}_z \neq \hat{H}$, i.e. the free motion of the particle is unchanged, but the confinement potential changes sign and breaks T-symmetry. Microscopically, since the spin is locked with the momentum, at each reflection there will be an extra phase due to the rotation of the spin. While for periodic orbits, if the period, or the number of bouncings at the boundary N , is even, then the accumulated phase along the orbit counterclockwise and clockwise are the same modulo 2π , thus both orientations will satisfy the quantization rule simultaneously, i.e. if one orientation is a solution of the system, the other orientation (time-reversed) will also be a solution. This is the same for non-relativistic quantum billiards without magnetic field. However, if N is odd, then the accumulated phase difference for two opposite orientations will be π modulo 2π . Thus if one orientation satisfies the quantization condition, i.e. the overall accumulated phase along the complete orbit is integer multiples of 2π , and is thus a solution of the system, the reversed orientation will have an extra π phase and will not satisfy the quantization condition, thus will not be a solution. This breaks the T-symmetry as it requires that the two orientations must be or be not solutions of the system simultaneously.

Although this effect is quite subtle, it can result in GUE, instead of GOE, spectral statistics [450], which has also been verified by solving the system using other numerical techniques such as direct discretization [536], conformal mapping [535], and extended boundary integral method [537] (see also figure 1).

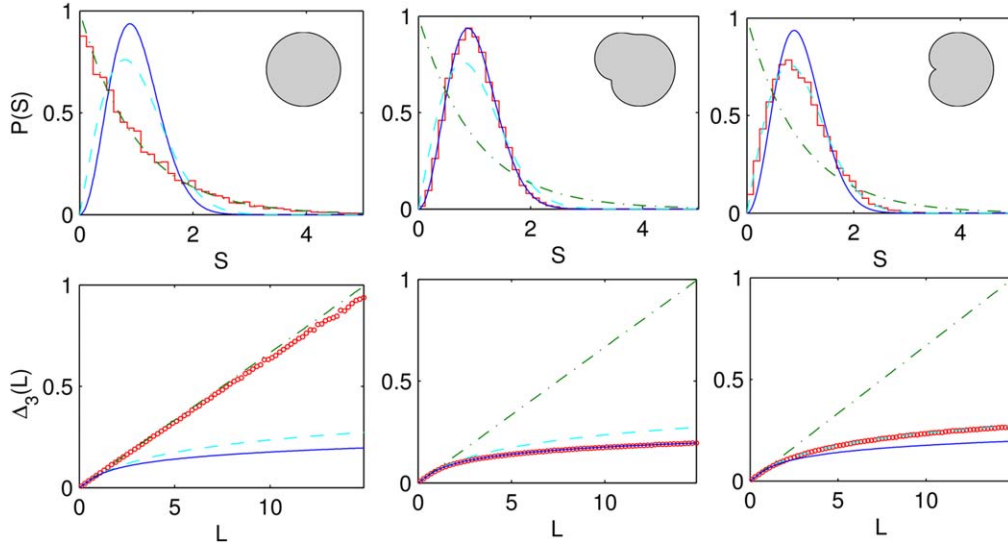


Figure 1. Level spacing statistics of the massless Dirac billiards with the boundary being a circle, the Africa shape, the heart shape for left, middle, and right, respectively. The first row shows the unfolded level-spacing distribution $P(S)$, and the second row shows the spectral rigidity $\Delta_3(L)$. The green dashed-dotted line, cyan dashed line, and blue solid line are for Poisson, GOE, and GUE. Red staircase curves and symbols are numerical results from 13000 energy levels for each shape by diagonalizing the operator \hat{H} given by equation (1). Adapted from [535] with permission.

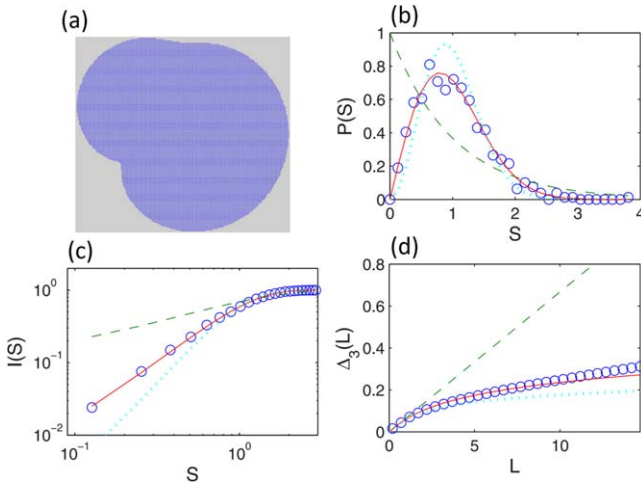


Figure 2. (a) Chaotic graphene billiard with Africa shape cut from a graphene sheet. The system has 42 505 carbon atoms. The outline is determined by the equation $x + iy = 70a(z + 0.2z^2 + 0.2z^3e^{i\pi/3})$, where z is the unit circle in the complex plane, $a = 2.46 \text{ \AA}$ is the lattice constant for graphene. The area is $A = 1117 \text{ nm}^2$. (b)–(d) are the level spacing distribution, integrated level spacing distribution, and the spectral rigidity, respectively, for 664 energy levels in the range $0.02 < E_n/t < 0.4$, where t is the hopping energy between nearest neighboring atoms. Dashed line is Poisson, solid line is GOE, and dotted line is GUE. The results show clear evidence of GOE. Adopted from [539] with permission.

Figure 1 shows the results for three billiards: the circular, the African, and the heart-shaped billiards. The circular billiard is integrable, leads to Poisson statistics. The African billiard is chaotic, leads to GUE. The heart-shaped billiard has a mirror symmetry, although it is not symmetric under the time-reversal operation for the corresponding Dirac billiard, it is symmetric under the joint parity and time-reversal operations. Thus again the GOE statistics are recovered. Since the pseudoparticles in

graphene follow the same 2D massless Dirac equation as in [450], it is quite natural to ask whether the graphene billiard follow the same GUE statistics. In this regard, the experimental work [462] by counting the resonance peaks in the transport measurement as approximations of intrinsic energy levels, obtained GUE statistics. However, subsequent numerical calculations provide concrete results of GOE statistics in chaotic graphene billiards in the absence of magnetic fields [534, 538, 539], see figure 2, which lead to further experimental investigations using artificial graphene with much higher accuracy and confirmed GOE statistics [540].

2.2. Chaotic graphene billiard

Numerically, the graphene billiard is a graphene sheet where the boundary is cut following a specific shape that carries desired classical dynamics. This is effectively an infinite potential well on the graphene sheet: the potential on the boundary is infinite, and the probability to find an electron on the boundary is zero. The general tight-binding Hamiltonian is given by

$$\hat{H} = \sum(-\varepsilon_i)|i\rangle\langle i| + \sum(-t_{ij})|i\rangle\langle j|, \quad (2)$$

where i and j are the indices of the atoms (or lattice sites), the first summation is over all the atoms within the billiard and the second summation is over pairs of all necessary neighboring atoms, which could be the nearest neighboring pairs, or the next or next-next nearest neighboring pairs, with their respective hopping energy t_{ij} 's. Note that hopping energies between atoms close to the boundary may be different from those far from the boundaries. For clean graphene the onsite energy ε_i is identical for all atoms, thus it is convenient to set it to zero. If there are static electric disorders, ε_i will be position dependent. In the atomic (or lattice site) basis $|j\rangle$, the Hamiltonian matrix element can be calculated as $H_{ij} = \langle i|\hat{H}|j\rangle$, which is given by $(-\varepsilon_i)$ for

the diagonal element H_{ii} and $(-t_{ij})$ for element H_{ij} . Once the Hamiltonian matrix is obtained, it can be diagonalized to yield the eigenenergies and the eigenstates. The results of spectral statistics for the African shaped graphene billiard is shown in figure 2, which assumes $\varepsilon_i = 0$ and uniform hopping energies $t_{ij} = t$ between only the nearest neighboring atoms. Thus the energy is in units of t , and it is convenient to use E_n/t for the values of the eigenenergies. It is clear that they follow GOE statistics. Non-idealities such as interactions beyond the nearest neighbors, lattice orientation, effect of boundary bonds and staggered potentials caused by substrates, etc. may have influence to the details of the system, but the GOE statistics are robust and persistent in these non-ideal situations [534].

This might be counterintuitive as one would expect that the graphene chaotic billiards should exhibit the same GUE level-spacing distribution as the massless Dirac billiard [450], since they obey the same equation. The reasoning is as follows. Graphene has two non-equivalent Dirac points (valleys). Quasiparticles in the vicinity of a Dirac point obey the same massless Dirac equation, but the abrupt edge termination in graphene billiard couples the two valleys. As a result, a full set of equations taking into account the effects of both the two nonequivalent Dirac points and the boundary conditions are thus necessary to describe the motion of the relativistic particle. The time-reversal operation for the massless Dirac particle interchanges the two valleys. Thus as a whole, the time-reversal symmetry is preserved [456], resulting in GOE statistics.

Spectral statistics of disordered graphene sheets has also been investigated extensively with both experiments [541] and numerical simulations [542–544], where GOE statistics have been identified in general. Reference [545] examined the level spacing statistics for the edge states only for energies close to the Dirac point. Since these states are localized, it was expected that the statistics may follow that of Poisson, but it turned out that the level spacing statistics was GOE, which can be attributed to the chiral symmetry that introduced long-range correlation between the edge states on different sides, and thus level repulsion. Indeed, when the symmetry is broken by non-zero next nearest neighbor hopping energies, the level spacing statistics becomes Poisson.

Much effort has been devoted to searching for GUE in graphene billiards, e.g. by decoupling the two valleys. A smooth varying mass term was added in [538], however, GUE statistics were not found, which was attributed to the residual inter-valley scatterings. Indications of GUE statistics were found in triangular graphene billiards with zigzag edges and smooth impurity potentials [544], and with an asymmetric strain [546] due to the induced pseudomagnetic field [547, 548].

In the spectral statistics, there is a series of works employing microwave artificial graphene, e.g. a manmade honeycomb lattice not for electrons, but for microwaves [253, 540, 549–563]. Due to the inherent similarity of the wave equations, the quasiparticles follow the same massless Dirac equation and behave similarly as those in graphene. Especially, the Darmstadt group of A. Richter used superconducting microwave cavities filling in photonic crystals, obtained spectra with unprecedentedly high accuracy, yielding convincing statistics [240, 253, 540, 562, 563].

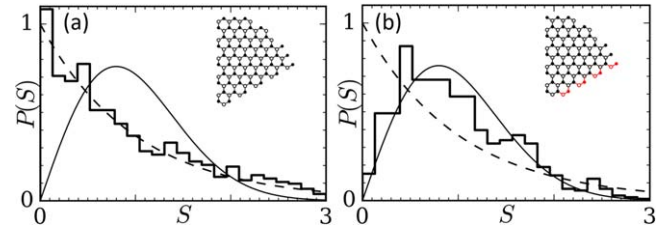


Figure 3. Level spacing statistics of graphene billiard with the shape of a 60° sector with armchair edges. (a) With perfect edges and 227 254 atoms. (b) With one row of atoms removed along one edge so the structure is no longer symmetric (as indicated by the red dots in the inset) and 226 315 atoms. The energy range is $0.02 < E_n/t < 0.2$. Dashed line is Poisson, solid line is GOE. Insets show the magnified view of the lattice structure close to the tip of the sector to illustrate the differences. Adopted from [564] with permission.

2.3. Beyond Berry and Tabor's conjecture

Berry and Tabor [113] proposed that for generically integrable systems, the energy levels are uncorrelated and the resulting statistics would be Poisson. This has been verified by extensive numerical and experimental studies. However, it is found that for graphene billiard with a sector shape where the corresponding classical dynamics are generically integrable, for energy levels close to the Dirac point, the spectral statistics are in general GOE, not Poisson [564]. Only close to the band edge ($E/t = \pm 3$) where the pseudoparticles follow the Schrödinger equation, the statistics become Poisson. The reason for this abnormal phenomenon is that when the energy is close to the Dirac point, the edges play an important role. Even for an ideal situation, say, 60° sector with both straight edges being armchair, the level spacing statistics could be Poisson, as figure 3(a) shows, but changing a few atoms around the tip, or adding or removing one line of atoms along one edge, as demonstrated in the insets, the level spacing statistic becomes that of the GOE (figure 3(b)). Thus the system is extremely sensitive to the imperfections of the boundary, and for sectors with arbitrary angles, the results are generally GOE [564]. An interesting question is that, is this result due to the particular lattice structure of graphene, or due to relativistic nature? A preliminary examination reveals that this might be caused by the complex boundary condition provoked by the multi-component spinor wavefunction. Concrete conclusion may require further investigation.

3. Quantum scars

Quantum scar has been an important pillar for quantum chaos. In the development of relativistic quantum chaos, one natural question is whether scars exist in relativistic quantum systems, and if so, are there any unique features that can distinguish them from the conventional quantum scars?

3.1. Relativistic quantum scars

The existence of scars in relativistic quantum systems was confirmed with a stadium shaped graphene billiard [565]. Scars in the Wimmer system (distorted circular) filled with

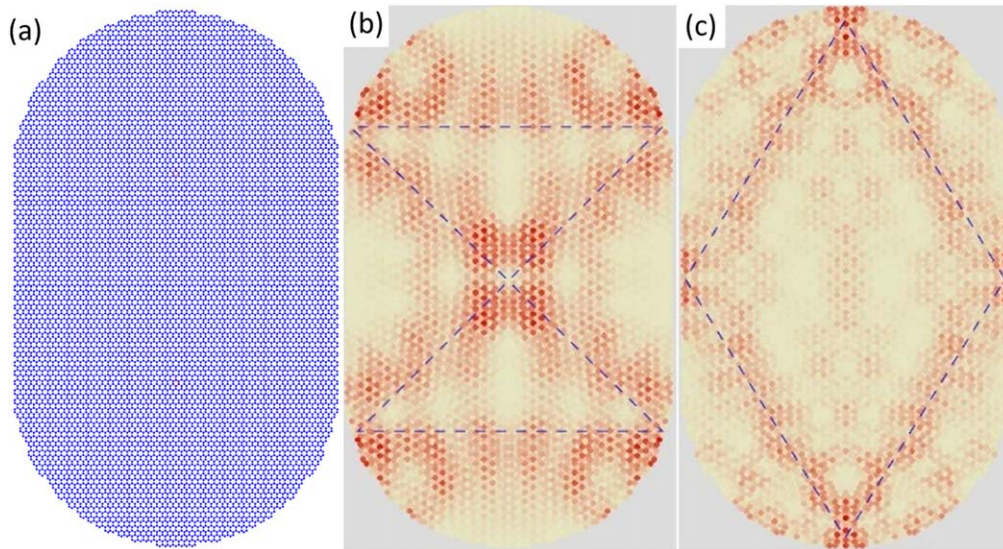


Figure 4. (a) A stadium shaped graphene billiard with 11814 atoms. (b) and (c) show $|\psi_n|^2$ with $E_n/t = 0.36358$ and 0.57665 , respectively. Adopted from [565] with permission.

graphene was also observed [566]. Employing the tight-binding Hamiltonian equation (2), the eigenstates ψ_n can be calculated. By examining the spatial distribution of $|\psi_n|^2$ for eigenstates close to Dirac point, unequivocal scars on periodic orbits are observed (figure 4). The scarring state can be formed when the particle, after traveling the orbit for a complete cycle, gains a global phase that is an integer multiple of 2π . Thus when there is a scar occurring at the wavenumber k_0 , as the wavenumber (or energy) is changed, there will be a scar again at (or close to) wavenumber k if $\Delta k \cdot L = 2n\pi$, where $\Delta k = k - k_0$, L is the length of the orbit and n is an integer. For two adjacent scarring states, one has $\Delta k = 2\pi/L$. This holds for both massless relativistic and non-relativistic quantum systems. Note that this does not hold for *massive* relativistic quantum billiard systems, as when varying k (or energy E), besides the $\Delta k \cdot L$ term, there will be an additional term that would lead to an extra phase depending on k , which would also need to be taken into account in the quantization formula. For massless relativistic and non-relativistic quantum systems, the key difference lies in the dispersion relation, with $E \propto k$ for the former and $E \propto k^2$ for the latter. Therefore, in terms of E , it will be either E or \sqrt{E} that will be equally spaced for recurrent scars, corresponding to massless relativistic (graphene) or nonrelativistic quantum cases. Figure 5 shows that for two representative scars as shown in the insets, the energy values where they occur versus the relative index. Despite small fluctuations, the linear relation is apparent, corroborating the massless relativistic predictions. In particular, for graphene, since $E = \hbar v_F k$, where $v_F = \sqrt{3}ta/(2\hbar)$ is the Fermi velocity, t is the hopping energy between the nearest neighbors, $a = 2.46 \text{ \AA}$ is the lattice constant, one has $\Delta E = \hbar v_F \Delta k = \hbar v_F/L$. For the scar shown in the left inset, the length of the orbit is $263a$, yielding $\Delta E = 0.0207t$; while from figure 5, the average ΔE equals to $0.0203t$, which agrees well. For the other scar, the length of the orbit is $275a$,

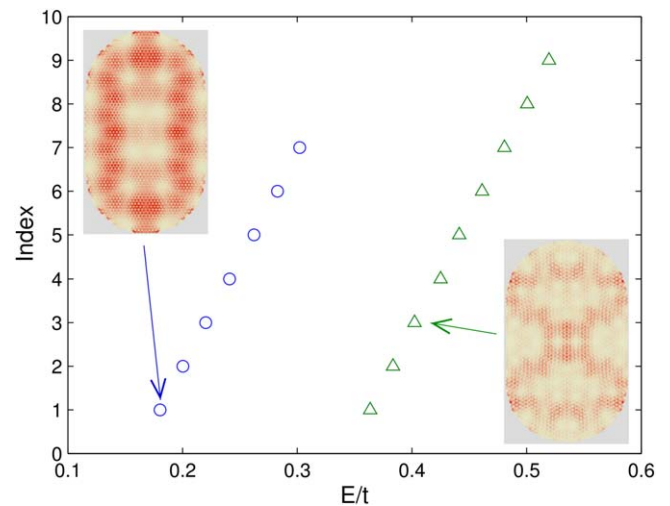


Figure 5. For two representative scars with orbit length $263a$ (left) and $275a$ (right), the energy values where it appears. Vertical axis shows the relative index of these scars. From [464] with permission.

leading to $\Delta E = 0.0198t$, agrees well with $0.0195t$ from figure 5.

Note that only when the energy is small, the dispersion relation is homogeneous and the pseudoparticles follow the massless Dirac equation. When the energy is large, the dispersion relation is no longer homogeneous but direction dependent, and the group velocity $\nabla_k E$ is concentrated in only three directions according to the symmetry of the honeycomb lattice. In this case, the motion of the pseudoparticles deviates from the massless Dirac equation. However, along these three directions, E is still approximately linear to $|k|$ even for E close to t . Since the scars are also constrained on orbits that are composed by straight lines only in these three directions, the relation $\Delta E = \hbar v_F/L$ holds for almost the whole range from 0 to t . This makes it easier to examine the scars and to verify this relation in experiments. Indeed, this

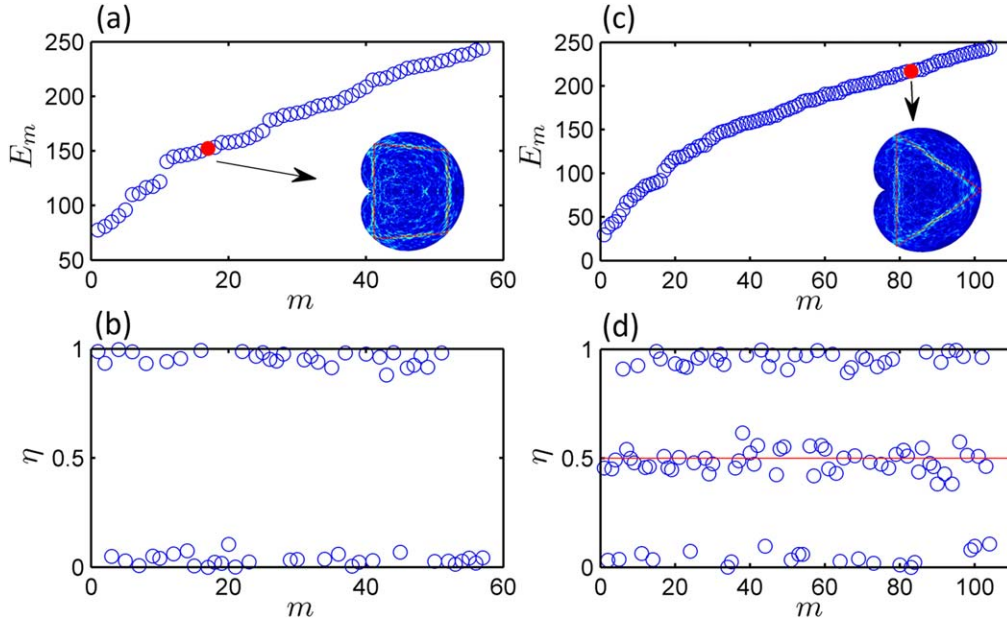


Figure 6. For scars on two representative orbits, period-4 for the left panels, and period-3 for the right panels, the upper panels show the corresponding eigenenergies of the scars, and the lower panels show η (see text) for these states. Adopted from [568] with permission.

feature of equal spacing of E in the recurring scarring states has been confirmed experimentally in a mesoscopic graphene ring system [567].

3.2. Chiral scars

Although pseudoparticles close to one Dirac point in graphene and Berry and Mondragon's 'neutrino' follow the same 2D massless Dirac equation, due to the coupling of the two Dirac points by the boundary, a complete description for the pseudoparticles in graphene will be different. Thus it is still intriguing to examine the scars in the 'neutrino' billiard and see how the time-reversal symmetry broken by the infinite mass boundary condition is revealed in scars. A direct discretization method was developed to solve the massless Dirac billiard in a confined region, where scars in an African billiard and a bow-tie shaped billiard were identified [536], but due to limited spatial resolution, recurrent rhythm can not be determined. Later, a conformal mapping method was developed where a huge number of eigenstates with extremely high spatial resolution can be obtained [568]. By solving the eigenproblem of a heart-shaped 2D massless Dirac billiard, scars on periodic orbits are identified. Furthermore, it is found that the properties of the scars depend on whether the orbit has even or odd bounces at the boundary, and the relation $k - k_0 = 2\pi n/L$ for recurring scars is no longer fulfilled for the odd orbits. Particularly, for a given reference point k_0 with pronounced scarring patterns, let $\delta k = 2\pi/L$ and define $\eta(m) = (k_m - k_0)/\delta k - [(k_m - k_0)/\delta k]$, where $[x]$ denotes the integer part of x and k_m is the eigenwavenumber of the m th identified scarring state on the same orbit, if the above relation is satisfied, then numerically, $\eta(m)$ will take values that are either close to zero or close to one. As shown in figure 6, this is indeed the case for period-4 orbits. But for period-3 orbits, η takes an extra value close to $1/2$ [568].

A complete understanding would involve many more details [569]. Here we would only provide the main arguments. The quantization condition is such that, following the orbit, after a complete cycle, the total phase accumulation should be integer multiples of 2π . For the massless Dirac billiard with a magnetic flux $\alpha\Phi_0/2\pi$ ($\Phi_0 \equiv h/e$ is the magnetic flux quanta) at the center of the billiard, the total accumulated phase after one complete cycle is

$$\Phi^\pm = \frac{1}{\hbar}S - \frac{\sigma\pi}{2} + \beta^\pm,$$

where \pm indicates whether the flow of the orbit is counterclockwise or clockwise. $\beta^\pm = \sum_i \delta_i^\pm$ is the extra phase due to spin rotation imposed by reflections at the boundary, e.g. see figure 7. For a given periodic orbit, at each reflection point, the angle δ_i^\pm can be calculated explicitly, which determines β^\pm unsuspectingly [569]. The Maslov index σ is the number of conjugate points along the orbit and is canonically invariant [17]. For the heart-shaped chaotic billiard, the value of σ is nothing but the number of reflections along a complete orbit [19]. The action is

$$S = \oint \mathbf{p} \cdot d\mathbf{q} = \hbar \oint \mathbf{k} \cdot d\mathbf{q} + e \oint \mathbf{A} \cdot d\mathbf{q} = k \cdot L \pm W\alpha,$$

where W is the winding number of the orbit with respect to the flux, i.e. how many times it circulates the flux. One thus has

$$\Phi^\pm = k \cdot L \pm W\alpha - \frac{\sigma\pi}{2} + \beta^\pm. \quad (3)$$

For semiclassically allowed states, the phase accumulation around one cycle should be an integer multiple of 2π , i.e. $\Phi^\pm = 2\pi n$ ($n = 1, 2, \dots$) so as to ensure that the wavefunction is single-valued. One thus has

$$k^\pm = (2\pi n \mp W\alpha + \frac{\sigma\pi}{2} - \beta^\pm)/L. \quad (4)$$

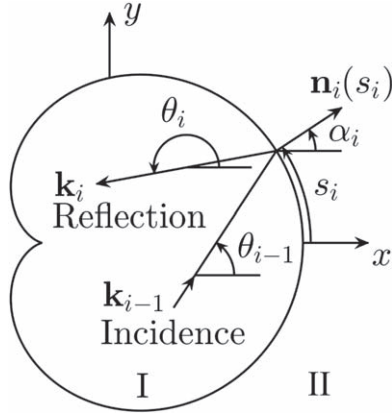


Figure 7. Definition of the angles. $\delta_i^+ = (\theta_i - \theta_{i-1})/2$ is the extra phase due to the rotation of the spin, where ‘+’ indicates counterclockwise orientation. Typically, the phase associated with the time reversed reflection, e.g. δ_i^- , from $-k_i$ to $-k_{i-1}$, would be different from δ_i^+ .

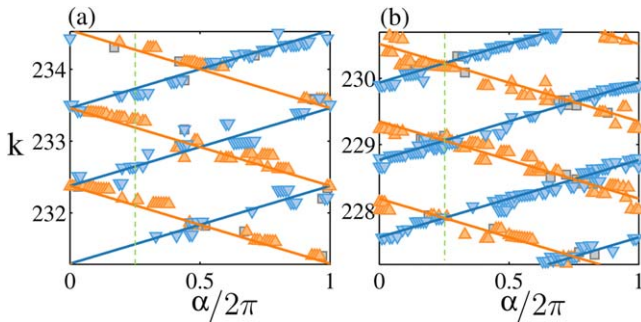


Figure 8. Validation of the quantization rule equation (4). Shown are the relations between wavenumber k and magnetic flux α , for (a) the period-4 scar in figure 6(a), and (b) the period-3 scar in figure 6(c). The orange up-triangles indicate scars with a counterclockwise flow, and the blue down-triangles are those with a clockwise flow. The gray squares mark the scars whose flow orientations cannot be identified, which typically occur close to the cross points of the two orientation cases. The solid lines are theoretical predictions of equation (4). Vertical lines indicate the position of $\alpha = \pi/2$. The step in the variation of α is 0.01. Adapted from [569] with permission.

This is the quantization rule for a scarring state on periodic orbit with length L , which tells at (or close to) which value of wavenumber k (or E) a scar can form. A comparison between this formula and numerical results for two representative orbits are shown in figure 8, which shows good agreement. The time-reversal symmetry is then imprinted with whether there are integers for both counterclockwise and clockwise orientations that could satisfy equation (4) simultaneously for the same value of k , and thus both states with counterclockwise and clockwise local current flows are solutions of the system. This requires $\Delta\Phi \equiv \Phi^+ - \Phi^- = 0$ modulo 2π , or $2W\alpha + \Delta\beta = 0$ modulo 2π , where $\Delta\beta = \beta^+ - \beta^-$. For systems without a magnetic flux, $\alpha = 0$, the condition becomes $\Delta\beta = 0$. It is surprising that $\Delta\beta$ only depends on whether the periodic orbit has even or odd number of bounces at the boundary: $\Delta\beta = 0$ modulo 2π for even orbits—

orbits with even number of bounces, and $\Delta\beta = \pi$ modulo 2π for odd orbits. Thus although each reflection breaks the time-reversal symmetry due to the polarization of the spin and the tangential current at each reflection point [569], even orbits, when considering the overall accumulated phase, preserve the T-symmetry, thus only odd orbits lead to T-symmetry broken. This provides an understanding of the behaviors of η in figure 6. For the period-4 orbit, at $\alpha = 0$, the values of k for scars with different orientation coincide with each other, as shown in figure 8(a) thus they are both allowed when k satisfies the quantization rule. The difference in neighboring k is then $2\pi/L$, leading to η to be either close to 1 or close to 0. However, at $\alpha = 0$, for the period-3 orbit, the values of k for scars with different orientation are interlaced, e.g. clockwise, counterclockwise, clockwise, and so on, as shown in figure 8(b), while for each orientation, the space between neighboring k is $2\pi/L$, but if one does not differentiate the orientations, the difference becomes π/L for the two neighboring k values corresponding to different orientations, leading to $\eta = 1/2$. Note that as the magnetic flux is varied, the system is periodic with $\alpha = 2\pi$. Furthermore, since $\Delta\Phi = 2W\alpha + \Delta\beta$, for period-4 orbit, $\Delta\beta = 0$, $W = 1$, thus when $\alpha = \pi/2$, as indicated by the vertical lines in figure 8, $\Delta\Phi$ will become π , which will be similar as that for period-3 at $\alpha = 0$. On the other hand, for period-3 orbit, $\Delta\beta = \pi$, thus when $\alpha = \pi/2$, $\Delta\Phi = 2\pi$, or 0 modulo 2π , which is similar to the case of periodic-4 at $\alpha = 0$. Thus by applying a magnetic flux of $\alpha = \pi/2$, the chirality interchanges for these two orbits.

3.3. Unification of chiral scar and nonrelativistic quantum scars

Recently we have developed quantization rule for scars in massive 2D Dirac billiards with infinite mass confinement [570]. Compare to the massless case, there is a new phase emerging during each reflection j . For the massless case, the reflection coefficient R_j is 1. While for the massive case, although the module of R_j is still 1, it has a non-trivial phase, i.e. $R_j^\pm = e^{i(\delta_j^\pm + 2\omega_j^\pm)}$, where $\delta_j^\pm = (\theta_j^\pm - \theta_{j-1}^\pm)/2$ is the same as in the massless case (figure 7), but $2\omega_j^\pm$ is a complicated function of the angles (θ_{i-1}, θ_j) , the mass m , and the wave-number k (or energy E) [570]. Let $\gamma^\pm = \sum_j 2\omega_j^\pm$ and in the absence of magnetic flux, i.e. $\alpha = 0$, the total phase accumulation around one complete cycle is then

$$\Phi^\pm = k \cdot L - \frac{\sigma\pi}{2} + 2\beta^\pm + \gamma^\pm. \quad (5)$$

Since $\text{mod}(\Delta 2\beta, 2\pi) = 0$, where $\beta^\pm = \sum_j \delta_j^\pm$, we then have $\Delta\Phi = \Delta\gamma \equiv \gamma^+ - \gamma^-$. Thus for massive Dirac billiards, the complex behavior can be all attributed to $\Delta\gamma$. We have found that when m goes to zero, $\Delta\gamma$ goes to 2π or π for even or odd orbits, respectively (see figure 9 when $m \rightarrow 0$), degenerating to the massless cases. When the mass m goes to infinity, $\Delta\gamma$ goes to zero for both even and odd orbits (see figure 9 when $k \rightarrow 0$): hence the difference between even and odd orbits diminish and the system becomes effectively a non-relativistic quantum billiard. Thus through the modulation of the extra phase in the reflection coefficients, the relativistic

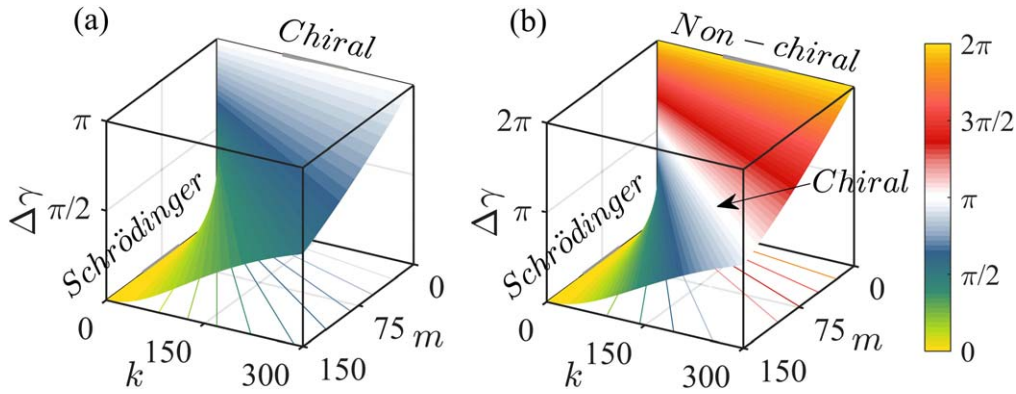


Figure 9. $\Delta\gamma$ between counterclockwise and clockwise scarring states on a period-3 orbit (a) and a period-4 orbit (b). The massless Dirac regime is $m \rightarrow 0$, while $k \rightarrow 0$ is effectively $m \rightarrow \infty$ and is the Schrödinger limit. Adapted from [570] with permission.

chiral scar and the nonrelativistic quantum scar can be unified as the two limiting cases of the massive Dirac billiards.

4. Scattering and tunneling

For open quantum systems, an important topic of quantum chaos is quantum chaotic scattering [50, 158–170]. In non-relativistic systems, a general observation is that, for classically mixed systems, the transmission (or conductance) of the corresponding quantum system exhibit many sharp resonances caused by the strongly localized states around the classically stable periodic orbits, while for classically chaotic system, the peaks are either broadened or removed. That is, chaos regularizes the quantum transport and makes the transmission curve smoother. Note that, for closed systems, such as a quantum billiard, although there are localized scarring state on the unstable periodic orbits for classically chaotic system, these states are unstable that once the system is opened up, due to the spanning chaotic sea in the phase space, they are typically washed out, leaving few or no localized states.

Similar investigation has been carried out for graphene/Dirac quantum dots with different classical dynamics [538, 571–577]. It has been found that, classical dynamics can indeed influence the quantum transport, e.g. by varying the boundary of the quantum dot to change the corresponding classical dynamics from mixed to chaotic, most of the sharp resonances are broadened or removed, however, there are residual sharp resonances, with still strong localized states on classically *unstable* periodic orbits that would not exist for nonrelativistic systems [573]. In particular, a cosine billiard [163] is adopted to demonstrate this phenomenon. The boundary is given by two hard walls at $y = 0$ and $y = W + (M/2)[1 - \cos(2\pi x/L)]$ for $0 \leq x \leq L$, with two semi-infinite leads of width W attached at the two openings of the billiard, whose length is L and the widest part is $(W + M)$. By changing the geometric parameters M , W , and L , the classical dynamics can be either mixed, e.g. for $W/L = 0.18$ and $M/L = 0.11$, or chaotic, e.g. $W/L = 0.36$ and $M/L = 0.22$. A tight-binding approach is employed, and

Green's function formalism is used to calculate the transmission and the local density of states (LDS) [578–581].

Assume the isolated dot region ($0 \leq x \leq L$) has Hamiltonian H_c with a set of eigenenergies and eigenfunctions $\{E_{0\alpha}, \psi_{0\alpha} | \alpha = 1, 2, \dots\}$. The effects of the semi-infinite leads can be incorporated into the retarded self-energy matrices, $\Sigma^R = \Sigma_L^R + \Sigma_R^R$, with the lower indices indicate whether it is due to the left or right leads. Then the whole Hamiltonian with the effects of the leads is $H_c + \Sigma^R$. Since Σ^R in general can be complex, and it is small that it can be regarded as a perturbation, the new set of eigenenergies becomes $E_\alpha = E_{0\alpha} - \Delta_\alpha - i\gamma_\alpha$, where Δ_α and γ_α are generally small. Δ_α represents a shift in $E_{0\alpha}$, and γ_α is the width of the resonance for the α 's state. $1/\gamma_\alpha$ can be regarded as the lifetime of the state [578]. For detailed formulas of calculating γ and the determining factors, please refer to [582].

The values of γ_α for four cases with mixed or chaotic dynamics and 2-DEG or graphene quantum dots are shown in figure 10. Note that smaller γ_α will result in sharper transmission resonances. For 2DEG quantum dots with mixed dynamics (figure 10(a)), there are many cases that γ_α takes very small values, in the order of 10^{-4} , indicating extremely sharp resonances. When the classical dynamics change from mixed to chaotic, beside the envelope, the small values in γ_α are almost all removed (figure 10(b)). For graphene quantum dots, when the classical dynamics is mixed, beside the smooth envelope for $\gamma_\alpha \sim 10^{-2}$, there is a cluster of points for small γ_α values (figure 10(c)). When the classical dynamics becomes chaotic (figure 10(d)), although the overall trend is that the small values are shifted upwards, there are still a big cluster of points take apparently smaller values than the envelope, indicating the persistence of the sharp resonances.

In addition, figure 11 shows the LDS for the most pronounced patterns in both the 2DEG and graphene quantum dots. (a) and (d) are for classically mixed dynamics, which show strong localizations on the stable periodic orbits in both cases. (b), (c) and (e), (f) are for classically chaotic dynamics. It is clear that for 2DEG cases, chaos ruined the localized states on the unstable periodic orbits that are present in the closed case, i.e. the scars; but for graphene quantum dot, localization on unstable periodic orbits still persists. Actually,

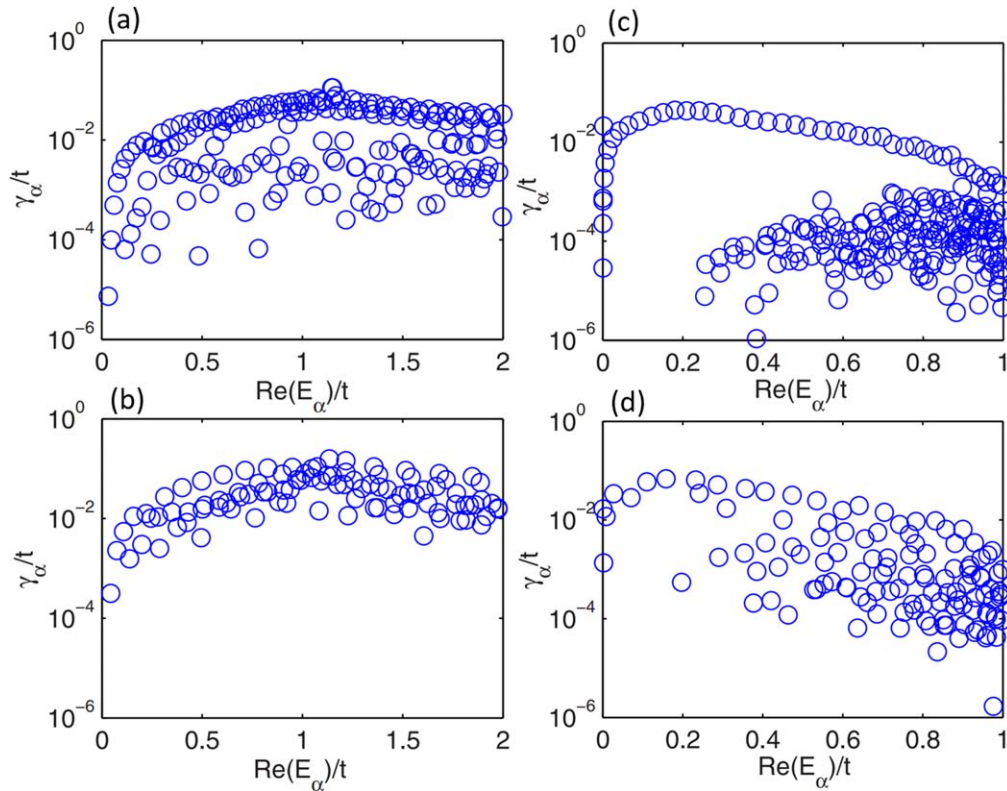


Figure 10. The imaginary part γ of the eigenenergies due to coupling between the dot and the leads, which is an effective indicator of the resonance width. The left panels are for 2DEG quantum dots, and the right panels are for graphene quantum dots. Upper panels are for the cases with mixed dynamics, and lower panels are for classically chaotic dynamics. Adapted from [573] with permission.

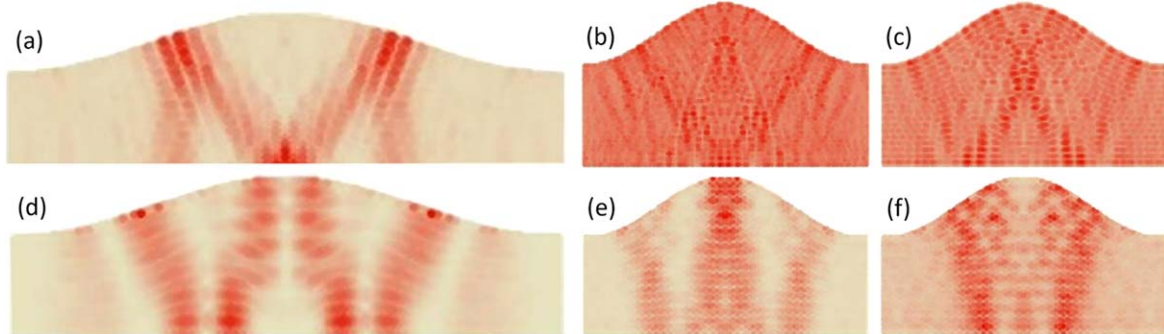


Figure 11. Typical local density of states for quantum dots with classically mixed (a), (d) and chaotic (b), (c), (e), (f) dynamics. The upper panels are for 2DEG quantum dots, and lower panels are for graphene quantum dots.

there are many such states, corroborating the results in figure 10. Although weaker, the effect that classical chaos can make the conductance fluctuation becomes smoother can be exploited to articulate a controlling scheme to modulate the conductance fluctuations in quantum transport through a quantum dot, by changing the underlying classical dynamics [583, 584]. In the presence of a strong magnetic field, the difference caused by the classical dynamics can be suppressed further [585].

The same phenomenon has also been observed in bilayer graphene [586]. The pseudoparticles in bilayer graphene follow the 2D massive Dirac equation. Thus this indicates that the suppression of the effect of eliminating sharp resonances by chaos also persists for massive Dirac systems. In addition,

when the pseudoparticle is traveling along the classical ballistic orbit, it tends to hop back and forth between the two layers, exhibiting a Zitterbewegung-like effect.

Besides scattering, there are other phenomena that the effect of chaos has been suppressed. For example, for regulation of tunneling rates by chaos [191, 192], it has been found that if 2DEG is replaced by graphene or the massless Dirac fermion, although the regularization effect persists, it is much weaker than the 2DEG case [193–195]. The same hold for persistent currents [587, 588] in Aharonov–Bohm (AB) rings [589]. Conventional metallic [590–593] or semiconductor [594] ring systems with a central AB magnetic flux may exhibit dissipationless currents, e.g. the persistent or permanent current. However, the current is quite sensitive that

small non-idealities such as boundary deformation or disorders may destroy the persistent current drastically [595–598]. While in a ring of massless Dirac fermions, due to the Dirac whispering gallery modes [599–601], the persistent current is quite robust against boundary deformations that even in the case where the classical dynamics become chaotic there is still a quite large amount of persistent currents [602–604]. Furthermore, recently, Han *et al* [605] investigated out-of-time-order correlator in relativistic quantum billiard systems and found that the signatures of classical chaos are less pronounced than in the nonrelativistic case. Here again the effect of chaos is suppressed.

5. Quantum chaos in pseudospin-1 Dirac materials

Dirac materials hosting pseudospin-1 quasiparticles with a conical intersection of triple degeneracy in the underlying energy band have attracted a great deal of attention [606–633]. The physics of these 2D Dirac materials is described by the generalized Dirac-Weyl equation for massless spin-1 particles [607, 608, 626]. Pseudospin-1 quasiparticles are different from Dirac, Weyl and Majorana fermions, and are of particular interest to the broad research community with diverse experimental realization schemes such as artificial photonic lattices [612, 616, 620, 621, 624], optical [622] and electronic Lieb lattices [631, 632], as well as superconducting qutrits [633]. A striking relativistic quantum hallmark of pseudospin-1 particles is super-Klein tunneling through a scalar potential barrier [608, 610, 623, 634, 635], where omnidirectional and perfect transmission of probability one occurs when the incident energy is about one half of the potential height. Generally, Klein tunneling defines optical-like, negatively refracted ray paths through the barrier interface via angularly resolved transmittance in the short wavelength limit [636–638].

A recent study [639] addressed the issue of confinement of quasiparticles in pseudospin-1 materials. When both super-Klein tunneling and chaos are present, one may intuitively expect severe leakage to predominantly occur so that trapping would be impossible. However, quite counterintuitively, an energy range was found in which robust wave confinement occurs in spite of chaos and super-Klein tunneling. Especially, the three-component spinor wave concentrates in a particular region of the boundary through strongly squeezed local current vortices generated there, whose pattern in physical space can be manipulated in a reconfigurable manner, e.g. by deforming the boundary shape or setting the direction of excitation wave. While these modes are distributed unevenly in physical space because of the irregular deformations, even fully developed chaos and super-Klein tunneling are not able to reduce their trapping lifetime. That is, these modes contradict the intuitive expectation that electrostatically confining relativistic type of carriers/particles to a finite chaotic domain is impossible due to the simultaneous presence of two leaking (Q-spoiling) mechanisms: chaos assisted tunneling and Klein tunneling. This phenomenon has no counterpart in nonrelativistic quantum or even in

pseudospin-1/2 systems. The resulting narrow resonances are also characteristically different from those due to scarring modes concentrating on periodic orbits in conventional wave chaotic scattering, in quantum dots [573, 640–645] or in open optical microcavities [646–648].

6. Discussions

Beside the above discussed few topics in relativistic quantum chaos, there are many other interesting topics that have been investigated in depth, such as quantum tunneling without [193, 194] and with electron–electron interactions [195], super-persistent currents that are robust to boundary deformations [602, 603] and the presence of disorders [604], relativistic quantum chimera states that electrons with different spins exhibit distinct scattering behaviors as they follow different classical dynamics [649], OTOC for relativistic quantum systems [605], anomalous entanglement in chaotic Dirac billiards [650], relativistic quantum kicked rotors [442, 510–512], kicked relativistic particle in a box [651], etc. More efforts are needed to gain deeper understandings of these interesting subjects. In addition, electron–electron interaction effects [652] in a chaotic graphene quantum billiard have also been considered and compared with scanning tunneling microscopy (STM) experiments, which could explain both the measured density of state values and the experimentally observed topography patterns [198]. Most of the understandings achieved so far for relativistic quantum chaos are for massless cases. Massive Dirac billiards have been considered only recently, where quantization formula for scarred states in confined 2D massive Dirac billiard has been proposed and validated numerically, and restoration of time-reversal symmetry in the infinite mass limit has been unveiled [570]. However, there are many other issues to be understood in the massive Dirac billiards, e.g. to what extent the intriguing observations for massless Dirac billiard persist in the massive case? Pseudo-spin one systems [607–609] have attracted much attention recently. Due to the flat band, it has many interesting properties regarding quantum chaotic scattering, such as superscattering that could even defy chaos Q-spoiling and Klein tunneling [635, 639, 653, 654]. There are still many open questions concerning pseudo-spin one system and quantum chaos.

Retrospecting the half-century development of quantum chaos, there are many subjects that would be interesting to extend into the relativistic quantum realm, such as the validity of the proposed indicators of universality corresponding to different classical dynamics, Loschmidt echo, many-body effects, quantum thermalization, etc. that have been discussed in section 1, as it is not straightforward to speculate what will happen when stepping into the relativistic regime. Efforts in trying to understand the behaviors of these subjects in relativistic quantum systems may not only advance the knowledge on the fundamental physics of relativistic quantum chaos, but may also bring new concepts of applications base on the state-of-art Dirac material technologies.

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