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Controlling chaotic dynamical systems

Celso Grebogi[a,*], Ying-Cheng Lai[b]

[a]Institut für Theoretische Physik und Astrophysik, Universität Potsdam, PF 601553, D-14415 Potsdam, Germany
[b]Departments of Physics and Astronomy and of Mathematics, The University of Kansas, Lawrence, KS 66045, USA

Abstract

We review the major ideas involved in the control of chaos. We present the Ott–Grebogi–Yorke (OGY) method of controlling chaos, which is a particular case of the pole placement technique, but which is the one leading to the shortest time to achieve the control of chaotic systems. Implementation using only measured time series in experimental settings is also described. © 1997 Elsevier Science B.V.

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1. Introduction

Besides the occurrence of chaos in a large variety of natural processes, chaos may also occur because one may wish to design a physical, biological or chemical experiment, or to project an industrial plant to behave in a chaotic manner. We argue herewith that chaos may indeed be desirable since it can be controlled by using small perturbation to some accessible parameter [17, 18] or to some dynamical variable of the system [9].

The major key ingredient for the control of chaos [17, 18] is the observation that a chaotic set, on which the trajectory of the chaotic process lives, has embedded within it a large number of unstable low-period periodic orbits. In addition, because of ergodicity, the trajectory visits or accesses the neighborhood of each one of these periodic orbits. Some of these periodic orbits may correspond to a desired system's performance according to some criterion. The second ingredient is the realization that chaos, while signifying sensitive dependence on small changes to the current state and henceforth rendering unpredictable the system state in the long time, also implies that the system's behavior can be altered by using small perturbations [17, 18]. Then, the accessibility of a chaotic system to many different periodic orbits combined with its sensitivity to small perturbations, allows for the control and the manipulation of the chaotic process. Specifically, the Ott–Grebogi–Yorke (OGY) approach is then as follows. One first determines some of the unstable low-period periodic orbits that are embedded in the chaotic set. One then examines the location and the stability of these orbits and chooses one which yields the desired system performance. Finally, one applies small control to stabilize this desired periodic orbit. However, all this can be done from data [17, 18] by using nonlinear time-series analysis for the observation, understanding and control of the system. This is particularly important since chaotic systems can be rather complicated and a detailed knowledge of the equations of the process is often unknown.

In this paper, we review the fundaments of the OGY ideas and present a method derived from it which can in principle be applied to high-dimensional chaotic systems [19]. The implementation of

* Correspondence address: Institute for Plasma Research, The University of Maryland, College Park, MD 20742, USA.

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the method utilizes the "pole placement technique" [16] developed in the field of engineering control. We also argue how the method can be applied to most experimental situations where the system's equations are not known.

2. Control of chaos

We consider the following discrete-time dynamical system,

$$x_{n+1} = F(x_n, p_n), \tag{1}$$

where $x_n \in \mathbb{R}^N$, F is a smooth vector function, p_n is an accessible parameter that can be externally perturbed. Continuous dynamical systems can be regarded as discrete maps on the Poincaré surface of section. Periodically driven dynamical systems have a natural Poincaré surface of section at the period of the driver. However, for autonomous dynamical systems such a section may not exist, or it may be singular if some of the trajectories take arbitrarily long time to return to it. One might need then, in order to discretize the dynamical process, to select some other kind of section whose choice typically depends on the particular system. We conceive using only small controls, so we restrict p to lie in some small interval,

$$|p_n - \bar{p}| < \delta, \tag{2}$$

where $\bar{p}$ is a nominal parameter value. If p_n is outside this interval, we set $p_n = \bar{p}$. Assuming that the dynamical system $F(x_n, \bar{p})$ possesses a chaotic attractor, our goal is to vary the parameter p_n within the range $(\bar{p} - \delta, \bar{p} + \delta)$ in such a way that for almost all initial conditions in the basin of the chaotic attractor, the dynamics of the system converges onto a desired time periodic orbit contained in the attractor. To do this we consider a small neighborhood of size comparable to δ of the desired periodic orbit. In this neighborhood, the dynamics is approximately linear. Since linear systems are stabilizable if the controllability assumption is obeyed, it is reasonable to assume that the chosen periodic orbit can be stabilized by feedback control. The ergodic nature of the chaotic dynamics guarantees that the state trajectory enters the neighborhood. Once inside, we apply the stabilizing feedback control law to keep the trajectory in the neighborhood of the desired orbit.

For simplicity we describe the method as applied to the case where the desired orbit is a fixed point of the map F. Consideration of periodic orbits of period larger than one is straightforward [19]. Let $x_*(\bar{p})$ be an unstable fixed point on the attractor. For values of p_n close to $\bar{p}$ and in the neighborhood of the fixed point $x_*(\bar{p})$, the map can be approximated by the linear map

$$x_{n+1} - x_*(\bar{p}) = A[x_n - x_*(\bar{p})] + B(p_n - \bar{p}), \tag{3}$$

where A is the $N \times N$ Jacobian matrix and B is an N-dimensional column vector,

$$\begin{aligned} A &= D_x F(x, p), \\ B &= D_p F(x, p). \end{aligned} \tag{4}$$

The partial derivatives in A and B are evaluated at $x = x_*$ and $p = \bar{p}$. To calculate the time-dependent parameter perturbation $(p_n - \bar{p})$, we assume that it is a linear function of x,

$$p_n - \bar{p} = -K^{\mathrm{T}}[x_n - x_*(\bar{p})], \tag{5}$$

where the $1 \times n$ matrix K^{T} is to be determined so that the fixed point x_* becomes stable. Substituting Eq. (5) into Eq. (3), we obtain,

$$x_{n+1} - x_*(\bar{p}) = (A - BK^{\mathrm{T}})[x_n - x_*(\bar{p})], \tag{6}$$

which shows that the fixed point will be stable if the matrix $(A - BK^{\mathrm{T}})$ is asymptotically stable; that is, all its eigenvalues have modulus smaller than unity.

The solution to the problem of determining K^{T}, such that the eigenvalues of the matrix $(A - BK^{\mathrm{T}})$ have specified values, is known from control systems theory as the "pole placement technique" [16]. In this context, the eigenvalues of the matrix $(A - BK^{\mathrm{T}})$ are called the "regulator poles". The following results give a necessary and sufficient condition for a unique solution of the pole placement problem to exist, and also a method for obtaining it (Ackermann's method) [16]: (1) The pole placement problem has a unique solution if and only if the $N \times N$ matrix

$$C = (B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \cdots \,\vdots\, A^{n-1}B),$$

is of rank N, where C is the controllability matrix; and (2) the solution of the pole placement problem is given by

$$K^{\mathrm{T}} = (\alpha_N - a_N, \ldots, \alpha_1 - a_1)T^{-1},$$

where $\boldsymbol{T} = \boldsymbol{CW}$ and,

$$\boldsymbol{W} = \begin{pmatrix} a_{N-1} & a_{N-2} & \cdots & a_1 & 1 \\ a_{N-2} & a_{N-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Here $\{a_1, \ldots, a_N\}$ are the coefficients of the characteristic polynominal of $\boldsymbol{A}$,

$$|s\boldsymbol{I} - \boldsymbol{A}| = s^N + a_1 s^{N-1} + \cdots + a_N,$$

and $\{\alpha_1, \ldots, \alpha_N\}$ are the coefficients of the desired characteristic polynomial $(\boldsymbol{A} - \boldsymbol{BK}^{\mathrm{T}})$.

The condition for the matrix $\boldsymbol{C}$ to be of rank N is too strong as far as stabilizability of a closed-loop system is concerned. In fact, the pole placement technique only requires a set of N points, placed symmetrically with respect to the real axis in the complex plane. Then there exists a feedback matrix $\boldsymbol{K}^{\mathrm{T}}$ such that the poles of the closed-loop system are the above set of points. It should be pointed out that there is a large class of control systems, in particular those arising in physical situations, which do not have a controllable linearization as indicated in Eq. (6). One has then to choose another control that obeys the controllability assumption if one wishes to use linear control. In particular, special care should be exercised when dealing with pole placement technique for nonautonomous systems. It should be noted that the control Eq. (5) is based on the linearized Eq. (3) and, therefore, it is only valid in the neighborhood of the desired fixed point $\boldsymbol{x}_*(\bar{p})$. The size of this valid neighborhood is determined by the limitation in the size of the parameter perturbation δ. Combining Eqs. (2) and (5), we obtain

$$|\boldsymbol{K}^{\mathrm{T}}[\boldsymbol{x}_n - \boldsymbol{x}_*(\bar{p})]| < \delta. \qquad (7)$$

This defines an invariant slab of width $2\delta/|\boldsymbol{K}^{\mathrm{T}}|$ in $\mathbb{R}^N$. We choose to activate the control according to Eq. (7) only when the trajectory falls into the slab, and we leave the control parameter at its nominal value $\bar{p}$ when the trajectory is outside this slab. It should also be noted that the matrix $\boldsymbol{K}^{\mathrm{T}}$ can be chosen in many different ways. In principle, a choice of regulator poles inside the unit circle, which do not violate the controllability condition, works [19]. The OGY method consists of setting the unstable poles equal to zero while leaving the stable ones as they are. With the OGY choice of regulator poles, the trajectory approaches the fixed point geometrically along the stable manifold after the control is turned on.

Since the control is turned on only when the trajectory enters the thin slab about the desired fixed point, one has to wait for some time for this to occur if the trajectory starts from a randomly chosen initial condition. Even then, because of nonlinearity not included in the linearized Eq. (3), the control may not be able to keep the trajectory in the vicinity of the fixed point. In this case the trajectory will leave the slab and continue to wander chaotically as if there was no control. *Since a chaotic trajectory on the uncontrolled chaotic attractor is ergodic, at some time it will eventually reenter the slab and also be sufficiently close to the fixed point so that control is achieved.* As a result, we create a stable orbit, which, for a typical initial condition, is preceded by a chaotic transient [8, 7] in which the orbit is similar to orbits on the uncontrolled chaotic attractor. Of course, there is a probability zero Cantor-like set of initial conditions which never enters the slab. The length τ of such a chaotic transient, or *the time to achieve control*, depends sensitively on the initial condition. For initial conditions randomly chosen in the basin of attraction, the probability distribution of the chaotic transient lengths is exponential [8] for large τ. The average transient length $\langle\tau\rangle$ is thus the average time to achieve control. It can be shown [17, 18] that $\langle\tau\rangle$ scales with δ algebraically: $\langle\tau\rangle \sim \delta^{-\gamma}$, where $\gamma > 0$ is the scaling exponent that is determined by the stable and unstable eigenvalues of the desired fixed point $\boldsymbol{x}_*(\bar{p})$. For a two-dimensional diffeomorphism [18], the scaling exponent is given by

$$\gamma = 1 + \frac{1}{2}\frac{\ln|\lambda_{\mathrm{u}}|}{\ln(1/|\lambda_{\mathrm{s}}|)},$$

where λ_{s} and λ_{u} are the stable and unstable eigenvalues of the periodic orbit being controlled. In Ref. [19], it is shown that the OGY choice for the regulator poles yields the shortest chaotic transient or, equivalently, the shortest average time to achieve control.

3. Use of delay coordinates

In most experimental situations a detailed knowledge of the system's equations is not known. One usually measures a time series of a single scalar state variable, say $u(t)$, and then utilizes delay coordinates [1, 2, 22] to represent the system state. A delay-coordinate vector in the m-dimensional embedding space can be formed as follows:

$$x(t) = (u(t), u(t - t_D), u(t - 2t_D), \ldots, u(t - (m-1)t_D)),$$

where t is the continuous time variable, and t_D is some conveniently chosen delay time. The embedding theorem [22] guarantees that for $m \geqslant 2N$, where N is the phase-space dimension of the system, the vector $\boldsymbol{x}$ is generically a global one-to-one representation of the system state. Since we only require $\boldsymbol{x}$ to be one-to-one in the small region near the fixed point, the requirement for the embedding dimension is actually $m = N - 1$ [17]. To obtain a map, one can take a Poincaré surface of section. For the often encountered case of periodically driven systems, one can define a "stroboscopic surface of section" by sampling the state at discrete time $t_n = nT + t_0$, where T is the driving period. In this case the discrete state variable is $\boldsymbol{x}_n = \boldsymbol{x}(t_n)$.

As pointed out in Ref. [5], in the presence of parameter variation, delay coordinates lead to a map of a different form than Eq. (1). For example, in the periodically forced case, since the components of $\boldsymbol{x}_n$ are $u(t - it_D)$ for $i = 0, 1, \ldots, (m-1)$, the vector $\boldsymbol{x}_{n+1}$ must depend not only on p_n, but also on all previous values of the parameter that are in effect during the time interval $(t_n - (m-1)t_D) \leqslant t \leqslant t_n$. In particular, let r be the smallest integer such that $mt_D < rT$. Then the relevant map is in general of the form,

$$\boldsymbol{x}_{n+1} = \boldsymbol{G}(\boldsymbol{x}_n, p_n, p_{n-1}, \ldots, p_{n-r}). \tag{8}$$

We now discuss how the OGY method can be applied to the case of delay coordinates. For simplicity we consider $r = 1$. In this case, we have,

$$\boldsymbol{x}_{n+1} = \boldsymbol{G}(\boldsymbol{x}_n, p_n, p_{n-1}). \tag{9}$$

Linearizing as in Eq. (3) and again restricting our attention to the case of a fixed point, we have,

$$\boldsymbol{x}_{n+1} - \boldsymbol{x}_*(\bar{p}) = \boldsymbol{A}[\boldsymbol{x}_n - \boldsymbol{x}_*(\bar{p})] + \boldsymbol{B}_a(p_n - \bar{p}) + \boldsymbol{B}_b(p_{n-1} - \bar{p}), \tag{10}$$

where $\boldsymbol{A} = \boldsymbol{D}_x\boldsymbol{G}(\boldsymbol{x}, p, p')$, $\boldsymbol{B}_a = \boldsymbol{D}_p\boldsymbol{G}(\boldsymbol{x}, p, p')$, $\boldsymbol{B}_b = \boldsymbol{D}_{p'}\boldsymbol{G}(\boldsymbol{x}, p, p')$, and all partial derivatives in $\boldsymbol{A}$, $\boldsymbol{B}_a$, and $\boldsymbol{B}_b$ are evaluated at $\boldsymbol{x} = \boldsymbol{x}_*(\bar{p})$ and $p = \bar{p} = p'$.

One can now define a new state variable with one extra component by,

$$\bar{x}_{n+1} = \begin{pmatrix} \boldsymbol{x}_{n+1} \\ p_n \end{pmatrix}, \tag{11}$$

and introduce the linear control law,

$$p_n - \bar{p} = -\boldsymbol{K}^{\mathrm{T}}[\boldsymbol{x}_n - \boldsymbol{x}_*(\bar{p})] - k(p_{n-1} - \bar{p}). \tag{12}$$

Combining Eqs. (10) and (12), we obtain

$$\bar{x}_{n+1} - \bar{x}_*(\bar{p}) = (\bar{A} - \bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{\mathrm{T}})[\bar{x} - \bar{x}_*(\bar{p})], \tag{13}$$

where

$$\bar{x}_*(\bar{p}) = \begin{pmatrix} \boldsymbol{x}_*(\bar{p}) \\ \bar{p} \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}_b \\ \boldsymbol{0} & 0 \end{pmatrix},$$

$$\bar{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{B}_a \\ 1 \end{pmatrix}, \quad \bar{\boldsymbol{K}} = \begin{pmatrix} \boldsymbol{K} \\ k \end{pmatrix}.$$

Since Eq. (13) is now of the same form as Eq. (6), the method of Section 2 can be applied. A similar result holds for any $r > 1$. Although the explicit form for the function $\boldsymbol{G}(\boldsymbol{x}_n, p_n, p_{n-1})$ is not known, the quantities required for computing the parameter perturbations in Eq. (13) can usually be extracted directly from the measurement [6]. The location of the periodic orbit is obtained by looking at recurrences in the embedded space [3, 14]. The matrix $\boldsymbol{A}$ in Eq. (10) and the corresponding eigenvalues and eigenvectors are obtained by looking at the same recurrences about the desired periodic orbit and fitting an affine transformation $\boldsymbol{x}_{n+1} = \boldsymbol{A}\boldsymbol{x}_n + \boldsymbol{b}$, since the dynamics is approximately linear close to the periodic orbit. The vectors $\boldsymbol{B}_a$ and $\boldsymbol{B}_b$ in Eq. (10) are obtained by perturbing the control parameter of the system [5, 4, 17].

4. Discussions

An important issue is the effect of noise. Noise can, in general, kick the controlled trajectory out of the neighborhood of the chosen periodic orbit where the control is activated. When this occurs, the trajectory wanders chaotically over the attractor until it falls in the controlled region again. Thus there are epochs where the orbit is kept near the desired orbit interspersed with epochs wherein the orbit wanders chaotically far from the desired orbit.

If the latter are, on average, relatively much shorter than the former, then one might still regard the control as being effective.

The transient phase where the orbit wanders chaotically before locking into a controlled orbit can be greatly shortened by applying a "targeting" technique [11, 23] so that a trajectory can be rapidly brought to a target region on the attractor by using small control perturbations. The idea is that, since chaotic systems are exponentially sensitive to perturbations, careful choice of even small control perturbations can, after some time, have a large effect on the trajectory location and can be used to guide it. Thus, the time to achieve control can, in principle, be greatly reduced by properly applying small controls when the orbit is far from the neighborhood of the desired periodic orbit.

In this paper, we have considered the case where there is only a single control parameter available for adjustment. While generically a single parameter is sufficient for stabilization of a desired periodic orbit, there may be some advantage to utilizing several control variables. Therefore, the single control parameter p becomes a vector. In particular, the added freedom in having several control parameters might allow better means of choosing the control so as to minimize the time to achieve control, as well as the effects of noise.

We emphasize that a full knowledge of the system dynamics is not necessary in order to apply the OGY idea [17, 18]. In particular, we only require the location of the desired periodic orbit, the linearized dynamics about the periodic orbit, and the dependence of the location of the periodic orbit on small variation of the control parameter. Delay-coordinate embedding has been successfully utilized in experimental studies to extract such information purely from observations of experimental chaotic orbits on the attractor without any a priori knowledge of the equations of the system, and such information has been utilized to control periodic orbits [6].

Finally, we mentioned that the OGY idea of controlling chaos gives flexibility. By switching the small control, one can switch the time asymptotic behavior from one periodic orbit to another. In some situations, where the flexibility offered by the ability to do such switching is desirable, it may be advantageous to design the system so that it is chaotic. In other situations, where one is presented with a chaotic system, the method may allow one to eliminate chaos and achieve greatly improved behavior at relatively low cost. The OGY idea can also be used to stabilize a desired chaotic trajectory, which has potential applications to problems such as synchronization of chaotic systems [12], conversion of transient chaos into sustained chaos [13], communication with chaos [9, 10, 20], and selection of a desired chaotic phase [15, 21].

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