

Riddling of Chaotic Sets in Periodic Windows

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Previous investigations of riddling have focused on the case where the dynamical invariant set in the symmetric invariant manifold of the system is a chaotic attractor. A situation expected to arise commonly in physical systems, however, is that the dynamics in the invariant manifold is in a periodic window. We argue and demonstrate that riddling can be more pervasive in this case because it can occur regardless of whether the chaotic set in the invariant manifold is transversely stable or unstable. Scaling behavior associated with this type of riddling is analyzed and is supported by numerical experiments.

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Recently, the phenomenon of riddling has attracted much attention [1–9]. The dynamical conditions for riddling to occur were first described in Ref. [2], where it was shown that for systems with an invariant manifold \mathcal{M} : (i) if there is a chaotic attractor in \mathcal{M} ; (ii) if a typical trajectory in the chaotic attractor is stable with respect to perturbations transverse to \mathcal{M} , then the basin of the chaotic attractor in \mathcal{M} can be riddled with holes that belong to the basin of another attractor off \mathcal{M} , provided that such an attractor exists. That is, for every initial condition that asymptotes to the chaotic attractor in \mathcal{M} , there are initial conditions arbitrarily nearby that asymptote to the attractor off \mathcal{M} . Riddling has become a topic of much investigation, because it is fairly common for systems with symmetry or for spatiotemporal chaotic systems such as those described by coupled maps or by coupled differential equations. Spatially coupled systems naturally possess an invariant manifold: the synchronization manifold on which all individual oscillators evolve chaotically and synchronously in time [5].

Previous research has been based on the assumption that conditions (i) and (ii) are necessary for riddling to occur. In this work, we argue that one of the conditions, condition (i)—the existence of a chaotic attractor in the invariant manifold \mathcal{M} , is sufficient but not necessary. It is necessary, however, that a chaotic invariant set does exist in \mathcal{M} [10]. It is thus sufficient to have a chaotic saddle in \mathcal{M} . These findings have significant dynamical consequences to important practical problems. While chaotic attractors are common in physical systems, it can easily disappear as a system parameter undergoes arbitrarily small changes [11]. For instance, the dynamics in the invariant manifold may be that of one of the infinite number of periodic windows near a parameter value at which the chaotic attractor is observed. Since periodic windows occupy intervals of parameter values which are apparently dense [12] in the parameter space, we expect this situation to be common in nonlinear systems. At a given periodic window, one finds the coexistence of a nonattracting chaotic saddle and an attracting periodic

orbit [13]. A question is then, how does riddling manifest itself when the dynamics in the invariant manifold is in a periodic window whose invariant sets are both a nonattracting chaotic saddle [necessary condition (i) for riddling] and an attracting periodic orbit? The purpose of this Letter is to address this question in view of the stated necessary conditions for riddling. We find that riddling, in fact, occurs in the transverse vicinity of the chaotic saddle. Moreover, the basin of attraction in the transverse vicinity of the stable periodic orbit also consists of open volumes (open areas in two dimensions). Globally, the basin of attraction of the stable periodic orbit is therefore of a mixed type: riddled basins and open volumes. A surprising finding is that this type of riddling occurs in a wide parameter region for both transversely stable or transversely unstable chaotic saddles. This is in contrast to riddling with chaotic attractors, where riddling occurs only when the attractor is transversely stable with some of the embedded unstable periodic orbits being transversely unstable. To quantify riddling, we investigate scaling laws for physically observable quantities such as the probabilities for a random initial condition to asymptote to different attractors. The main implication of our results is that riddling is a robust dynamical phenomenon, regardless of the nature of the attracting set in the invariant manifold, in so far as there is chaos (attracting or nonattracting) in the system.

We begin by presenting a qualitative argument for the condition under which riddling can be observed. Let \mathcal{M} be the invariant manifold in which there is a nonattracting chaotic saddle S and an attracting periodic orbit \mathcal{O} , and let \mathcal{A} be an attractor off \mathcal{M} . Assume that the periodic attractor \mathcal{O} is transversely stable so that there is a boundary between the basins of attraction of \mathcal{O} and \mathcal{A} . Since \mathcal{O} is stable both in \mathcal{M} and in the transverse direction, the basin of attraction in the transverse vicinity of \mathcal{O} is an open set containing \mathcal{O} with finite volume. Now consider the situation where all unstable periodic orbits embedded in the chaotic saddle S are transversely stable. In this case, almost

all initial conditions in the vicinity of \mathcal{M} asymptote to the periodic attractor in \mathcal{M} . There is no riddling in this case. As a system parameter p changes through a critical value p_c , one of the unstable periodic orbits in S becomes transversely unstable and, as a consequence, a set of an infinite number of tongues opens at the locations of the periodic orbit and all its preimages [7]. This is the riddling bifurcation that marks the onset of riddling for $p > p_c$ [7]. There is, however, a practically important difference in the dynamics for $p > p_c$ between the case treated in Ref. [7], where the invariant set in \mathcal{M} is a chaotic attractor, and our case here. It was shown in Ref. [7] that for $p > p_c$, trajectories in the vicinity of \mathcal{M} can typically spend an extremely long time near \mathcal{M} before asymptoting to the attractor \mathcal{A} —a superpersistent chaotic transient. The lifetime of the transient scales with $\Delta p \equiv |p - p_c|$ as $\tau_A \sim \exp[K(\Delta p)^{-\gamma}]$, where $\gamma > 0$ and $K > 0$. In our case, this type of supertransient still occurs [14], but there is another lifetime τ_O associated with the chaotic saddle, which is typically much shorter than the supertransient lifetime: $\tau_O \ll \tau_A$. Note that τ_O is the lifetime of transient chaos around the saddle [15]. That is, a trajectory in the basin of the periodic attractor \mathcal{O} approaches this attractor in a time that is typically shorter than the time τ_A . As a practical consequence, no riddling can be observed readily, say, in numerical experiments, after the riddling bifurcation, until $\tau_O \sim \tau_A$. This, of course, does not imply that the basin is not riddled until $\tau_O \sim \tau_A$. As Δp increases, τ_A decreases and becomes smaller than or equivalent to τ_O . In this case, riddling can be observed. Mathematically, the basin of attraction of \mathcal{A} consists of an open dense set of tongues off the invariant manifold \mathcal{M} but these tongues are restricted to the set S . Insofar as unstable periodic orbits in \mathcal{M} can be both transversely stable and transversely unstable, which can occur regardless of whether the chaotic saddle itself is transversely stable or unstable, riddling can occur. This is different from riddling of a chaotic attractor in \mathcal{M} , in which case riddling disappears when the attractor becomes transversely unstable [1–9]. In this sense, we expect riddling to be more pervasive when the dynamics in \mathcal{M} is in a periodic window.

We consider the following general class of dynamical systems: $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, r) + \text{higher order terms of } \mathbf{y}_n$ and $\mathbf{y}_{n+1} = g(\mathbf{x}_n, a)\mathbf{y}_n + \text{higher order terms of } \mathbf{y}_n$, where $\mathbf{x} \in R^{N_s}$ ($N_s \geq 1$), $\mathbf{y} \in R^{N_r}$ ($N_r \geq 1$), $\mathbf{f}(\mathbf{x}_n, r)$ is a map possessing an infinite number of periodic windows, $g(\mathbf{x}_n, a)$ is a scalar function, and r and a are parameters. The invariant manifold is defined by $\mathbf{y} = 0$ because for initial $\mathbf{y}_0 = \mathbf{0}$, trajectories have $\mathbf{y}_n = \mathbf{0}$ for all times. We choose the parameter r in the map $\mathbf{f}(\mathbf{x}_n, r)$ so that it is in a periodic window of period m . Let Λ_T^S and Λ_T^O be the largest transverse Lyapunov exponents of the chaotic saddle and of the periodic attractor, respectively. To be concrete, we study the following two-dimensional map:

$$x_{n+1} = rx_n(1 - x_n) + by_n^2 \text{ and } y_{n+1} = ax_ny_n + cy_n^3, \quad (1)$$

where $f(x) = rx(1 - x)$ is the logistic map, and a , b , and c are parameters. We choose $r = 3.84$ so that the logistic map is in a period-3 window with an attracting periodic orbit of period 3 coexisting with a chaotic saddle. We are interested in the case where Λ_T^O remains negative. There are thus two attractors in the system: the period-3 attractor in the invariant manifold $y = 0$ and the attractor at $|y| = +\infty$. We find, for instance, for $b = 0.1$ and $c = 1.0$, Λ_T^O is negative in a wide range of values of parameter a , while Λ_T^S passes through zero from the negative side at a_c as a is increased, where $1.7 < a_c < 1.8$. Numerically, we find that the basin of the period-3 attractor is riddled, regardless of whether the chaotic saddle in $y = 0$ is transversely stable or transversely unstable, and the basin of the attraction at infinity is an open dense set of tongues in the transverse neighborhood of S . However, due to $\tau_O \ll \tau_A$, it is numerically difficult to observe riddling when Λ_T^S is more negative, say, for $a < 1.5$.

To quantify riddling, we focus on the scaling behaviors of some physical observables. For example, we can consider the probability for an initial condition chosen randomly from a line at $y = \epsilon$ near the invariant manifold $y = 0$ to asymptote to the attractor at infinity. Denote this probability by $F^+(\epsilon)$. Figures 1(a) and 1(b) show $F^+(\epsilon)$ vs ϵ on a logarithmic scale for $a = 1.7$ and $a = 1.8$, respectively. Apparently, we have, for both cases, the following algebraic scaling law:

$$F^+(\epsilon) \sim \epsilon^\gamma, \quad (2)$$

where the scaling exponent is $\gamma \approx 1.73$ and $\gamma \approx 0.92$ for Figs. 1(a) and 1(b), respectively. Note that the scaling exponent in Fig. 1(a) is significantly larger than that for the case where there is a chaotic attractor in the invariant manifold with similar transverse Lyapunov exponent. In that case, the scaling exponent is proportional to $|\Lambda_T^S|$ which is close to zero [3,8]. The largeness of the scaling exponent γ for $|\Lambda_T^S| \geq 0$, regardless of whether Λ_T^S is positive or negative, is a general feature of riddling in our case, in contrast to riddled basins of chaotic attractors. Physically, such a large exponent means that it is significantly more difficult for trajectories with random initial conditions near the invariant manifold to asymptote to the attractor off the invariant manifold. The dynamical reason lies in the finite lifetime τ_O of the chaotic saddle: A trajectory will approach the periodic attractor in a time given by τ_O . When $|\Lambda_T^S| \geq 0$, it usually takes a long time for trajectories to escape the finite-time transverse attraction of the chaotic saddle in order to asymptote to the attractor off the invariant manifold.

To better understand riddling and the scaling law Eq. (2), we construct a simple analyzable model that captures the main feature of our numerical model Eq. (1).

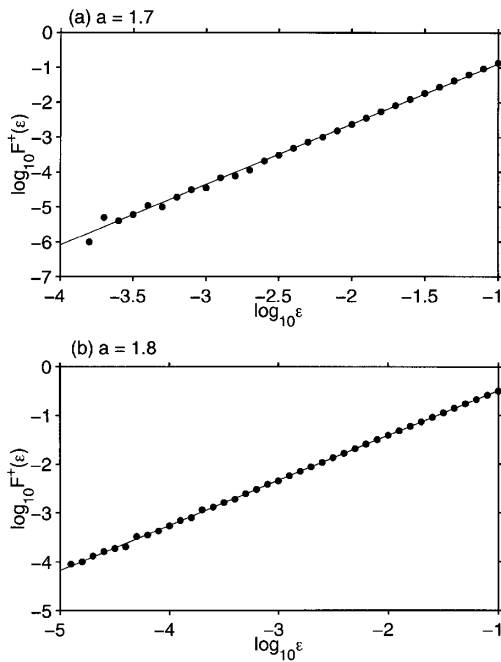


FIG. 1. $F^+(\epsilon)$ vs ϵ on a logarithmic scale for (a) $a = 1.7$, and (b) $a = 1.8$.

The model is the following two-dimensional map defined for $-\infty < x < \infty$ and $0 \leq y < \infty$:

$$\begin{aligned}
 x_{n+1} &= \begin{cases} h(x_n), & x_n < 0, \\ \frac{1+q}{p}x_n, & 0 < x_n < p, \\ \frac{1+q}{1-p}(1-x_n), & x_n > p, \end{cases} \\
 y_{n+1} &= \begin{cases} e^{-\Gamma}y_n, & x_n < 0 \text{ and } 0 \leq y_n < 1, \\ cy_n, & 0 < x_n < p < 1/2 \text{ and } 0 \leq y_n < 1, \\ dy_n, & x_n > p \text{ and } 0 \leq y_n < 1, \end{cases}
 \end{aligned} \tag{3}$$

where $q \geq 0$, $0 < p < 1$, $c > 1$, $0 < d < 1$, and

$$\bar{P}(Y, s) = \frac{1}{D(P_+ - P_-)} [-e^{P_-Y_0 - P_+Y} + \Theta(Y_0 - Y)e^{P_-(Y_0 - Y)} + \Theta(Y - Y_0)e^{-P_+(Y - Y_0)}], \tag{4}$$

where $P_{\pm} = (-\nu \pm \sqrt{\nu^2 + 4sD})/2D$, and $\Theta(x)$ is the Heaviside step function. Since $\int_0^\infty P(Y, t) dY$ is the probability that the walker is still undergoing diffusion at time t , the probability for a trajectory to asymptote to the attractor **A** in time τ_O is $F^+(\epsilon) = 1 - \int_0^\infty P(Y, \tau_O) dY$. As a crude approximation, we assume that walkers who are still diffusing for time $t > \tau_O$ asymptote to the fixed-point attractor **O**. We thus obtain

$$\begin{aligned}
 F^+(\epsilon) &= 1 - \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds e^{s\tau_O} \int_0^\infty dY \bar{P}(Y, s) \\
 &= \frac{1}{2\pi i} \int_\infty^\infty \frac{d\omega}{\omega} e^{i\omega\tau_O} \exp\left[\frac{\ln\epsilon}{\sqrt{D}} \sqrt{i\omega + \frac{\nu^2}{4D}}\right].
 \end{aligned} \tag{5}$$

Figure 2(b) shows $\log_{10}F^+(\epsilon)$ vs $\log_{10}\epsilon$, where the integration in Eq. (5) is done numerically, and the parameters

$\Gamma > 0$. The map $h(x)$ is chosen such that it has a stable fixed point **O** in $x < 0$, as shown in Fig. 2(a). Since $q \geq 0$, we see that the x dynamics has a chaotic saddle in $(0, 1)$ with lifetime $\tau_O = [\ln(1 - \Delta)^{-1}]^{-1} = 1/\ln(1 + q) \approx 1/q$, and almost all initial conditions eventually asymptote to the fixed-point attractor **O**. This is the dynamics in the invariant manifold $y = 0$. The y dynamics is described by a simple expansion-contraction process for $0 \leq y < 1$, and we assume there is another attractor **A** located at $y > 1$ and any trajectory with $y > 1$ asymptotes to it rapidly. The transverse Lyapunov exponents of the fixed-point attractor and the chaotic saddle are $\Lambda_T^O = -\Gamma < 0$ and $\Lambda_T^S = (p/q)\ln c + [(1 - p)/q]\ln d$, respectively. Choosing p as the bifurcation parameter, we see that Λ_T^S crosses zero from the negative side when p passes through the critical point $p_c = |\ln d|/(\ln c + |\ln d|)$. Letting $Y = -\ln y$, we have $Y_{n+1} = \alpha_n + Y_n$, where $\alpha_n = -\ln c < 0$ if $0 < x_n < p$ and $\alpha_n = -\ln d > 0$ if $p < x_n < 1$. Since the x dynamics is chaotic for $0 < x < 1$ in time τ_O , we see that the dynamics in Y_n is a finite-time random walk. Focusing on the situation where $p \approx p_c$ so that $\Lambda_T^S \approx 0$, the random walk dynamics can be described by the following drift-diffusion equation in time τ_O : $\frac{\partial P}{\partial t} + \nu \frac{\partial P}{\partial Y} = D \frac{\partial^2 P}{\partial Y^2}$, where $P(Y, t)$ stands for the probability distribution for Y , $\nu = -\Lambda_T^S \approx 0$ is the average drift, and $D = \frac{1}{2}\langle(\delta Y - \langle\delta Y\rangle)^2\rangle$ is the diffusion coefficient. Choosing initial conditions from a line at $Y_0 = 1/\ln\epsilon$ ($y = \epsilon$), we have $P(Y, 0) = \delta(Y - Y_0)$ as the initial condition for the diffusion equation. Since trajectories having $y > 1$ are lost to the attractor **A** at $y > 1$, we have an absorbing boundary at $Y = 0$ ($y = 1$): $P(0, t) = 0$ for $t < \tau_O$. Solving the diffusion equation with these conditions by using the standard Laplace-transform technique [16], we obtain $\bar{P}(Y, s)$, the Laplace transform of $P(Y, t)$, as follows:

are chosen to mimic the parameter setting in our numerical example [Eq. (1)]: $\nu = 0.05$, $D = 1$, and $\tau_O = 20$. We see that the algebraic scaling law (2) holds well, in agreement with Figs. 1(a) and 1(b). The dashed straight line in Fig. 2(b) denotes the case where $\tau_O \rightarrow \infty$, for which a closed expression for $F^+(\epsilon)$ can be obtained

$$\begin{aligned}
 F^+(\epsilon) &\sim \epsilon^{\nu/D} \left[\frac{1}{2} - \frac{1}{\tau_O \sqrt{\pi}} \sqrt{\frac{D}{\nu^3}} |\ln\epsilon| \right. \\
 &\quad \left. \times \exp\left(\frac{-\tau_O^2 \nu^3}{4D|\ln\epsilon|}\right) \right].
 \end{aligned} \tag{6}$$

We see that indeed, the decay behavior for finite τ_O is generally faster than that for the case of a chaotic attractor ($\tau_O = \infty$) in the invariant manifold, in agreement with our numerical experiments of Eq. (1). Note that Eq. (6) includes the case of a chaotic attractor ($\tau_O = \infty$)

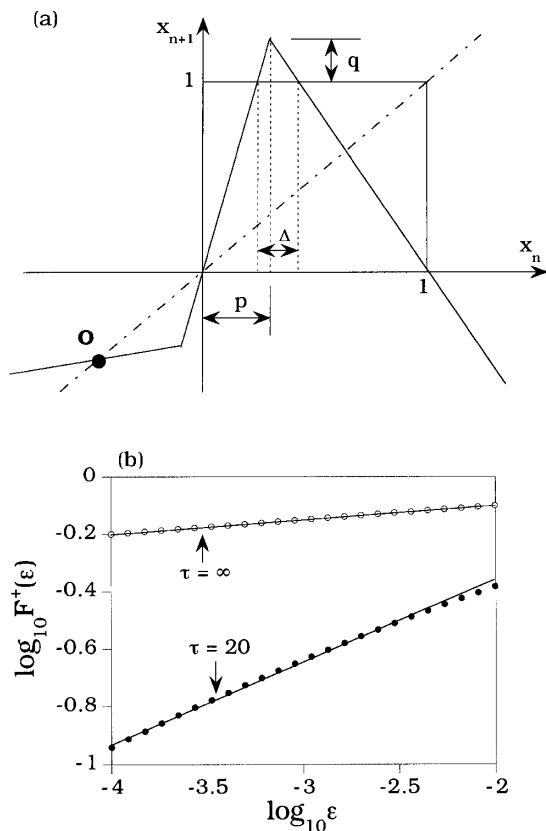


FIG. 2. (a) The x dynamics in the theoretical model. (b) Theoretically predicted and numerically computed scaling relations between $F^+(\epsilon)$ and ϵ .

as a special case where it was found previously that $F^+(\epsilon) \sim \epsilon^{\nu/D}$ [3].

In summary, our study indicates that, due to the dynamical interplay between the nonattracting chaotic saddle and the stable periodic attractor, riddling can then be *more pervasive* than that studied previously. Our scaling analysis generalizes the existing ones which focus exclusively on chaotic attractors in the invariant manifold. Since periodic windows are structurally stable, dense in parameter, and therefore are extremely common in nonlinear systems, we expect our results to be relevant to a large variety of problems including synchronization of nonlinear oscillators [17].

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