Modeling of Coupled Chaotic Oscillators

Ying-Cheng Lai[1] and Celso Grebogi[2]
[1]*Departments of Physics and Astronomy and of Mathematics, University of Kansas, Lawrence, Kansas 66045*
[2]*Institute for Plasma Research, Department of Mathematics, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742*
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Chaotic dynamics may impose severe limits to deterministic modeling by dynamical equations of natural systems. We give theoretical argument that severe modeling difficulties may occur for high-dimensional chaotic systems in the sense that no model is able to produce reasonably long solutions that are realized by nature. We make these ideas concrete by investigating systems of coupled chaotic oscillators. They arise in many situations of physical and biological interests, and they also arise from discretization of nonlinear partial differential equations. [S0031-9007(99)09331-X]

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Coupled oscillators are relevant to a large variety of physical and biological phenomena [1]. Arrays of Josephson junctions [2] and coupled solid state lasers [3] are well known examples in physics. In biology, vital organs such as hearts, auditory, visual, and central nervous systems are complex networks of many small oscillators such as cells and neurons. Systems of coupled equations can also arise from spatial discretization of nonlinear partial differential equations such as the Navier-Stokes equation in fluid dynamics. The dynamics of the fundamental elements, or the individual oscillators in the network, can be either regular or complicated. Typically, the collective behavior of all the oscillators in the network can be extremely rich, ranging from steady state or periodic oscillations to chaotic or turbulent motions. Systems of coupled oscillators have thus become an area of great interest for physicists, mathematicians, and biologists.

Mathematically, a natural system of N coupled oscillators can be modeled using either continuous-time flows or a lattice of coupled maps; the latter can be written as follows:

$$\mathbf{x}_{n+1}^{i} = \mathbf{F}(\mathbf{x}_n^i) + \epsilon \sum_{j=1}^{N} g_{ij}\mathbf{H}(\mathbf{x}_n^j), \qquad i = 1, \ldots, N, \tag{1}$$

where i and j denote the lattice site, n is the discrete time, $\mathbf{x}_i$ is a D-dimensional vector, $\mathbf{F}$ and $\mathbf{H}$ are D-dimensional vector functions, ϵ is a parameter characterizing the coupling strength, and g_{ij} denotes the elements of the coupling matrix. For such a system, of great importance is the synchronization state (manifold) defined by $\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N$. If the elements of the coupling matrix satisfy $\sum_j g_{ij} = 0$, then the synchronization state is a solution of Eq. (1). In this case, if the system is initialized in the synchronization manifold, the state of the system remains synchronized in the absence of random noise. The synchronization manifold is thus an *invariant* manifold for the system. It has dimension D, while the full dynamics lies in a manifold of dimension $N \times D$.

There have been a number of recent papers addressing various issues on the stability of the synchronization manifold for coupled chaotic oscillators [4]. The purpose of this Letter is to show that, by looking at the stability of the synchronization manifold, one can establish to what extent a natural system of coupled *chaotic* oscillators can be modeled *deterministically* by Eq. (1). Our principal result is that there are parameter regimes of positive measure for which Eq. (1) is not able to model deterministically the *natural* system of coupled oscillators in the sense that no trajectory of reasonable length [5] of Eq. (1) is close to any trajectory of the natural chaotic system of coupled oscillators that Eq. (1) is supposed to model. More specifically, say one constructs a *physical* system of coupled chaotic oscillators in a laboratory, and one measures a trajectory. Then no trajectory of reasonable length from the mathematical model of the *physical* system, as given by Eq. (1), is close to the measured physical trajectory [6,7]. We stress that subtleties and difficulties of numerical calculations of chaotic systems are not the issue here. The difficulty to model the natural process is a consequence of the inexactitude of the model given by the inevitable random disturbances and imperfections of the model such as various approximations used in the model-building process and intrinsic dynamical properties of the physical systems under consideration. We argue then in this Letter that, if the model is an approximation to the natural process, as indeed it is, due to imperfections of the natural system, no model can produce trajectories of reasonable length that are close to trajectories of the actual system of coupled oscillators. Moreover, we show that this obstruction to modeling occurs already for small, but nonzero, coupling. Thus, one should exercise some care when studying and interpreting results from models of coupled chaotic oscillators. Often, the only long-term meaningful results one can trust are the statistical invariants obtained by simulating a large number of trajectories of the model [7]. The implication is that in laboratory experiments involving coupled chaotic oscillators, it might only make sense to work *directly* with measured time series instead of a mathematical model similar to Eq. (1), when attempting to understand the long-term behavior of the system, even if the model is built upon physical laws and is considered to be reasonable.

The property that allows us to make such strong statements about the difficulties in modeling coupled chaotic oscillators is the notion of *unstable dimension variability* [8], a type of nonhyperbolicity believed to arise commonly in high-dimensional chaotic systems [9]. Roughly, unstable dimension variability means as the trajectory evolves in the chaotic invariant set, it experiences a distinct number of unstable directions in different regions of the phase space. This is a situation violating a basic requirement for hyperbolicity of dynamical systems [10]. The fundamental reason for unstable dimension variability is the different number of unstable directions exhibited by the unstable periodic orbits embedded in the chaotic set. The consequence of unstable dimension variability may be severe: due to modeling error, no model trajectory of reasonable length is close to any trajectory of the actual system [7–9,11,12].

To address modeling of coupled chaotic systems, it thus suffices to search for unstable dimension variability in such systems, which can be qualitatively seen as follows. Consider the chaotic attractor (usually low-dimensional) in the synchronization manifold of dimension D. Since the synchronization manifold is invariant under the dynamics, the infinite number of unstable periodic orbits embedded in the attractor are the periodic orbits of Eq. (1). Coupling of the N individual oscillators immediately introduces $N - 1$ additional D-dimensional eigenspaces to each one of these periodic orbits, subspaces which are *transverse* to the synchronization manifold. In these transverse subspaces, the unstable periodic orbits in the synchronization manifold can have a different number of unstable directions due to coupling. As a result, these unstable periodic orbits in the space of dimension ND for the coupled oscillator system have a different number of unstable directions, a situation characterizing unstable dimension variability.

To explicitly demonstrate unstable dimension variability in models of coupled chaotic oscillators, we study a model system of N coupled Hénon maps [13] on a circle (periodic boundary condition) for which unstable periodic orbits in the invariant manifold can be computed systematically:

$$\{x_{n+1}^i, y_{n+1}^i\} = \Big\{a - (x_n^i)^2 + b y_n^i + \frac{\epsilon}{2}(2x_n^i - x_n^{i+1} - x_n^{i+1}), x_n^i\Big\}, \quad i = 1, \dots, N, \tag{2}$$

where the coupling is assumed to be nearest-neighbor-type, ϵ is the coupling strength, and a and b are the local parameters in the Hénon map [denoted by $\mathbf{F}(\mathbf{x})$]. Writing $\mathbf{X} \equiv \{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^N\}$, where $\mathbf{x}^i \equiv (x^i, y^i) \in \mathbf{R}^2$ and $\mathbf{X} \in \mathbf{R}^{2N}$, we have for the synchronization manifold: $\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N \equiv \mathbf{x}$, which is invariant and a two-dimensional plane. Unstable periodic orbits of the Hénon map $\mathbf{F}(\mathbf{x})$ can be computed by the method of Ref. [14].

We now argue, quantitatively, that unstable dimension variability occurs when ϵ is nonzero. The starting point is to study the stability of the Hénon unstable periodic orbits in the $2N$-dimensional space of coupled oscillators, which are embedded in the synchronization manifold. To do this, we make use of the variational formalism developed by Pecora and Carroll [15]. For Eq. (2), an infinitesimal vector $\delta\mathbf{X}$ evolves in the tangent space according to $\delta\mathbf{X}_{n+1} = [\mathbf{I}_N \otimes \mathbf{DF}(\mathbf{x}) - \frac{\epsilon}{2}\mathbf{g} \otimes \mathbf{H}] \cdot \delta\mathbf{X}_n$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix, "$\otimes$" denotes direct product, $\mathbf{DF}(\mathbf{x})$ is the Jacobian matrix of the Hénon map, and the elements of the matrix $\mathbf{g}$ are zero except for the following: $g_{ii} = -2$ $(i = 1, \dots, N)$, $g_{i,i-1} = g_{i,i+1} = 1$ $(i = 2, \dots, N)$, $g_{12} = g_{1N} = g_{N1} = g_{N,N-1} = 1$, and $H_{11} = 1$ is the only nonzero element in the matrix $\mathbf{H}$. In order to find the stability of each unstable periodic orbit embedded in the invariant Hénon plane, it is necessary to diagonalize the matrix $[\mathbf{I}_N \otimes \mathbf{DF}(\mathbf{x}) - \frac{\epsilon}{2}\mathbf{g} \otimes \mathbf{H}]$. This can be done by diagonalizing the matrix $\mathbf{g}$, which does not influence the block-diagonal matrix $\mathbf{I}_N \otimes \mathbf{DF}(\mathbf{x})$. Let γ_k $(k = 0, 1, \dots, N - 1)$ be the eigenvalues of the matrix $\mathbf{g}$, where $\gamma_0 = 0$ (because $\sum_j g_{ij} = 0$). As a result of the diagonalization, we obtain the following N variational equations in the plane:

$$\delta\mathbf{x}_{n+1}^k = \left[\mathbf{DF}(\mathbf{x}) - \frac{\epsilon}{2}\gamma_k\mathbf{H}\right] \cdot \delta\mathbf{x}_n^k = \begin{pmatrix} -2x - \frac{\epsilon}{2}\gamma_k & b \\ 1 & 0 \end{pmatrix} \cdot \delta\mathbf{x}_n^k, \quad k = 0, 1, \dots, N - 1. \tag{3}$$

The first equation, corresponding to $\gamma_0 = 0$, gives the stability of an orbit in the invariant Hénon plane. The remaining $N - 1$ equations determine then the stabilities of the orbit in the $2(N - 1)$-dimensional *transverse* space, which is made up of $N - 1$ transverse planes. Let $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p)$ be one of the periodic orbits of period-p in the invariant Hénon plane, whose stability in the $N - 1$ transverse planes is determined by the following product of p matrices:

$$\prod_{i=1}^{p}\begin{pmatrix} -2x_i - \frac{\epsilon}{2}\gamma_k & b \\ 1 & 0 \end{pmatrix}, \qquad k = 1, 2, \dots, N - 1. \tag{4}$$

In the transverse planes, for typical eigenvalues $\gamma_k \neq 0$, a finite ϵ can cause a shift in the transverse eigenvalues of the Hénon unstable periodic orbits. Consider all the unstable periodic orbits of period-p (p large) in the Hénon chaotic attractor. By definition of the Lyapunov exponent, the probability distribution of the unstable eigenvalues of these Hénon orbits is approximately a Gaussian centered at $e^{\lambda p}$ and it has a finite width, where $\lambda > 0$ is the Lyapunov exponent of the Hénon chaotic attractor [16]. Thus, there can be a set of periodic orbits whose eigenvalues are larger than but close to unity. As one examines the stability of these orbits in the $N - 1$ transverse planes, for $\epsilon \gtrsim 0$, it is likely that the eigenvalues in some of the transverse planes cross the unit circle inward becoming less than 1, leading to the loss of a few unstable directions, among $N - 1$ of them when $\epsilon \neq 0$. Unstable dimension variability thus occurs for $\epsilon \gtrsim 0$ due to the existence of Hénon orbits whose eigenvalues are close to 1 if one requires small coupling strength.

We have undertaken a series of eigenvalue computations to demonstrate unstable dimension variability in the model given by Eq. (2) for $N = 5$ (just for an illustrative purpose). The full dynamics is hence in $\mathbf{R}^{10}$, but the invariant synchronization plane is $\mathbf{R}^2$. We first compute all the periodic orbits of period up to 28 for the Hénon chaotic attractor at $a = 1.4$ and $b = 0.3$ in the synchronization plane of Eq. (2). The Lyapunov spectra in each transverse plane for all the Hénon unstable periodic orbits up to $p = 28$ are then computed for $0 \leq \epsilon \leq 1.6$. Since $N = 5$, each periodic orbit has five degenerate unstable directions with equal eigenvalues when $\epsilon = 0$. The matrix $\mathbf{g}$ has the following set of eigenvalues for $N = 5$: $\gamma_0 = 0$, $\gamma_1 = \gamma_2 = -1.382$, and $\gamma_3 = \gamma_4 = -3.618$. Thus, as ϵ is increased from zero, periodic orbits begin to lose unstable directions in pairs. That is, for ϵ fixed but positive, the Hénon periodic orbits can have five, three, or one unstable directions, corresponding to four, two, or zero transversely unstable directions. Figures 1(a) and 1(b) show the histograms of the two largest transverse Lyapunov exponents ($\lambda_T^1 = \lambda_T^2$) for all periodic orbits of period 28 (there are 16 031 distinct ones) at $\epsilon = 0.4$ and $\epsilon = 0.8$, respectively. It can be seen that for $\epsilon = 0.4$, almost all orbits of period 28 have at least two transversely unstable directions; while for $\epsilon = 0.8$, a small fraction of these orbits are transversely stable, i.e., they have no transversely unstable directions, and, hence, these orbits have only one unstable direction, the one in the invariant manifold. Periodic orbits can also have two or four transversely unstable directions. This is shown in Figs. 1(c) and 1(d), where the histograms of the third and the fourth largest transverse Lyapunov exponents ($\lambda_T^3 = \lambda_T^4$) of all period-28 orbits are shown for $\epsilon = 0.4$ and $\epsilon = 0.8$, respectively. We see that for $\epsilon = 0.4$, a small but finite fraction of orbits have negative values of λ_T^3 and λ_T^4, indicating that these orbits can have at most two transversely unstable directions. For $\epsilon = 0.8$, a larger fraction of the period-28 orbits have this behavior. These results thus clearly indicate unstable dimension variability in model Eq. (2) for $\epsilon \neq 0$.

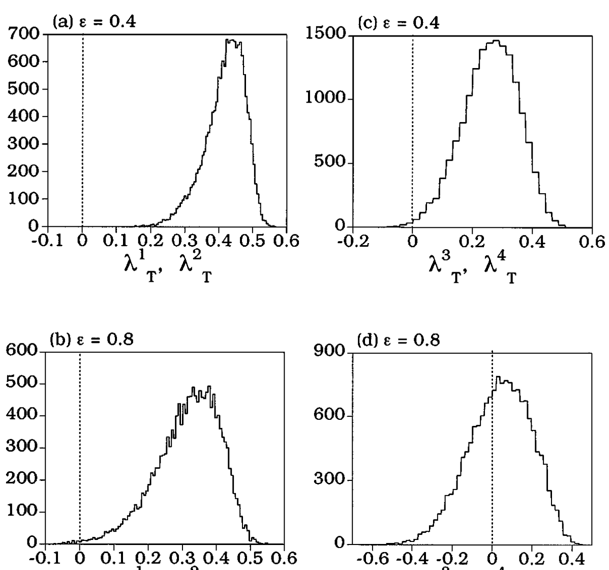

FIG. 1. For $N = 5$, $a = 1.4$, and $b = 0.3$ in Eq. (2). (a) and (b): Histograms of λ_T^1 and λ_T^2 ($\lambda_T^1 = \lambda_T^2$) for all periodic orbits of period 28 at $\epsilon = 0.4$ and $\epsilon = 0.8$, respectively; (c) and (d): histograms of λ_T^3 and λ_T^4 ($\lambda_T^3 = \lambda_T^4$) for all period-28 orbits for $\epsilon = 0.4$ and $\epsilon = 0.8$, respectively.

How large should the coupling parameter be for unstable dimension variability to occur? To address this question, we compute, for a given period p, $\epsilon_{\min}$, the minimum value of the coupling for which unstable dimension variability occurs for *all periodic orbits of period less than or equal to* p, as shown in Fig. 2(a) for $p \leq 28$. As p increases, $\epsilon_{\min}$ is a nonincreasing function of p. This implies that $\epsilon_{\min} \rightarrow \epsilon_c \gtrsim 0$ as $p \rightarrow \infty$, where ϵ_c is a small constant. Thus, unstable dimension variability may occur at small coupling strength. To understand to what extent one encounters unstable dimension variability for periodic orbits of a given (large) period, we compute the fractions of all period-28 orbits which have four, two, and zero transversely unstable directions as functions of ϵ. The results are plotted in Fig. 2(b) for $0 \leq \epsilon \leq 1.6$. The fraction of orbits with four unstable directions decreases linearly as ϵ is increased from zero, as shown in the inset of Fig. 2(b) for $0 \leq \epsilon \leq 0.5$. The linear behavior for $\epsilon \gtrsim 0$ can be understood from the histograms shown in Figs. 1(a)–1(d). For small ϵ, almost all period-28 orbits have at least two transversely unstable directions [Fig. 1(a)] and, hence, the fraction of orbits with four transversely unstable directions is proportional to the area of the histograms of λ_T^3 and λ_T^4 on the positive side. This area decreases approximately linearly both as the mean of the histogram is translated towards the negative direction when ϵ is increased and as the mean of the histogram is far from zero. However, for larger ϵ, as the means of each histogram of λ_T^3 and λ_T^4 gets close to zero, the fraction of orbits with four transversely unstable directions decreases sharply.

In summary, we presented theoretical justification and computational evidence for the occurrence of unstable dimension variability in systems of coupled chaotic oscillators. We gave support to our conjecture that unstable dimension variability and, consequently, severe modeling difficulties may arise as the coupling parameter is increased from zero. We expect these results to be quite general since the number of unstable directions of any unstable periodic orbits is determined by the local chaotic dynamics

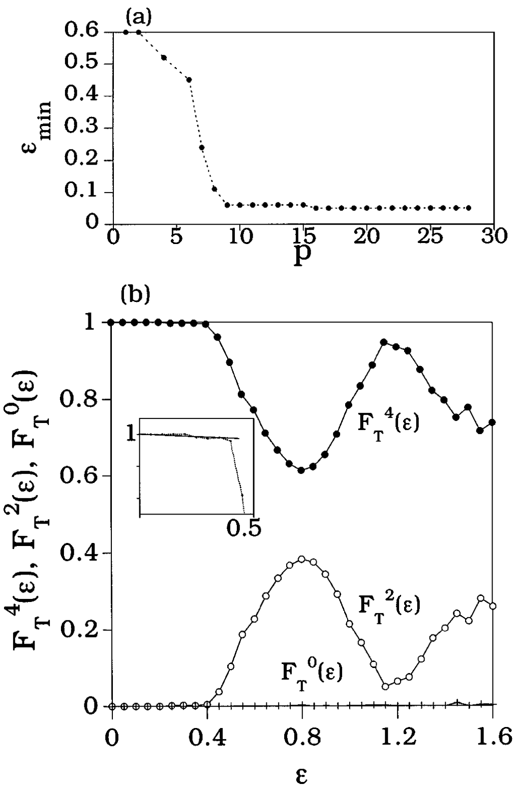

FIG. 2. (a) For $N = 5$, $a = 1.4$, and $b = 0.3$ in Eq. (2), $\epsilon_{\min}$ versus the period p. (b) Fractions of all period-28 orbits with four, two, and zero transversely unstable directions versus ϵ for $0 < \epsilon < 1.6$. Blowup in the range $0 < \epsilon \leq 0.5$ is shown in the inset.

and the coupling strength, regardless of the specific coupling scheme [Eq. (3)]. These conclusions also have deep consequences for the integration of partial differential equations using discretization, a procedure that yields a system of coupled ordinary differential equations and is used so commonly in physics and engineering. It means that even if the laws of physics are exact in terms of partial differential equations (e.g., Navier-Stokes equation), severe modeling difficulties arise as soon as one discretizes the equation.

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[1] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994), and references therein.

[2] See, for example, P. Hadley, M. R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988); S. Watanabe and S. H. Strogatz, Physica (Amsterdam) **74D**, 197 (1994).

[3] R. Roy and K. S. Thornburg, Jr., Phys. Rev. Lett. **72**, 2009 (1994).

[4] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. Lett. **73**, 3528 (1994); J. F. Heagy, L. M. Pecora, and T. L. Carroll, Phys. Rev. Lett. **74**, 4185 (1995).

[5] Typically, in low-dimensional chaotic systems, one expects τ, the length of valid trajectories from the model, to scale with the model uncertainty ϵ as $\tau \sim \epsilon^{-1/2}$. Here τ can be taken as the reasonable length of any valid model trajectory [C. Grebogi, S. M. Hammel, and J. A. Yorke, Bull. Am. Math. Soc. **19**, 465 (1988); C. Grebogi, S. M. Hammel, J. A. Yorke, and T. Sauer, Phys. Rev. Lett. **65**, 1527 (1990)].

[6] A necessary requirement for a model is robustness under small perturbations. For chaotic systems, the outcome of the system is sensitively dependent on the initial conditions. In view of this, we consider a model to be robust if the sets of all possible outcomes of the two slightly different versions of the model, say *A* and *B*, are very similar. Successful modeling would require that the set of all possible outcomes from model *A* agrees closely with the set of all possible outcomes from model *B*. Difficulties appear when there are trajectories of *A* that do not closely follow *any* trajectory of *B* (or vice versa) for all but short periods of time, because, if trajectories from the closely related models do not agree, either model is presumably useless in representing the *physical system.* The hierarchy of difficulty levels (L. Poon, C. Grebogi, T. Sauer, and J. A. Yorke, report) that can obstruct modeling shadowability and hence impede the ability to model certain physical processes are as follows: (i) mild: simple chaos or sensitive dependence on initial conditions, (ii) moderate: nonhyperbolicity due to quadratic tangencies of stable and unstable manifolds, and (iii) severe: unstable dimension variability, as in the system of coupled chaotic oscillators discussed in this Letter.

[7] Y.-C. Lai, C. Grebogi, and J. Kurths, Phys. Rev. E **59**, 2907 (1999).

[8] R. Abraham and S. Smale, Proc. Symp. Pure Math. **14**, 5 (1970).

[9] S. P. Dawson, C. Grebogi, T. Sauer, and J. A. Yorke, Phys. Rev. Lett. **73**, 1927 (1994); S. P. Dawson, Phys. Rev. Lett. **76**, 4348 (1996); E. J. Kostelich, I. Kan, C. Grebogi, E. Ott, and J. A. Yorke, Physica (Amsterdam) **109D**, 81 (1997).

[10] D. V. Anosov, Proc. Steklov Inst. Math. **90**, 1 (1967); R. Bowen, J. Differ. Eq. **18**, 333 (1975).

[11] T. Sauer, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **79**, 59 (1997).

[12] Unstable dimension variability is reflected and quantified by finite-time Lyapunov exponents fluctuating about zero [9,11]. It is possible to show [11] that model trajectories remain close to trajectories of the natural process only up to time $\tau \sim \delta^{-2m/\sigma^2}$, where δ is the modeling error, m and σ^2 are the mean and variance, respectively, of the time-one Lyapunov exponent closest to zero. Thus, the model solutions are not expected to reflect trajectories of the natural system for times longer than τ.

[13] M. Hénon, Commun. Math. Phys. **50**, 69 (1976).

[14] O. Biham and W. Wenzel, Phys. Rev. Lett. **63**, 819 (1989).

[15] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).

[16] C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. A **37**, 1711 (1988).