Phase Characterization of Chaos

Tolga Yalçınkaya* and Ying-Cheng Lai†
Department of Physics and Astronomy, The University of Kansas, Lawrence, Kansas 66045
(Received 29 May 1997)

The phase of a chaotic trajectory in autonomous flows is often ignored because of the wide use of the extremely popular Poincaré surface-of-section technique in the study of chaotic systems. We present evidence that, in general, a chaotic flow is practically composed of a small number of intrinsic modes of proper rotations from which the phase can be computed via the Hilbert transform. The fluctuations of the phase about that of a uniform rotation can be described by fractional Brownian random processes. Implications to nonlinear digital communications are pointed out. [S0031-9007(97)04534-1]

PACS numbers: 05.45.+b, 05.40.+j, 89.70.+c

Many physical, chemical, and biological processes in nature are described by a set of coupled first-order autonomous differential equations, or autonomous flows. A widely used technique in the study of these systems is the Poincaré surface-of-section technique [1]. On a Poincaré surface of section, the dynamics can be described by a discrete map whose phase-space dimension is one less than that of the original continuous flow. This sectioning technique thus provides a natural link between continuous flows and discrete maps. With a tremendous facilitation in analysis, numerical computation, and visualization, maps also capture many fundamental dynamical properties of flows. These advantages have made the Pioncaré surface-of-section technique one of the most popular analysis tools in nonlinear dynamics and chaos.

Despite its usefulness, the Pioncaré surface-of-section technique has a fundamental drawback: The discrete map produced by it contains no information about the *phase or timing* of the underlying flow. Consider, for instance, the case of a chaotic attractor. Trajectories on the attractor have the property of recurrence. That is, a chaotic flow starting from a point in the phase space must return to some arbitrarily small neighborhood of the starting point infinitely often but never exactly repeat the initial state. The trajectory can thus be considered as going through an infinite number of rotationlike motions. The orbit of the rotation, of course, never closes on itself due to the nature of chaos (otherwise, the asymptotic motion would be a limit cycle). A natural physical variable to describe rotation is the phase, i.e., some generalized angle associated with the rotation. Being an important physical quantity, the phase associated with a chaotic flow can bear fundamental information about the dynamics of the system. But unfortunately, the phase dynamics is totally disregarded when a Poincaré surface of section is employed to study the system.

The phase dynamics of some particular chaotic flows for which the phase can be naturally defined has recently been studied by Rosenblum *et al.* in the context of synchronization [2]. To our knowledge, the phase characterization of *general* chaotic flows remains an open question. The purpose of this Letter is to address this question. But before we present our main results, we wish to point out that the phase dynamics of chaotic flows is fundamental to at least one important technological application of recent interest: *multichannel nonlinear digital communications.* The topic deals with digital communication using encoded chaotic signals. Specifically, it has been demonstrated that chaotic systems can be manipulated, via arbitrarily small time-dependent perturbations, to generate controlled chaotic orbits whose symbolic representation corresponds to the digital representation of a desirable message [3,4]. Imagine a chaotic oscillator that generates an apparently random signal. To encode a message into the chaotic signal, one often utilizes the discrete map obtained on some Poincaré surface of section to define a good symbolic dynamics. In practical applications, multichannel communications are desirable. In order to transmit information through multiple channels, one must integrate a large number of chaotic oscillators into a single communication system. A potential difficulty is that the time intervals for reading off digital information bits (symbols) from a chaotic oscillator are random. That is, when one examines the Poincaré surface of section to determine the symbolic dynamics, one typically observes that the time periods for successive piercing of the trajectory through the section are randomly distributed. An integrated multichannel communication system demands that all the chaotic oscillators be paced. That is, the time intervals for generating symbols in the symbolic dynamics need to be identical for all oscillators. It is, thus, essential to pace the chaotic oscillators so that a trajectory passes through the Poincaré surface of section in equal time intervals. The solution to this chaos-pacing problem relies on a good knowledge about the phase of the chaotic flow.

In this Letter, we investigate the phase dynamics of continuous chaotic flows. A fundamental difficulty in defining a proper phase variable [denoted by $\phi(t)$ herein] for the flow is the requirement that there be a definite direction (e.g., either clockwise or counterclockwise) and a unique center of rotation. That is, the instantaneous angular velocity (or frequency) $d\phi(t)/dt$ of the rotation must not be negative, as shown schematically in Fig. 1(a). We call the rotation in Fig. 1(a) a *proper rotation.* A

0031-9007/97/79(20)/3885(4)$10.00

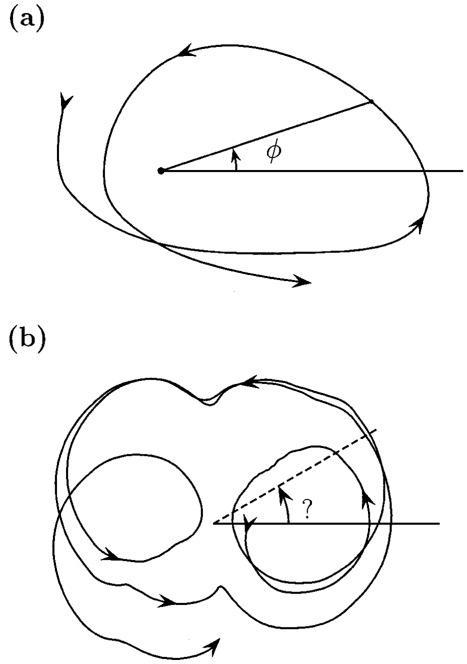

FIG. 1. Schematic illustration of (a) a proper rotation on which a phase can be defined, and (b) a rotation with multiple centers for which a phase cannot be defined.

chaotic flow, however, typically exhibits more than one center of rotation, as shown schematically in Fig. 1(b). To overcome this difficulty, we utilize a scheme to decompose a chaotic signal into a number of intrinsic modes, each corresponding to a proper rotation. By the nature of chaos, an orbit on each rotation never closes on itself. Nonetheless, the phase variable associated with the rotation can now be properly defined. Our main result is that a typical chaotic flow is supported practically by a *small number* of intrinsic modes of proper rotation. By utilizing the Hilbert transform to generate an analytical signal [5] corresponding to each rotation, we obtain the phase $\phi(t)$ for that rotation. Furthermore, we find that the fluctuations of the phase of each mode from that of a uniform rotation are a fractional Brownian-type of random process. We obtain universal scaling laws governing the phase fluctuations. Based on our results, we conjecture that the phase organization of chaotic flows by proper rotations and the fractional Brownian-type of random phase dynamics are general.

We first review the procedure [2,5] to compute the phase of a chaotic flow, *if the signal corresponds to a proper rotation as in Fig. 1(a).* Consider a chaotic system described by an N-dimensional flow $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$, where $\mathbf{x} \in \mathbf{R}^N$, and assume that the motion of trajectories occurs on a chaotic attractor. Arbitrarily choosing one state variable, say $x(t)$, we obtain a chaotic time series (or a chaotic signal). To compute the phase function $\phi(t)$ corresponding to $x(t)$, we first perform the Hilbert transform of $x(t)$ to obtain

$$\tilde{x}(t) = \text{P.V.}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(t')}{t-t'}\,dt'\right], \tag{1}$$

where P.V. stands for the Cauchy principal value for integral. The following analytic signal is then constructed to yield an amplitude function $A(t)$ and a phase function $\phi(t)$,

$$\psi(t) = x(t) + i\tilde{x}(t) = A(t)e^{i\phi(t)}. \tag{2}$$

If one plots $\tilde{x}(t)$ against $x(t)$ in the complex plane of $\psi(t)$, or equivalently, in the polar coordinate $[A(t), \phi(t)]$, one obtains a proper rotation.

The major difficulty for a general chaotic flow is that the trajectory in the complex plane of $\psi(t)$ may have multiple centers of rotation. Take, for an example, the chaotic attractor of the Lorenz system [6]: $dx/dt = -10(x-y)$, $dy/dt = -xz + 28x - y$, and $dz/dt = xy - (8/3)z$. Figure 2(a) shows the trajectory in the complex plane of $\psi(t)$, which is the analytic signal corresponding to $y(t)$. Clearly, no proper phase function $\phi(t)$ can be defined. It is, thus, necessary to preprocess the chaotic signal to decompose it into proper rotations. Our procedure is as follows [7]. Notice that for a proper rotation in the complex plane of the analytic signal, the real part (the original chaotic signal) usually possesses the property that the number of extrema is equal to the number of zero-crossing points, if the center of the rotation is taken to be the origin of the coordinate system. Thus, given a

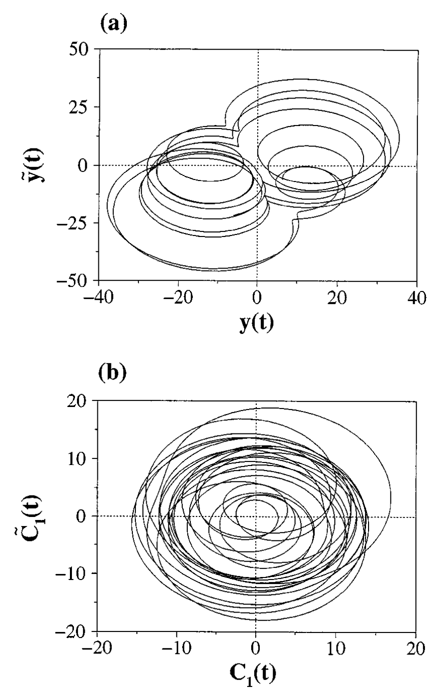

FIG. 2. (a) A trajectory in the complex plane of the analytic signal from the y component of the Lorenz equation. The motion exhibits multiple centers of rotation so that a proper phase cannot be defined. (b) A similar trajectory from the first intrinsic mode $C_1(t)$ of $y(t)$ of the Lorenz system. In this case, the phase $\phi_1(t)$ can be properly defined.

chaotic signal $x(t)$, we can extract signals that have this property. To achieve this, we adopt the empirical-mode-decomposition method by Huang *et al.* [7]: (i) Construct two smooth splines connecting all the maxima and minima, respectively, to get $x_{\max}(t)$ and $x_{\min}(t)$; (ii) compute $\Delta x(t) \equiv x(t) - [x_{\max}(t) + x_{\min}(t)]/2$; and (iii) repeat steps (i) and (ii) for $\Delta x(t)$ until the resulting signal corresponds to a proper rotation. Denote the resulting signal by $C_1(t)$, which is the first intrinsic mode. We then take the difference $x_1(t) \equiv x(t) - C_1(t)$ and repeat steps (i)–(iii) to obtain the second intrinsic mode $C_2(t)$. The procedure continues until the mode $C_M(t)$ shows no apparent variation [8]. Summarizing these steps, we have decomposed the original chaotic signal $x(t)$ in the following manner: $x(t) = \sum_{j=1}^{M} C_j(t)$, where the functions $C_j(t)$'s are nearly orthogonal to each other [7]. By construction, each mode $C_j(t)$ generates a proper rotation in the complex plane of its own analytic signal $\psi_j(t) = A_j(t)e^{i\phi_j(t)}$, and the average rotation frequencies $\omega_j \equiv \langle d\phi_j(t)/dt \rangle$ obey the following order $\omega_1 \geq \omega_2 \cdots \geq \omega_M$ because the sifting procedure picks the component with the fastest variation embedded in the original signal first and that with the slowest variation last [7]. Figure 2(b) shows the trajectory in the complex plane of the analytic signal of the first component $C_1(t)$ from the Lorenz system. Clearly, now there is a unique rotation center for which the phase of $C_1(t)$ can be properly defined.

In general, the number of fundamental modes M required to capture the rotationlike motions in the signal is *small*. Figure 3 shows the phase functions $\phi_j(t)$ corresponding to different modes for the Lorenz flow. We see that there is a clear separation between the average rotation frequencies (the average slopes) of the various modes, with ω_1 being the largest. We also see that $\omega_j \approx 0$ for $j \geq 7$, indicating that the eighth mode and up are insignificant. In principle, an infinitely long chaotic signal can be decomposed into an infinite number of proper rotations. However, we find that the amplitudes and the average rotation frequencies decrease rapidly as higher-order modes are examined. Thus, a few proper rotations are sufficient to represent the phase of the chaotic signal, but we have no rigorous assurance of this. We stress that while the sifting procedure is necessary for chaotic flows that exhibit multiple centers of rotation in the complex plane of its analytic signal, there are systems in which the flow apparently already has a unique center of rotation. In this case, the sifting procedure is not necessary and one can define the phase associated with the flow directly from the analytic signal. Flows on the Rössler attractor appear to belong to this category [2].

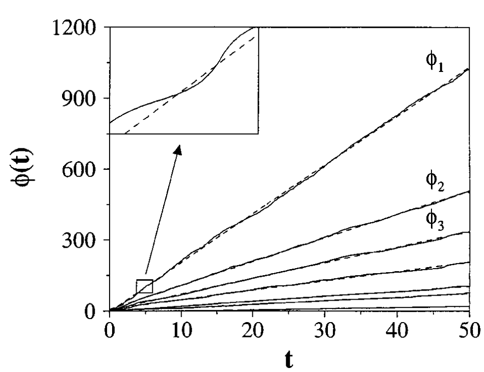

FIG. 3. $\phi_j(t)$ versus t for $j = 1, \ldots, 8$. Note that the phase variation in $\phi_8(t)$ is already approximately zero, indicating that further components are insignificant. The frequencies are $\omega_1 \approx 20.68$, $\omega_2 \approx 10.01$, $\omega_3 \approx 6.61$, $\omega_4 \approx 3.95$, $\omega_5 \approx 2.068$, $\omega_6 \approx 1.48$, and $\omega_7 \approx 0.46$. Roughly, we have $\omega_2 \approx \omega_1/2$, $\omega_3 \approx \omega_1/3$, $\omega_4 \approx \omega_1/5$, $\omega_5 \approx \omega_1/10$, $\omega_6 \approx \omega_1/14$, and $\omega_7 \approx \omega_1/45$.

We now ask, what are the characteristics of the proper rotations that constitute a chaotic flow? Since the flow is chaotic, we expect to see fluctuations of the phases $\phi_j(t)$ from those of the uniform rotations $\omega_j t$ $(j = 1, \ldots, M)$. In general, we can write $\phi_j(t) = \omega_j t + F_j[A_j(t)]$, $j = 1, \ldots, M$, where $A_j(t)$ is the amplitude of the analytic signal associated with the jth mode $C_j(t)$, and $F_j(t)$ is a smooth function. By the nature of chaos, $A_j(t)$'s are essentially random variables if one examines them in time scales $\gtrsim 1/\omega_j$, respectively. Thus, compared with a uniform rotation on a circle, a chaotic rotation $\phi_j(t)$ not only has a random instantaneous radius but it also rotates faster or slower than the uniform rotation in a random fashion. We concentrate on the fluctuations of the chaotic phase function $\phi_j(t)$ about the phase $\omega_j t$ of the uniform rotation, i.e., $\Delta\phi_j(t) \equiv \phi_j(t) - \omega_j t = F_j[A_j(t)]$. Figure 4(a) shows, for the Lorenz signal in Fig. 2, $\Delta\phi_1(t)$ versus t. The plot resembles that of a fractional Brownian motion [9]. To confirm this, we perform a scaling analysis. Specifically, we examine the probability distribution of the first-return time τ of the random process $\Delta\phi_j(t)$. It was shown in Ref. [10] that for a fractional Brownian motion, the first-return-time distribution obeys the following scaling laws:

$$P(\tau) \sim \tau^{H-2}, \tag{3}$$

where H is the Hurst exponent [11]. Figure 4(b) shows $\log_{10} P(\tau)$ versus $\log_{10} \tau$ for $\Delta\phi_1(t)$ in Fig. 4(a). To obtain this figure, 30 000 random initial conditions were chosen, each yielding a time series for $0 \leq t \leq 600$ with integration step $h = 0.01$ (after regarding a sufficiently long chaotic transient). For each time series, $\Delta\phi_j(t)$'s $(j = 1, \ldots, 8)$ were computed using a combination of the sifting procedure and the Hilbert transform. The plot in Fig. 4(b) can be fitted by a straight line of slope -1.26 ± 0.03, indicating a robust algebraic scaling law Eq. (3) with a Hurst exponent of approximately 0.74. Computation using the mean-square displacement of $\Delta\phi_j(t)$ yields essentially the identical exponent [11]. While this exponent is obtained by using the time series $y(t)$ from the Lorenz equation, computation using $x(t)$ and $z(t)$ of the Lorenz equation yields similar scaling laws. These results, thus,

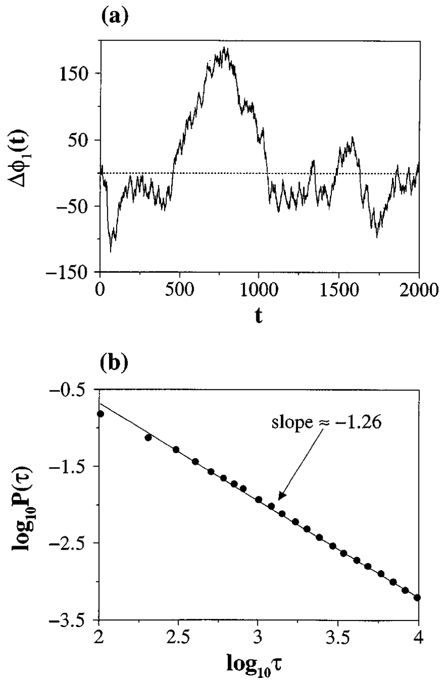

FIG. 4. (a) Plot of $\Delta\phi_1(t)$, fluctuations of the phase about that of a uniform rotation of the analytic signal of $C_1(t)$. (b) Probability distribution of the first-return time of $\Delta\phi_1(t)$ (on a logarithmic scale). The Hurst exponent is $H \approx 0.74$.

strongly suggest that *the fluctuations of the chaotic phases of the intrinsic rotations about these of uniform rotations are fractional Brownian random processes.*

In summary, we have discovered an interesting organization of chaos in autonomous flows. That is, a chaotic signal can be practically decomposed into a small number of intrinsic modes of proper rotation. Each mode has a unique center of rotation so that the phase associated with the rotation can be properly defined. We find that the fluctuations of the phase in each mode about that of a uniform rotation are a fractional Brownian-type of random process. These results appear quite general because they hold for all cases of chaotic systems which we have examined: the Lorenz system, the Rössler system [12], and an experimentally measured signal from a chaotic laser. Thus, although chaos is always associated with a broad-band Fourier spectrum and is commonly considered as complicated, the phase of chaos is actually quite simple: It consists of *only a small number* of nontrivial proper rotations. Besides its apparent application to important technological problems such as nonlinear digital communication [3], we believe that the study of the phase of chaotic flows can provide us with new insights into the fundamental organization of chaos.

We thank Dr. N. Huang for sending us Ref. [7] before its publication, and Professor A. Pikovsky for valuable discussions. This work was supported by AFOSR under Grant No. F49620-96-1-0066, by NSF under Grant No. PHY-9722156, and by the University of Kansas.

*Electronic address: tolga@math.ukans.edu
†Also Department of Mathematics, University of Kansas, Lawrence, Kansas 66045
Electronic address: lai@poincare.math.ukans.edu

[1] See, for example, K.T. Alligood, T.D. Sauer, and J.A. Yorke, *Chaos: An Introduction to Dynamical Systems* (Springer, New York, 1996).
[2] M.G. Rosenblum, A.S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996); **78**, 4193 (1997); A.S. Pikovsky, M.G. Rosenblum, G. Osipov, and J. Kurths, Physica (Amsterdam) **104D**, 219 (1997); A.S. Pikovsky, G. Osipov, M.G. Rosenblum, and J. Kurths, Phys. Rev. Lett. **79**, 47 (1997).
[3] S. Hayes, C. Grebogi, and E. Ott, Phys. Rev. Lett. **70**, 3031 (1993); S. Hayes, C. Grebogi, E. Ott, and A. Mark, Phys. Rev. Lett. **73**, 1781 (1994); E. Bollt and M. Dolnik, Phys. Rev. E **55**, 4 (1997); E. Rosa, S. Hayes, and C. Grebogi, Phys. Rev. Lett. **78**, 1247 (1997).
[4] An advantage of this type of communication strategy is that the nonlinear chaotic oscillator that generates the wave form for transmission can remain simple and efficient while all the necessary electronics controlling encoding of the signal remain at a low-powered microelectronic level [3].
[5] D. Gabor, J. IEE London **93**, 429 (1946); P. Panter, *Modulation, Noise, and Spectral Analysis* (McGraw-Hill, New York, 1965).
[6] E.N. Lorenz, J. Atmos. Sci. **20**, 130 (1963). For the Lorenz system, an alternative phase can be defined via a particular change of variables [2].
[7] N. Huang *et al.*, Proc. R. Soc. London A (to be published).
[8] It should be noted that the sifting procedure generates modes $C_M(t)$ with the degree of time variations in a decreasing order. That is, $C_1(t)$ has the fastest time variations, $C_2(t)$ contains the second fastest time variations, etc. The procedure involves only simple subtraction of successive averages from the original time series and, hence, the modes $C_M(t)$ are naturally defined. Given a time series, the sifting procedure continues until the mode generated shows no apparent time variations, that is, the mode is nearly a constant in time. Our numerical criterion is that the last mode contains fewer than two local extrema.
[9] B.B. Mandelbrot and J.R. Wallis, Water Res. **4**, 909 (1968); B.B. Mandelbrot and J.W. Van Ness, SIAM Rev. **10**, 422 (1968); B.B. Mandelbrot, *Fractal Geometry of Nature* (Freeman, New York, 1983).
[10] M. Ding and W. Yang, Phys. Rev. E **52**, 207 (1995).
[11] Given a random process $X(t)$, if the correlation function $C_X(t)$ decays sufficiently fast, then the mean-square displacement $\langle X^2(t)\rangle$ scales with time t as $\langle X^2(t)\rangle \sim t$ (t large). If, however, $\int_0^t C_X(t')\,dt'$ converges to a finite value but $\int_0^t t' C_X(t')\,dt'$ diverges as $t \to \infty$, or if the decay rate of $C_X(t)$ is slower than that required for $\int_0^t C_x(t')\,dt'$ to remain finite, then the scaling of $\langle X^2(t)\rangle$ generalizes to $\langle X^2(t)\rangle \sim t^{2H}$, where H is the Hurst exponent which must satisfy $0 < H < 1$ [9]. For ordinary Brownian motion, $H = 1/2$. Deviation of the value of the Hurst exponent from $1/2$ signifies a fractional Brownian motion.
[12] O.E. Rössler, Phys. Rev. Lett. **71A**, 155 (1979).