Characterization of the Natural Measure by Unstable Periodic Orbits in Chaotic Attractors

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(Received 23 April 1997)

The natural measure of a chaotic set in a phase-space region can be related to the dynamical properties of the unstable periodic orbits embedded in that set. This result has been proven to be valid for hyperbolic chaotic systems. We test the goodness of such a periodic-orbit characterization of the natural measure for nonhyperbolic chaotic systems by comparing the natural measure of a typical chaotic trajectory with that computed from unstable periodic orbits. Our results suggest that the unstable periodic-orbit formulation of the natural measure is typically valid for nonhyperbolic chaotic systems. [S0031-9007(97)03650-8]

PACS numbers: 05.45.+b

In studying chaotic systems, one is often interested in long term statistics such as averages, Lyapunov exponents, dimensions, and other invariants of the probability density or the measure. Both these statistical quantities are physically meaningful only when the measure being considered is the one generated by a typical trajectory in phase space. This measure is called the natural measure [1] and it is invariant under the evolution of the dynamics. Therefore, it is of paramount physical importance to be able to understand and to be able to characterize the natural measure [2] in terms of fundamental dynamical quantities. And there is nothing more fundamental than to express the natural measure in terms of the periodic orbits embedded in a chaotic attractor.

A key contribution along these lines was made in Ref. [3] in which the authors obtained an expression for the invariant natural measure in terms of the magnitude of the eigenvalues of the unstable periodic orbits embedded in the chaotic attractor. They proved [3] the correctness of their expression but only for the special case of a hyperbolic dynamics [4]. The validity of their results for physical, which are typically nonhyperbolic, situations remained, however, only a conjecture. The purpose of this Letter is to provide strong evidence for the applicability of the results of Grebogi *et al.* [3] to nonhyperbolic chaotic systems and, hence, validating their conjecture.

The long-time probability density or the natural measure generated by typical trajectories of chaotic dynamical systems is generally highly singular. A trajectory originated from a random initial condition in the basin of attraction of a chaotic attractor visits different parts of the attractor with drastically different probabilities. Call regions with high probabilities "hot" spots and regions with low probabilities "cold" spots. Such hot and cold spots in the attractor can be interwoven on arbitrarily fine scales. In this sense, chaotic attractors are said to possess a multifractal structure, which is a property of the natural measure. To obtain the natural measure, one covers the chaotic attractor with a grid of cubes and examines the frequency with which a typical trajectory visits these cubes in the limit that both the length of the trajectory goes to infinity and the size of the grid goes to zero [5]. Except for an initial condition set of Lebesgue measure zero in the basin of attraction of the chaotic attractor, these frequencies in the cubes are the same for different choices of the initial condition $\mathbf{x}_0$, and they are called the natural measure. Specifically, let $f(\mathbf{x}_0, T, \epsilon_i)$ be the amount of time that a trajectory from a random initial condition $\mathbf{x}_0$ in the basin of attraction spends in the ith covering cube C_i of edge length ϵ_i in a time T. The natural measure of the attractor in the cube C_i is

$$\mu_i = \lim_{T\to\infty} \frac{f(\mathbf{x}_0, T, \epsilon_i)}{T}. \tag{1}$$

The spectrum of an infinite number of fractal dimensions quantifies the singular behavior of the natural measure [6].

Hyperbolic chaotic attractors have embedded within themselves an infinite number of unstable periodic orbits. The same thing we expect to occur for nonhyperbolic chaotic attractors arising from physical processes [7]. The unstable periodic orbits are *atypical* in the sense that they form a Lebesgue measure zero set. Invariant measures produced by unstable periodic orbits are thus atypical, and there are an infinite number of such atypical invariant measures embedded in a chaotic attractor. The hot and cold spots are a reflection of these atypical measures. The natural measure, on the other hand, is typical in the sense that it is generated by a trajectory originated from any one of the randomly chosen initial conditions in the basin of attraction. A typical trajectory visits a fixed neighborhood of any one of the different periodic orbits from time to time. Thus, chaos can be considered as being organized with respect to the unstable periodic orbits since these orbits support the natural measure [8]. An interesting question is then how the natural measure can be quantified in terms of the infinite number of atypical invariant measures embedded in the attractor. This question was systematically addressed in Ref. [3], in which a formula was obtained relating the natural measure

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of a hyperbolic chaotic set in phase space to the expanding eigenvalues of all the periodic orbits enclosed in the set. Specifically, consider a d-dimensional map $\mathbf{M}(\mathbf{x})$. Let $\mathbf{x}_{ip}$ be the ith fixed point of the p-times iterated map, i.e., $\mathbf{M}^p(\mathbf{x}_{ip}) = \mathbf{x}_{ip}$. Thus each $\mathbf{x}_{ip}$ is on a periodic orbit whose period is either p or a factor of p. The natural measure of a chaotic attractor in C_i is given by $\mu_i = \lim_{p\to\infty} \mu_i(p)$, where

$$\mu_i(p) = \sum_{\mathbf{x}_{ip}\in C_i} \frac{1}{L_1(\mathbf{x}_{ip})}. \qquad (2)$$

In Eq. (2), $L_1(\mathbf{x}_{ip})$ is the magnitude of the expanding eigenvalue of the Jacobian matrix $\mathbf{DM}^p(\mathbf{x}_{ip})$, and the summation is taken over all fixed points of $\mathbf{M}^p(\mathbf{x})$ in C_i. Equation (2) is theoretically significant because it provides a fundamental link between the natural measure and various atypical invariant measures embedded in a chaotic attractor.

To provide strong evidence for the applicability of Eq. (2) to nonhyperbolic chaotic systems, we consider two-dimensional invertible maps, which in principle can be obtained from a system of differential equations through a surface of section. We cover the chaotic attractor with a fine grid of boxes and compute the natural measure μ_i in each nonempty box C_i according to Eq. (1). We then compute $\mu_i(p)$ from *all* the fixed points of the p-times iterated map contained in each box C_i according to Eq. (2). Let $\Delta\mu(p) \equiv \sqrt{\sum_{i=1}^{N}[\mu_i(p) - \mu_i]^2/N}$, where N is the number of boxes C_i with nonzero natural measure. We find that $\Delta\mu(p)$ decreases exponentially as the period p increases, thereby validating Eq. (2).

To obtain Eq. (2) [3], we first cover the chaotic attractor with a grid of partitioning boxes, each being confined by segments of the stable and unstable manifolds. If the boxes are small compared with the size of the phase-space region in which the chaotic set lies, each box can be regarded as being rectangular, as shown in Fig. 1(a), where the horizontal and vertical sides are segments of the stable and unstable manifolds, respectively. Now imagine that we choose a large number of initial conditions according to the natural measure. The natural measure contained in the box C_i is the fraction of trajectories that come back to C_i in the limit where the number of iterations $n \to \infty$. Let $\mathbf{x}_0$ be an initial condition in the box C_i in Fig. 1(a). Because of recurrence or ergodicity, the trajectory from $\mathbf{x}_0$ comes back to some point $\mathbf{x}_p$ in C_i, say, after p iterations, as shown in Fig. 1(a). Let ab be the horizontal line segment through $\mathbf{x}_0$ ending at the two unstable-manifold segments, and $c'd'$ be the vertical line segment through $\mathbf{x}_p$ ending at the two stable-manifold segments, as shown in Fig. 1(b). Since ab is parallel to the stable-manifold segments and, since $\mathbf{x}_0$ maps to $\mathbf{x}_p$ after p iterations, the image of ab under the p-times iterated map $\mathbf{M}^p(\mathbf{x})$ is a shorter horizontal line segment $a'b'$ straddling $\mathbf{x}_p$. Similarly, the pth preimage of $c'd'$ is a shorter vertical line segment straddling $\mathbf{x}_0$.

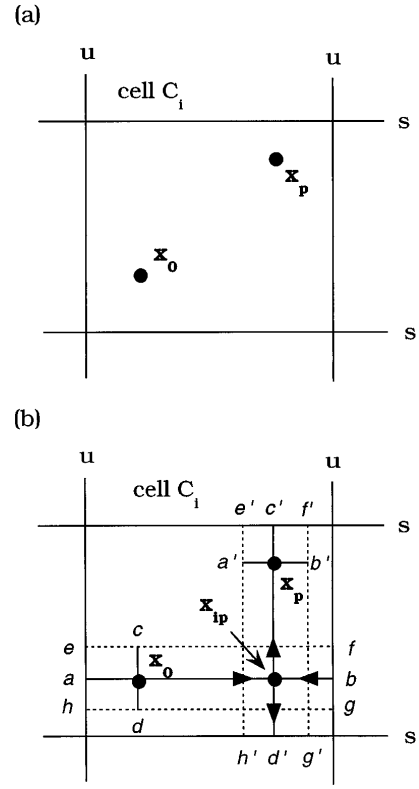

FIG. 1. (a) An initial condition $\mathbf{x}_0$ in the cell C_i and the point $\mathbf{x}_p$ that comes back to C_i after p iterations. (b) The rectangle $efgh$ maps to the rectangle $e'f'g'h'$ after p iterations. There must be then a fixed point $\mathbf{x}_{ip}$ of the p times iterated map in C_i.

Now construct two rectangles $efgh$ and $e'f'g'h'$ with side lengths (ab, cd) and $(a'b', c'd')$, respectively, as shown in Fig. 1(b). We see that the rectangle $efgh$ maps to the rectangle $e'f'g'h'$ under $\mathbf{M}^p(\mathbf{x})$. Consequently, there must be an unstable fixed point $\mathbf{x}_{ip}$ of $\mathbf{M}^p(\mathbf{x})$ in the box C_i. Assuming that $c'd'$ has a length ϵ, we have $\epsilon/L_1(\mathbf{x}_{ip})$ for the length of cd, where $L_1(\mathbf{x}_{ip})$ is the unstable (expanding) eigenvalue of the fixed point $\mathbf{x}_{ip}$. Since the natural measure is uniform along the unstable direction, we see that associated with the unstable fixed point $\mathbf{x}_{ip}$, the fraction of trajectories that come back to C_i in p iterations is $[\epsilon/L_1(\mathbf{x}_{ip})]/\epsilon = 1/L_1(\mathbf{x}_{ip})$. Taking into consideration all the unstable fixed points contained in C_i and taking the limit $p \to \infty$, we obtain Eq. (2).

The above argument applies to situations where a good partition of the phase space exists such that the shorter line segments $a'b'$ and cd in Fig. 1(b) are completely contained in the box C_i. For hyperbolic systems, such a partition exists, which is the Markov partition [9]. Therefore, Eq. (2) is rigorously valid for hyperbolic dynamical systems [3]. The argument becomes problematic for nonhyperbolic systems. A grid of boxes in which each box C_i looks like the box in Fig. 1 cannot be constructed because of the set of an infinite number of tangency points between the stable and unstable manifolds [4]. Therefore, the applicability of Eq. (2) to nonhyperbolic systems is only a conjecture.

We have undertaken a series of numerical experiments to test the goodness of the conjecture given by Eq. (2). We choose the Hénon map, $(x, y) \to (a - x^2 + by, x)$. It is one of the very few model systems for which there is a numerical algorithm to compute, in principle, all unstable periodic orbits of arbitrarily high periods [10]. We study $a = 1.4$ and $b = 0.3$, a parameter setting for which it apparently possesses a chaotic attractor. The attractor is also apparently nonhyperbolic because a rigorous computation of the stable and unstable manifolds [11] points towards the existence of an infinite number of tangency points of these manifolds on the attractor. The periodic orbits up to period 31 are computed using the procedure in Ref. [10]. To compute the natural measure of the attractor in different phase-space regions, we use a 128×128 grid to cover the region $-2 \leq (x, y) \leq 2$ and then use a trajectory of length 10^7 (after disregarding 10^4 initial iterations) from a randomly chosen initial condition. There are 909 nonempty boxes to which the trajectory visits. The quantity μ_i in Eq. (2) in each nonempty box C_i is approximately the fraction of time that the trajectory visits the box. Next, we compute, in each nonempty box, all the fixed points $\mathbf{x}_{ip}$ of the p-times iterated map and their associated expanding eigenvalues $L_1(\mathbf{x}_{ip})$ to obtain the quantity $\mu_i(p)$ in Eq. (2). Figure 2(a) shows $\ln \Delta\mu(p)$ versus p for $6 \leq p \leq 31$. We observe the following scaling relation:

$$\Delta\mu(p) \sim \exp(-\alpha p), \tag{3}$$

where $\alpha \approx 0.14$ is the scaling exponent. Thus, we see that the quantitative characterization of the natural measure of the chaotic attractor by unstable periodic orbits becomes exponentially accurate as the period p increases. Asymptotically, we have $\Delta\mu(p) \to 0$, indicating the applicability of Eq. (2) to nonhyperbolic chaotic sets. It is interesting to note that the somewhat large fluctuations in Fig. 2(a) are partly due to the fact that there are fewer periodic orbits of lower period p, since their number increases with p exponentially, where the exponential rate is the topological entropy. Figure 2(b) shows the total period-p natural measure $\mu_S(p) \equiv \sum_{C_i} \mu_i(p)$ versus p. It can be seen that $\mu_S(p)$ approaches unity rapidly as p increases. The dashed line in Fig. 2(a) is $\ln \Delta\mu(p)$ versus p but the quantity $\mu_i(p)$ is rescaled by $\mu_S(p)$. The rescaled plot has a similar slope as the unscaled one (the solid line), but the fluctuations are smaller. We find that Eq. (3) appears to hold regardless of the fineness of the grid used to cover the attractor. For instance, almost identical plots as in Fig. 2(a) are obtained when grids 64×64 and 256×256 are used. Thus, we expect Eq. (2) to be valid for any phase-space region containing part of the chaotic set in nonhyperbolic systems.

To understand the *exponential* scaling law Eq. (3), we utilize a simple one-dimensional analyzable model: the doubling transformation $x_{n+1} = 2x_n \bmod(1)$. All periodic orbits of period p of this map have the same eigenvalue 2^p. Divide the unit interval into N bins so that the size of each bin is $\epsilon = 1/N$. The natural measure contained in each bin is ϵ because it is uniform in the unit interval. There are $(2^p \pm 1)/N$ fixed points of the pth-fold map in each bin so that $\mu_i(p) = [(2^p \pm 1)/N]/2^p = \epsilon(1 \pm 2^{-p})$. Thus, we have $\Delta\mu(p) = |\mu_i(p) - \epsilon| \sim 2^{-p} = \exp(-p \ln 2)$. Notice that the scaling exponent for the doubling transformation is $\ln 2$, which is the topological entropy. This is due to the fact that the natural measure is uniform and all periodic-orbit points have the same eigenvalue. For more complicated nonhyperbolic systems, such as the one in our numerical example, the natural measure is highly nonuniform and the positive Lyapunov exponents of all the period-p orbits are not the same but obeys some probability distribution with width proportional to $\sqrt{p}$ [12]. Thus, the scaling exponent in Eq. (3) is less than the topological entropy. We have also checked the scaling Eq. (3) for another hyperbolic map, the Kaplan-Yorke map [13]

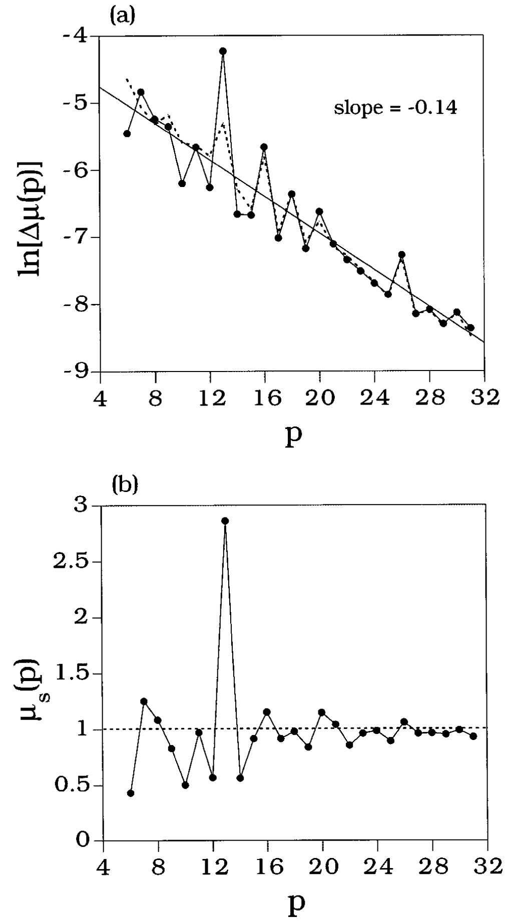

FIG. 2. For the Hénon chaotic attractor, (a) $\ln \Delta\mu(p)$ versus p. We have, approximately, $\Delta\mu(p) \sim e^{-0.14p}$. (b) The total period-$p$ natural measure $\mu_S(p)$ computed from Eq. (2) using all the period-p orbits. The total measure approaches to unity as p increases. The dashed line in (a) is $\ln \Delta\mu(p)$ versus p but the quantity $\mu_i(p)$ is rescaled $\mu_S(p)$.

and have found that the exponent is approximately the topological entropy. It is thus interesting to note that nonhyperbolicity makes the scaling exponent deviate from the topological entropy, but nonetheless the scaling law is still exponential.

In summary, we have presented evidence for the validity of the theory that relates the natural measure to unstable periodic orbits for nonhyperbolic chaotic sets. Our conclusion is that such a theory, while previously shown to be valid for hyperbolic systems [3], is apparently correct for nonhyperbolic chaotic systems. Unstable periodic orbits play a pivotal role in determining the dynamics on chaotic sets. These orbits are the fundamental building blocks of chaotic sets since they support the natural measure, apparently even for nonhyperbolic sets as indicated by our numerical investigations herewith. Dynamical invariants such as the Lyapunov exponents, topological entropy, and even the spectrum of fractal dimensions of a chaotic set, hyperbolic or not, can now then be determined based on the natural measures expressed in terms of the unstable periodic orbits embedded in the set. The periodic-orbit theory is conceptually appealing and is potentially useful for further theoretical or even practical developments [14].

This work was supported by AFOSR under Grant No. F49620-96-1-0066, by NSF under Grant No. DMS-962659, and by the University of Kansas. This work was also supported by the Department of Energy (Mathematical, Information and Computational Sciences Division, High Performance Computing and Communication Program).

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