Blowout Bifurcation Route to Strange Nonchaotic Attractors

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Strange nonchaotic attractors are attractors that are geometrically strange, but have nonpositive Lyapunov exponents. We show that for dynamical systems with an invariant subspace in which there is a quasiperiodic torus, the loss of the transverse stability of the torus can lead to the birth of a strange nonchaotic attractor. A physical phenomenon accompanying this route to strange nonchaotic attractors is an extreme type of intermittency. [S0031-9007(96)01861-3]

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Strange nonchaotic attractors are attractors that are geometrically complicated, but typical trajectories on these attractors exhibit no sensitive dependence on initial conditions *asymptotically* [1–11]. Here the word *strange* refers to the complicated geometry of the attractor: A strange attractor is not a finite set of points, and it is not piecewise differentiable. The word *chaotic* refers to a sensitive dependence on initial conditions: trajectories originating from nearby initial conditions diverge exponentially in time. Strange nonchaotic attractors occur in dissipative dynamical systems driven by several *incommensurate* frequencies (quasiperiodically driven systems) [1–11]. For example, it was demonstrated that in two-frequency quasiperiodically driven systems, there exist regions of finite Lebesgue measure in the parameter space for which there are strange nonchaotic attractors [3,4]. More recent work demonstrates that a typical trajectory on a strange nonchaotic attractor actually possesses positive Lyapunov exponents in finite time intervals, although asymptotically the exponent is negative [7]. Strange nonchaotic attractors can arise in physically relevant situations such as quasiperiodically forced damped pendula and quantum particles in quasiperiodic potentials [2], and in biological oscillators [4]. These exotic attractors have been observed in physical experiments [8,9].

While the existence of strange nonchaotic attractors was firmly established, a question that remains interesting is how these attractors are created as a system parameter changes through a critical value, i.e., what the possible routes to strange nonchaotic attractors are. One route was investigated by Heagy and Hammel [5] who discovered that, in quasiperiodically driven maps, the transition from two-frequency quaisperiodicity to strange nonchaotic attractors occurs when a period-doubled torus collides with its unstable parent torus [5]. Near the collision, the period-doubled torus becomes extremely wrinkled and develops into a fractal set at the collision, although the Lyapunov exponent remains negative throughout the collision process. Recently, Feudel *et al.* found that the collision between a stable torus and an unstable one at a dense set of points leads to a strange nonchaotic attractor [10]. A renormalization-group analysis was also devised for the transition to strange nonchaotic attractors in a particular class of quasiperiodically driven maps [11].

In this paper, we present at route to strange nonchaotic attractors in dynamical systems with a symmetric low-dimensional invariant subspace **S** in the phase space. Since **S** is invariant, initial conditions in **S** result in trajectories which remain in **S** forever. We consider the case where *there is a quasiperiodic torus in* **S** [12]. Whether the torus attracts or repels initial conditions in the vicinity of **S** is determined by the sign of the largest transverse Lyapunov exponent Λ_T computed for trajectories in **S** with respect to perturbations in the subspace **T** which is *transverse* to **S**. When Λ_T is negative, **S** attracts trajectories transversely in the phase space, and the quasiperiodic torus in **S** is also an attractor of the full phase space. When Λ_T is positive, trajectories in the vicinity of **S** are repelled away from it, and, consequently, the torus is transversely unstable, and it is hence not an attractor of the full phase space. Assume that a system parameter changes through a critical value a_c; Λ_T passes through zero from the negative side. This bifurcation is referred to as the "blowout bifurcation" [13–15]. Our main result is that blowout bifurcation can lead to the birth of a strange nonchaotic attractor. A physical phenomenon associated with this route to strange nonchaotic attractors is that the dynamical variables in the transverse subspace **T** exhibit an extreme type of temporally intermittent bursting behavior: on-off intermittency [16]. Thus, our work also demonstrates that on-off intermittency can occur in quasiperiodically driven dynamical systems, whereas, to our knowledge, these intermittencies have been reported only for systems that are driven either randomly or chaotically.

We consider the following class of N-dimensional dynamical systems,

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, \mathbf{z}, p)\,, \qquad \frac{d\mathbf{z}}{dt} = \omega\,, \tag{1}$$

where $\mathbf{x}$ is N_x dimensional, $\mathbf{z}$ is N_z dimensional, $N_x + N_z = N$, $\omega \equiv (\omega_1, \omega_2, \ldots, \omega_{N_z})$ is a frequency vector, and p is a bifurcation parameter. The function $\mathbf{F}$ satisfies $\mathbf{F}(0, \mathbf{z}, p) = 0$ so that $\mathbf{x} = 0$ defines the invariant subspace $\mathbf{S}$ in which the dynamics is described by [17] $d\mathbf{z}/dt = \omega$. The frequencies $(\omega_1, \omega_2, \ldots, \omega_{N_z})$ are incommensurate so that the $\mathbf{z}$ dynamics gives a quasiperiodic torus. The largest transverse and nontrivial overall Lyapunov exponents of the systems are given by $\Lambda_T = \lim_{t\to\infty} \frac{1}{t} \ln[|\delta\mathbf{x}(t)|/|\delta\mathbf{x}(0)|]$ and $\Lambda = \lim_{t\to\infty} \frac{1}{t} \times \ln[|\delta\mathbf{X}(t)|/|\delta\mathbf{X}(0)|]$, respectively, where the infinitesimal vectors $\delta\mathbf{x}(t)$ and $\delta\mathbf{X}(t)$ are evolved according to $d\delta\mathbf{x}(t)/dt = (\partial\mathbf{F}/\partial\mathbf{x})|_{\mathbf{x}=0}$. $\delta\mathbf{x}$ and $d\delta\mathbf{X}(t)/dt = (\partial\mathbf{F}/\partial\mathbf{x}) \cdot \delta\mathbf{X}$ for random initial vectors $\delta\mathbf{x}(0)$ and $\delta\mathbf{X}(0)$ [13]. To illustrate our findings, we consider a physical example, mathematically described by the following version of Eq. (1),

$$\frac{dx}{dt} = y\,,$$
$$\frac{dy}{dt} = -\kappa y - \gamma x^3 + (\beta + f_1 \sin z_1 + f_2 \sin z_2) \times \sin(2\pi x)\,, \tag{2}$$
$$\frac{dz_1}{dt} = \omega_1, \qquad \frac{dz_2}{dt} = \omega_2\,,$$

where the invariant subspace $\mathbf{S}$ is given by $(x, y) = (0, 0)$, ω_1 and ω_2 are two incommensurate frequencies so that there is a two-frequency quasiperiodic torus in $\mathbf{S}$ (z_1, z_2) for which the largest Lyapunov exponent is zero. In Eq. (2), κ (dissipation), γ, β, f_1, and f_2, are parameters. Equation (2) is a slightly modified version of the experimental model used by Zhou, Moss, and Bulsara, which is relevant to the radio-frequency-driven superconducting quantum interference device (SQUID) [9]. In our numerical experiments, we arbitrarily choose κ as the bifurcation parameter and fix other parameters at $\gamma = 2.0$, $\beta = -1.1$, $\omega_1 = 2.25$, $\omega_1/\omega_2 = \frac{1}{2}(\sqrt{5} + 1)$ (the golden mean), $f_1 = 3.5$, and $f_2 = 5.0$. Numerically we observe that a blowout bifurcation occurs at $\kappa_c \approx 4.17$, where $\Lambda_T > 0$ (< 0) for $\kappa < \kappa_c$ $(>\kappa_c)$.

Figure 1(a) shows the (x, y) projection of a trajectory of 50 000 iterations (after 10 000 preiterations) on the stroboscopic surface of section defined by $z_1(t_n) = 2n\pi$ $(n = 1, 2, \ldots)$ for $\kappa = 4.1$ $(\Lambda_T \approx 0.019)$. The largest nontrivial Lyapunov exponent of the system is $\Lambda \approx -0.134$. Thus, there is no positive Lyapunov exponent for this parameter setting. The geometric shape of the attractor, however, appears strange, as can be seen from Fig. 1(a). Qualitatively, the strangeness can be understood as follows. For $\kappa \lesssim \kappa_c$, the transverse Lyapunov exponent is slightly positive. Thus, a typical trajectory on the torus in $\mathbf{S}$ is transversely unstable. There are time intervals during which the trajectory, when it is in the vicinity of $\mathbf{S}$, is repelled from $\mathbf{S}$. As can be verified numerically, at this parameter setting there are apparently no other attractors in the phase space [18]. Thus, the trajectory comes back to the neighborhood of $\mathbf{S}$ intermittently in the course of time evolution. Since the trajectory is bounded in both x and y, the asymptotic attractor in the full phase space develops a strange shape: It apparently consists of an infinite number of points, and it is not piecewise differentiable. Since Λ is negative, despite $\Lambda_T \gtrsim 0$, the attractor is nonchaotic. Thus, as κ decreases through the critical value κ_c, a strange nonchaotic attractor is born: The positiveness of the transverse Lyapunov exponent Λ_T gives rise to strangeness of the attractor, and the negativeness of the largest nontrivial Lyapunov exponent Λ guarantees the attractor is nonchaotic.

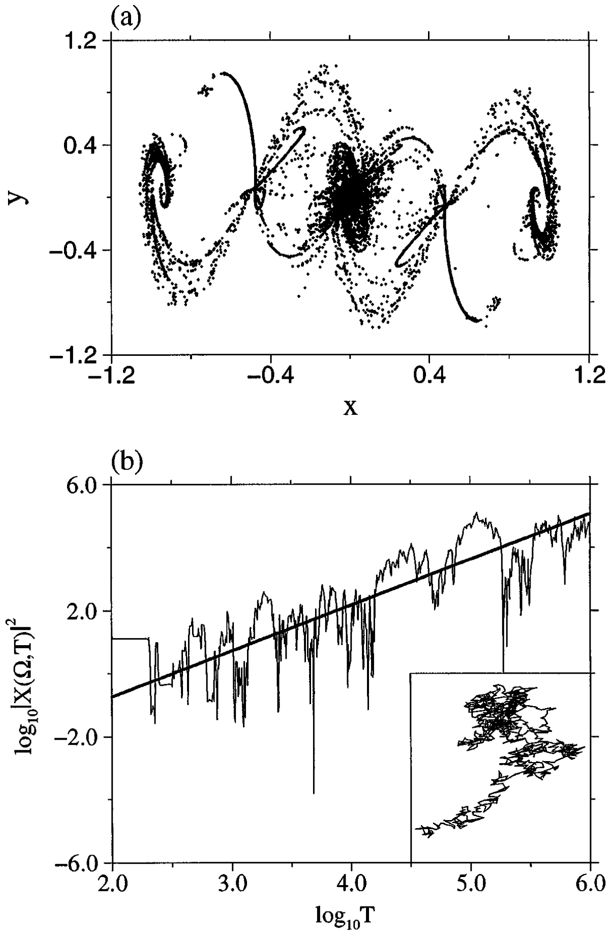

FIG. 1. (a) A trajectory of 50 000 points on the stroboscopic section for Eq. (2) at $\kappa = 4.1$ (see text for other parameter values). Apparently, the attractor is geometrically strange. (b) Singular-continuous spectrum analysis of the time series $\{x_n\}$: shown is $\log_{10}|X(\Omega, T)|^2$ versus $\log_{10} T$. We have $|X(\Omega, T)|^2 \sim T^{1.26}$. The subplot shows the corresponding path $(\mathrm{Re}\, X, \mathrm{Im}\, X)$.

To qualitatively verify that the attractor for $\kappa \lesssim \kappa_c$ is strange nonchaotic, we perform a singular-continuous spectrum analysis that was first proposed in the investigation of models of quasiperiodic lattices and quasiperiodically forced quantum systems [19], and was later applied to strange nonchaotic attractors by Pikovsky and Feudel [6]. Take a time series $\{x_n\}$ or $\{y_n\}$ on the surface of section. Compute the following Fourier sum: $X(\Omega, T) = \sum_{n=1}^{T} x_n e^{i2\pi n\Omega}$, where Ω is proportional to the ratio of the two incommensurate frequencies of the quasiperiodic driving. When T is regarded as time, the quantity $X(\Omega, T)$ defines a path in the complex plane $(\mathrm{Re}\,X, \mathrm{Im}\,X)$. For discrete spectrum, we have $|X(\Omega,T)|^2 \sim T^2$, while for continuous spectrum, the path is described by Brownian motion and, hence, $|X(\Omega,T)|^2 \sim T$. The spectrum associated with strange nonchaotic attractors falls somewhat in between these two categories. It was demonstrated [6] that for strange nonchaotic attractors, the quantity $X(\Omega, T)$ has the following two distinct features: (1) $|X(\Omega,T)|^2 \sim T^\alpha$, where $\alpha \neq 1, 2$; and (2) the path $(\mathrm{Re}\,X, \mathrm{Im}\,X)$ is fractal. Figure 1(b) shows, for $\Omega = (\sqrt{5}+1)/2$, $\log_{10}|X(\Omega,T)|^2$ versus $\log_{10} T$ and the path $(\mathrm{Re}\,X, \mathrm{Im}\,X)$ (subplot) computed from the time series $\{x_n\}$ in Fig. 1(a). We have $\alpha \approx 1.26$ and, the path $(\mathrm{Re}\,X, \mathrm{Im}\,X)$ apparently exhibits a fractal self-similar structure. These results strongly suggest that the attractor in Fig. 1(a) is indeed strange nonchaotic.

A feature associated with the birth of the strange nonchaotic attractor is the occurrence of on-off intermittency [16] when $\kappa \lesssim \kappa_c$ (Λ_T being slightly positive). This is shown in Fig. 2, where the time series $\{y_n\}$ is plotted for $\kappa = 4.1$. We see that there are time intervals when y_n stays near $y = 0$ (the "off" state), but there are also intermittent bursts of y_n (the "on" state) away from the off state. This is a typical consequence of the blowout bifurcation [13–15]. Note that the on-off intermittency in Fig. 2 is, in fact, produced by a quasiperiodic driving to the transverse dynamics.

To understand why Λ remains negative in parameter regimes after the blowout bifurcation where $\Lambda_T \gtrsim 0$, we consider the following analyzable two-dimensional map that captures the essential feature of physical model Eq. (2) on the surface of section

$$\begin{aligned} x_{n+1} &= \frac{1}{2\pi}(a\cos z_n + b)\sin(2\pi x_n), \\ z_{n+1} &= (z_n + 2\pi\omega)\,\mathrm{mod}(2\pi), \end{aligned} \tag{3}$$

where a and b are parameters, and $\omega \in (0,1)$ is an irrational number so that the z dynamics is a map on the circle that generates a quasiperiodic torus with uniform invariant density $\rho(z) = 1/(2\pi)$ in $z \in [0, 2\pi]$ (two-frequency quasiperiodicity). The one-dimensional invariant subspace is $x = 0$. The transverse Lyapunov exponent is $\Lambda_T = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\ln|a\cos z_j + b| = \frac{1}{2\pi} \times \int_0^{2\pi}\ln|a\cos z + b|\,dz$. We obtain $\Lambda_T = \ln|b| - \ln(2/\{1 + [1-(a/b)^2]\})$ if $a \le b$, and $\Lambda_T = \ln|b| + \ln\frac{a}{2b}$ if $a > b$. We have, for example, $a_c = 2$ for the case $a > b > 0$, where $\Lambda_T \le 0$ for $a \le a_c$ and $\Lambda_T > 0$ for $a > a_c$. The nontrivial Lyapunov exponent is $\Lambda \approx \Lambda_T + \lambda$, where $\lambda \equiv \int \ln|\cos(2\pi x)|\rho(x)\,dx < 0$ and $\rho(x)$ is the invariant density of x for $a > a_c$. Note that for $a < a_c$ we have $\lambda = 0$ and hence $\Lambda = \Lambda_T$ because in this case x approaches zero asymptotically. For $a \gtrsim a_c$ where $\Lambda_T \gtrsim 0$, it is possible to have $\Lambda \lesssim 0$ because $\lambda < 0$. Thus a strange nonchaotic attractor is born at the blowout bifurcation with on-off intermittency, as can also be verified numerically. We stress that it is essential to have quasiperiodicity in the invariant subspace. When frequencies are locked on the torus (corresponding to ω's being rational), trajectories only have a finite number of possible z values. Numerical analysis indicates that the attractor does not appear to be strange in the phase space.

We now describe two issues related to the bifurcation.

(1) Variation of the Lyapunov exponent after the blowout bifurcation.—We address the following question: how does the strange nonchaotic attractor become a chaotic attractor as κ decreases from κ_c? To answer this question, notice that $\Lambda \approx \Lambda_T + \lambda$, where $\lambda < 0$. Thus, Λ becomes positive when $\Lambda_T > |\lambda|$. As κ decreases, Λ_T increases monotonically near κ_c, but λ is not monotonic. This is shown in Fig. 3, where Λ_T, Λ, and λ versus κ are plotted for 2000 values of κ in [3.2, 4.8]. For each value of κ, Λ_T, Λ, and λ are computed with 50 000 iterations and 10 000 preiterations on the surface of section. When Λ_T and $|\lambda|$ have comparable magnitudes, Λ can change

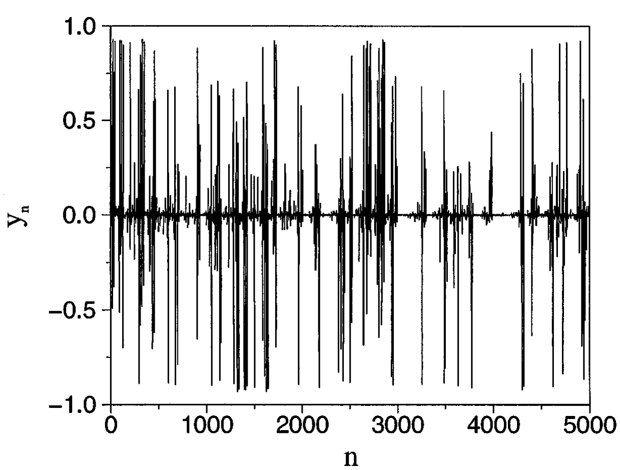

FIG. 2. On-off intermittency at $\kappa = 4.1$.

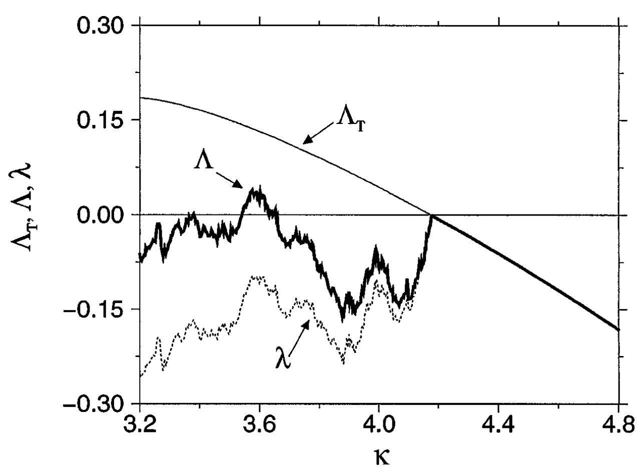

FIG. 3. Λ_T, Λ, and λ versus the parameter κ.

from negative to positive and vice versa. Consequently, there exist several parameter intervals for chaotic attractors ($\Lambda > 0$) which are interspersed with parameter intervals for strange nonchaotic attractors ($\Lambda < 0$). This behavior has actually been observed in other physical systems such as the quasiperiodically forced pendulum [3]. We believe that the existence of two competing exponents, i.e., Λ_T and λ, is responsible for the alternation of strange nonchaotic and chaotic attractors. We have observed that when κ is decreased further through some critical value, Λ_T is sufficiently large so that $\Lambda_T < |\lambda|$ does not occur, the system possesses a positive Lyapunov exponent Λ and, consequently, strange nonchaotic attractors are no longer possible.

(2) Transient on-off intermittent behavior preceding the blowout bifurcation.—Before the birth of the strange nonchaotic attractor, a typical trajectory exhibits transient on-off intermittent behavior before finally approaching the invariant subspace $x = y = 0$. For a given parameter value κ, the average transient lifetime, $\tau(\kappa)$, depends on the parameter difference $|\kappa - \kappa_c|$. Numerically, we find the following scaling law: $\tau(\kappa) \sim |\kappa - \kappa_c|^{-1}$. This can be understood by noting that $\tau(\kappa) \sim 1/\Lambda_T$, and for κ near κ_c, we have $\Lambda_T \sim |\kappa - \kappa_c|$.

In summary, we have shown that two distinct dynamical phenomena, strange nonchaotic attractor and on-off intermittency, commonly thought as arising in very different contexts in the study of nonlinear systems, can actually be closely related. The link is the blowout bifurcation that destabilizes, transversely, the quasiperiodic torus in the invariant subspace. Our study thus demonstrates that blowout bifurcation can occur even if the driving is not chaotic or random but quasiperiodic. As a consequence, on-off intermittency can arise in quasiperiodically driven dynamical systems. We have presented a physical example for which the blowout bifurcation route to strange nonchaotic attractor can be observed numerically, and we believe that this route can be tested in physical experiments.

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