Critical Exponent for Gap Filling at Crisis

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A crisis in chaotic dynamical systems is characterized by the conversion of a nonattracting, Cantor-set-like chaotic saddle into a chaotic attractor. The gaps in between various pieces of the chaotic saddle are densely filled after the crisis. We give a quantitative scaling theory for the growth of the topological entropy for a major class of crises, the interior crisis. The theory is confirmed by numerical experiments. [S0031-9007(96)01224-0]

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The nature of the asymptotic dynamics of a physical system depends on its parameters. As a system parameter changes, there can be qualitative changes in the asymptotic set of the system, which can usually be characterized by quantitative and universal scaling behaviors.

Typically, in nonlinear system, one finds parameter regions in which chaos is present. In this region there is an infinite number of unstable periodic orbits in the chaotic set and this chaotic set can be either attracting or repelling. The latter occurs in *periodic windows.* The unstable periodic orbits that existed in the large chaotic attractor before the creation of a given periodic window (by a saddle-node bifurcation) are in the repelling chaotic set when the window is created [1]. Hence, at a given periodic window, one finds the coexistence of both this repelling chaotic set *and* the attracting periodic orbit. This orbit bifurcates in an infinite period-doubling cascade leading to a small chaotic attractor. The collision of this small chaotic attractor with the coexisting repelling chaotic set marks the end of the window and the recovery of the large chaotic attractor in an interior crisis [2] which has been observed in many experiments [3]. The purpose of this Letter is to establish *quantitatively* a universal scaling for this attractor enlargement. The scaling, valid after the interior crisis, is due to the emergence of new orbits connecting the two colliding sets. Because there are an infinite number of periodic windows, believed to be dense in parameter space, our scaling occurs in the neighborhood of an infinite number of parameter values. Therefore, this scaling is of fundamental importance to understanding chaotic systems, whether in the study of theoretical models or in laboratory experiments.

As an example, Fig. 1(a) shows a bifurcation diagram for the Hénon map [4], $(x_{n+1}, y_{n+1}) \mapsto (a - x_n^2 + 0.3y_n, x_n)$, where the parameter a varies near the crisis value $a_c \approx 1.272$. In this crisis, a small seven-piece attractor changes suddenly into a single, much larger chaotic attractor for $a > a_c$. Figure 1(b) illustrates (for $a = 1.27 < a_c$) that besides the chaotic attractor there also exists a chaotic saddle in the surrounding region. The chaotic saddle apparently has a Cantor-set-like fractal structure with finite size *gaps* along its unstable foliation. As a comparison, Fig. 1(c) shows that after the crisis (at $a = 1.28 > a_c$) the enlarged attractor fills these gaps *completely.* Note that this *gap filling* happens abruptly at the crisis. We emphasize that the example shown in Figs. 1(a)–1(c) for the Hénon map is paradigmatic: The same phenomenon occurs in any physical system exhibiting crisis, such as in the driven pendulum experiment by Leven and Selent [5], but has not been analyzed in detail.

In this Letter, we present qualitatively that gap filling is accomplished by the creation of a large number of *new unstable periodic orbits* that are not present before the crisis, yet they provide the support for the dense filling of the gaps after the crisis. The creation of the gap-filling orbits thus provides the primary mechanism for the structure development of chaotic attractors. Quantitatively, this process leads to an increase in the topological entropy h of the chaotic attractor. We find that for parameter values a beyond the crisis value a_c, the topological entropy obeys the following algebraic scaling law

$$h(a) - h(a_c) \sim (a - a_c)^{\chi}, \quad \text{with } \chi = h(a_c)/\Lambda, \quad (1)$$

where Λ denotes the Lyapunov exponent of the unstable periodic orbit mediating the crisis. We call χ the *gap-filling exponent.* In what follows we derive Eq. (1) for chaotic maps by a diagram technique based upon a scaling argument. We also provide numerical results that support our theoretical prediction.

We call the confined phase space region where the original chaotic attractor resides the *band region* (B) and the space in between these bands the *surrounding region* (S), respectively. For parameter values below the crisis, every unstable periodic orbit is contained either in the small attractor or in the coexisting chaotic saddle, i.e., every periodic orbit is restricted strictly to the band or to the surrounding region. The boundary between the band and surrounding regions is formed by the stable manifold of an unstable periodic orbit [2], the so-called *mediating periodic orbit* (M) [6], which belongs to the

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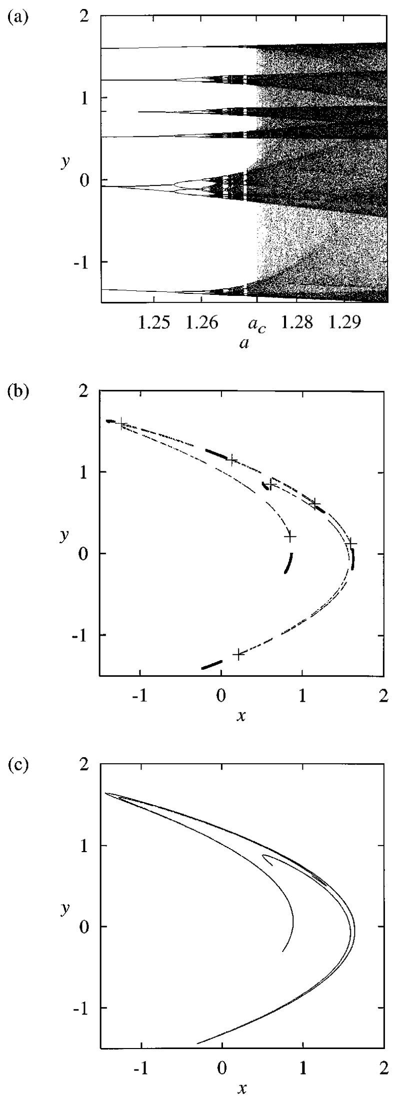


FIG. 1. (a) Bifurcation diagram $x(a)$ of the Hénon map near the interior crisis $a_c \approx 1.272$. Because of the projection on the x axis, the seven pieces of the attractor appear as six bands. (b) The seven-piece chaotic attractor (heavy dots) and the chaotic saddle (light dots) before the crisis. $+$'s indicate the period-7 mediating orbit. (c) The single enlarged attractor after the crisis.

chaotic saddle below the crisis. Hence the mediating orbit has the same period, say p, as the number of pieces of the small attractor [e.g., period 7 in Fig. 1(b)]. The crisis occurs when the chaotic saddle collides with this boundary. (Here we assume that the mediating orbit is hyperbolic with a single repulsive direction and the topology of the invariant manifolds changes smoothly with the control parameter [7].) After the crisis there are periodic orbits that visit both regions. We call them *coupling orbits* since they have components in both regions. The orbit components in the surrounding region are called *bursts* [2]. For parameter values slightly above the crisis, an orbit can escape from the band region, as schematically illustrated in Fig. 2. Here W_M^s and W_M^u denote the stable and unstable manifolds of the mediating orbit, respectively; the region above W_M^s is the surrounding region. The lobes $L_1, \ldots, L_4, \ldots$ are pieces of the unstable foliation of the remnant of the former small chaotic attractor. These lobes penetrate the surrounding region above the crisis; thus the former small attractor is *converted into a chaotic saddle within the band region* [6]: A trajectory which is on the closure of the unstable foliation is injected into the surrounding region through the lobe L_1. The depth of this protruding lobe is proportional to $a - a_c$. For subsequent iterations for the p-fold map, L_1 maps to L_2, L_2 to L_3, and so on, moving towards M and stretching along W_M^u at a rate determined by the Lyapunov exponent Λ of the mediating orbit. The time $m(a)$ that the trajectory spends in the vicinity of M before spreading over the surrounding region can be estimated by requiring that the length of the lobes increases up to an order of 1 after m iterations. Thus we obtain

$$m(a) \approx -[\ln(a - a_c)]/\Lambda \,. \tag{2}$$

We note that in the limit $a \to a_c + 0$, $m(a)$ diverges, indicating that very close to the crisis from above, trajectories escaping from the band region spend a very long time in the vicinity of M, in the course of which the motion is practically indistinguishable from the behavior of the mediating orbit. Thus *every burst starts with an approximately periodic motion* whose duration m is much longer than the period p of M. Consequently, since every coupling orbit must contain at least one burst, Eq. (2) also gives an *asymptotic scaling for the minimum length of coupling periodic orbits.*

As the parameter increases beyond the crisis value, $m(a)$ decreases, indicating the appearance of new, shorter and shorter coupling orbits. The creation of the new coupling orbits leads to an increase in the *topological entropy h* of the enlarged attractor, defined via $N(n) \sim e^{hn}$, where $N(n)$ is the number of period-n orbits embedded in the set. According to the thermodynamical description of dynamical systems [8], $N(n)$ can also be interpreted as a formal "partition sum." This lead us to the idea of using the following diagram technique to represent and calculate the topological entropy of the chaotic set.

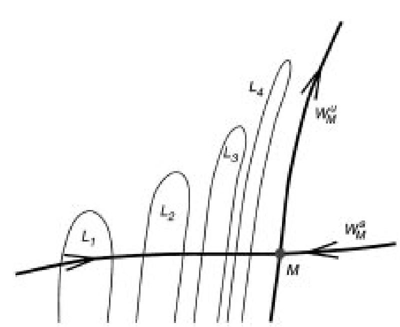
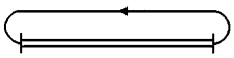


FIG. 2. Schematic diagram of the topology of the stable and unstable manifolds of the p-fold iterated map in the vicinity of the p-mediating orbit M above the crisis.

Let the diagram

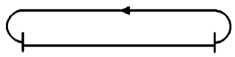
$$\sim e^{hn} =: r^n \, . \tag{3}$$

represent the number of unstable periodic orbits of length n on the attractor after the crisis, i.e., represent the partition sum $N(n)$ itself. Let



$$\sim e^{h^{[B]}n} =: b^n \tag{4}$$

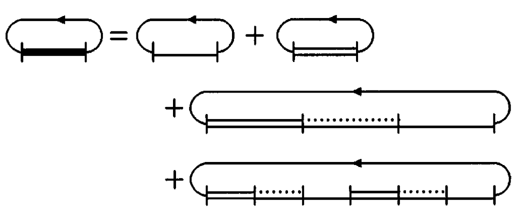
and



$$\sim e^{h^{[S]}n} =: s^n \tag{5}$$

denote the number of periodic orbits of length n located *entirely* within the band and the surrounding regions, respectively. Note that these two sets of periodic orbits form two chaotic saddles, both being embedded in the enlarged attractor. The respective partial topological entropies $h^{[B]}$ and $h^{[S]}$ of these chaotic saddles determine the growth rates of the two latter diagrams. The exponentials of the topological entropies in the representations (4) and (5), b and s, can be regarded as the *propagators* for the corresponding diagrams. These propagators take into account the contributions to the partition sum $N(n)$ from the periodic orbits in the band and in the surrounding regions. Similarly, r in (3) acts as the propagator representing the number of every allowed periodic orbit.

In determining r, we also have to take into account the contribution of the coupling orbits to $N(n)$. In fact the essence of the gap-filling phenomenon is *the growth in number of the coupling orbits*, being made up of various combinations of the orbits that already existed before the crisis. The counting of the total number of orbits of length n, with $0, 1, 2, \ldots$ bursts during their period, can be expressed by the following diagram equation:



$$+ \cdots \tag{6}$$

The first two terms correspond to the periodic orbits that never escape from the band and from the surrounding regions, as represented by diagrams (4) and (5), while the additional terms correspond to the coupling orbits. The approximately periodic components of the coupling orbits invoking the bursts give only a constant contribution to the partition sum (since they always closely follow the same mediating orbit) corresponding thus to a zero topological entropy. We incorporate this feature into Eq. (6) by inserting the dotted "interaction" diagram with the corresponding propagator $k \equiv 1$ at the beginning of each burst. Thus the number of dotted insertions is equal to the number of escapes. Note that before each burst the trajectory must spend at least one step in the band region and, due to Eq. (2), at least m steps to go over the mediating orbit. This implies that the lengths of the double-line and the dotted propagators is at least 1 and m, respectively. The full length of each diagram term is n.

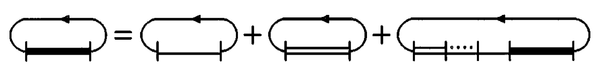
For instance, the number of the simplest coupling orbits (with exactly one burst), i.e., the third diagram on the right-hand side of Eq. (6), is given by

$$\begin{aligned} N_1(n) &\approx \sum_{n_b=1}^{n-m} \sum_{n_m=m}^{n-n_b} b^{n_b} k^{n_m} s^{n-n_b-n_m} \\ &= Ck^m[s^{n-m} + Bb^{n-m} + Ak^{n-m}] \end{aligned} \tag{7}$$

for $n > m \gg 1$. Here the asymptotic A, B, and C coefficients depend on the control parameter and the values of the propagators and reflect the effect of short range correlations between subsequent orbit segments. Equation (7) describes the simplest interaction, or coupling, between the two chaotic saddles in the band and in the surrounding regions. The number of possible orbit combinations characterizes the strength of the coupling.

Equation(6) can be viewed as a perturbation series, with Eq. (7) being its first "loop order" term and the subsequent diagrams accounting for the higher loop order terms.

In general, before the crisis the topological entropy of the chaotic set (the small attractor) in the band region is smaller than that of the coexisting chaotic saddle in the surrounding region [6]. This behavior persists after the crisis. Thus we have, for the above propagators, $r \gtrsim s > b > k$, which, for large n, implies $r^n > s^n \gg b^n \gg k^n$. In the limit $n \gg m$ the simplest coupling term in Eq. (7) gives $N_1(n) \approx C(k/s)^m s^n$, yielding a "coupling strength" proportional to $s^{-m} \ll 1$. We assume that s and g depend continuously on the control parameter at a_c. Then according to (2), as $a \rightarrow a_c + 0$ the value of m diverges, meaning that the number of the combinational possibilities decreases drastically, and the coupling becomes weak. This fact guarantees the convergence of the perturbation series Eq.(6), which can be rewritten in the following self-consistent form,



$$. \tag{8}$$

By neglecting the effect of long range correlations we can substitute the propagators and the expression for the first loop order term, Eq. (7), to obtain $N(n) \approx r^n = s^n + b^n + \sum_{n_1=m}^{n} N_1(n_1)N(n - n_1)$. By taking the relation among the propagators into consideration, the solution of the implicit equation for r in the scaling region $1 \ll m \ll n \rightarrow \infty$ is $r \approx s[1 + Ck^m s^{-m} + \mathcal{O}(Ck^m s^{-m})^2]$. Therefore, close to the crisis the topological entropy is

$$h \approx h^{[S]} + Ce^{-h^{[S]}m}. \tag{9}$$

By using the property that at a_c the topological entropy of the enlarged attractor coincides with that of the chaotic saddle, and combining Eqs. (2) and (9), we obtain our main result, which is Eq. (1).

To test the theoretical prediction of Eq. (1), we have undertaken a series of numerical experiments using two-dimensional maps. Here we shall present results for the Hénon map for the interior crisis shown in Figs. 1(a)–1(c). The largest eigenvalue of the period-7 mediating orbit is approximately 10.87, which corresponds to the Lyapunov exponent $\Lambda \approx 0.34$. We have used the method developed in Ref. [9] to determine the topological entropy by monitoring how the length of an infinitesimal curve grows under the action of the map. In particular, we randomly choose an infinitesimal curve straddling a point on the chaotic attractor along the unstable manifold. As the map is iterated forward in time, its length grows exponentially. The positive exponential growth rate of the curve length is taken to be the topological entropy [9].

To obtain the scaling relation, Eq. (1), it is necessary to estimate $h(a_c)$. We choose 40 values of a uniformly distributed on the base-10 logarithmic scale in the small interval $[a_c + 10^{-4}, a_c + 10^{-3.6}]$ and compute the topological entropy h for all these 40 values of a. The average value of these 40 h's is then taken to be an approximation of $h(a_c)$. We obtain $h(a_c) \approx 0.38$. The scaling exponent from the theory is then $\chi \approx 0.38/0.34 \approx 1.12$. Figure 3 shows $h(a) - h(a_c)$ versus $a - a_c$, on a logarithmic scale, for 100 values of a in $[10^{-2}, 10^{-1}]$. The data can be fitted by a straight line with slope 1.13 ± 0.11, which agrees with the theoretical exponent. It can be seen that the fluctuations in $h(a) - h(a_c)$ increase as $a - a_c$ decreases. This is caused by the numerical precision in the computation of h. The typical confidence interval for h is about 10^{-3}. This indicates that the values of h are indistinguishable for variations of the parameter a less than approximately $10^{-2.6}$, given that $\chi \approx 1.12$. Our numerical computation indicates that no reliable scaling behavior can be obtained for $a - a_c < 10^{-2}$, imposing the smallest scale of confidence in the parameter variation, above which our computation (Fig. 3) gives a scaling exponent which agrees with the theoretical prediction.

In summary, in this work we have given a qualitative explanation for the fundamental phenomenon of gap-filling accompanying crises in chaotic dynamical systems. In addition, we have obtained the quantitative scaling relation, Eq. (1), providing a new scaling exponent as a characteristic quantity of this phenomenon [10]. The scaling relation is valid for systems related to one- and two-dimensional maps with smooth control parameter dependence, since our qualitative arguments and the diagram equations hold regardless of the details of the system. Our approach and results can be adapted for higher dimensional maps [7] and other types of crises as well.

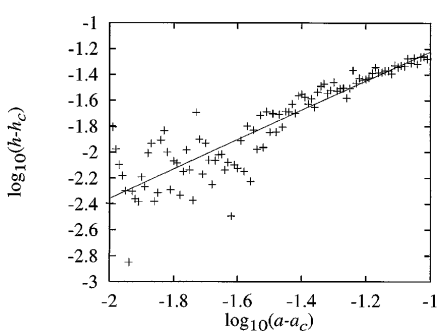


FIG. 3. Numerically obtained scaling of the excess topological entropy near the crisis shown in Figs. 1(a)–1(c). The numerical gap-filling exponent is $\chi = 1.13 \pm 0.11$, which agrees with the theoretical prediction $\chi \approx 1.12$.

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