Riddled Parameter Space in Spatiotemporal Chaotic Dynamical Systems

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We demonstrate that some chaotic parameter subsets of a class of spatiotemporal chaotic systems modeled by globally coupled maps are riddled. That is, for every point in the chaotic parameter subset, there are parameter values arbitrarily nearby that lead to nonchaotic attractors. A consequence is an extremely sensitive parameter dependence characterized by a significant probability of error in numerical computation of asymptotic attractors, regardless of the precision with which parameters are specified.

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The asymptotic behavior of a dynamical system is determined both by initial conditions in phase space and by choice of parameters specifying the system. In particular, chaotic dynamical systems exhibit sensitive dependence on initial conditions. These dependencies are of two types. The first is characterized by exponential separation of trajectories originating from nearby initial conditions. The second type of dependence can be observed in chaotic systems with multiple attractors. Grebogi *et al.* first demonstrated that for some systems basins of attraction are separated by a fractal set called the fractal basin boundary [1]. It is impossible to predict, with certainty, the asymptotic attractor for initial conditions in the neighborhood of this set. Systems with multiple attractors may also exhibit "riddled basins" [2], in which case at least one of the basins of attraction has the property that any neighborhood about each point within that basin contains points belonging to another basin of attraction. Finally, there can exist an extreme type of riddled basin, the so-called intermingled basin [2], in which all basins of attraction are riddled.

In this Letter, we present the first evidence of "riddled" parameter sets in chaotic dynamical systems. A consequence of such riddled parameter sets is an extreme sensitive dependence on parameters. The dependence is sufficiently strong so that no matter how precisely *a parameter value* is specified, there is a significant uncertainty in numerical computation of the asymptotic attractor. This uncertainty in parameter space means that statistical properties [3] of asymptotic attractors, such as Lyapunov exponents and dimensions, cannot be computed reliably for specific parameter values.

Sensitive parameter dependence in dynamical systems was first demonstrated by Farmer [4] using the one-dimensional quadratic map $x_{n+1}=r(1-2x_n^2)$. This map exhibits a unique attractor starting from almost all initial conditions in $(-1,1)$ for any given value of the parameter r [5]. Attractors are of two types: chaotic and periodic. Farmer [4] demonstrated that the set of r values generating chaotic attractors (the chaotic parameter set) is a fractal set with positive Lebesgue measure [6] and box-counting dimension [7] one. Such sets have come to be known as "fat fractals" [4,8]. Most importantly, Farmer demonstrated that because of the fractal nature of the chaotic set, arbitrarily small perturbations ϵ about parameter values r drawn from this set yield parameters $r+\epsilon$ with nonzero probability of producing asymptotic attractors with completely different properties than those generated using parameter r. Nonetheless, this parameter sensitivity is sufficiently "weak" so that specification of parameters to a precision achievable on digital computers makes possible the reliable prediction of statistical quantities of asymptotic attractors such as the Lyapunov exponent [4].

To quantify sensitive parameter dependence, one can use the scaling exponent β introduced by Farmer [4,8,9]. Grebogi, Ott, and Yorke [10] noted subsequently that β is equivalent to the uncertainty exponent α first introduced by Grebogi *et al.* to characterize fractal basin boundaries [1]. The exponent α (or β) can be calculated as follows. Randomly choose a parameter value r_0 in the fat fractal set. Define $r'=r_0+\epsilon$, where ϵ is a small perturbation. Determine whether the asymptotic dynamics of the system using these two parameters is qualitatively different (chaotic versus periodic). Estimate the probability $P(\epsilon)$ that parameters r_0 and r' yield different asymptotic dynamical behavior by repeating the experiment for many random choices of r_0 in the parameter range of interest. In practice, $P(\epsilon)$ decreases with decreasing ϵ, typically scaling with ϵ as $P(\epsilon)\sim\epsilon^\alpha$ [1,11].

In numerical simulation of orbits, ϵ can be viewed as the precision with which the parameter r is specified. Then the scaling exponent α determines the probability $[1-P(\epsilon)]$ that the computed asymptotic behavior accurately reflects the true dynamics of the system. If $\alpha>1$, reducing ϵ can improve the probability of correct computation of the final state significantly. If $\alpha=1$, then improvement in ϵ results in an equal improvement in the probability of correct computation of the final state. If $\alpha<1$, then reduction of ϵ will result in only a small reduction of $P(\epsilon)$. In particular, in the extreme case where $\alpha\approx 0$, improvement in the precision ϵ with which r is specified (even over many orders of magnitude) may result in only an incremental improvement in the ability to predict the asymptotic state correctly. For the class of quadratic maps, Farmer found that the fatness exponent

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$\beta \approx 0.45$ [4], while Grebogi *et al.* found that the uncertainty exponent $\alpha \approx 0.41$ [11]. To appreciate the meaning of $\alpha = 0.41$, assume that the parameter r can be determined to a precision of 10^{-14}. Then $P(\epsilon) \sim 10^{-6}$ and, hence, the probability of error in numerical prediction of the final state of the quadratic map is roughly 1 in 10^6. This means that computer simulations are generally reliable for this class of systems, hence our use of the phrase "weak dependence" on parameters.

In this Letter, we study a class of spatiotemporal chaotic systems modeled by coupled map lattices [12]. For this class of systems the uncertainty exponent α defined above is determined numerically to be close to zero (it in fact cannot be distinguished from zero), which indicates strongly that the parameter space is riddled. Specifically, we consider globally coupled Hénon maps. Our motivations for studying this class of systems are as follows: (1) Globally coupled maps are approximations of spatiotemporal dynamical systems described by nonlinear partial differential equations or coupled ordinary differential equations [13,14]; and (2) the Hénon map [15] is one of the most extensively studied chaotic systems. The model is expressed as follows:

$$x_{n+1}^i = a - \left[(1-\delta)x_n^i + \frac{\delta}{N-1}\sum_{j,j\neq i} x_n^j\right]^2 + b y_n^i, \quad (1)$$

$$i = 1, \ldots, N,$$

$$y_{n+1}^i = x_n^i, \quad (2)$$

where i denotes discrete spatial sites, n denotes iteration number, a and b are the parameters of the Hénon map, and δ is a parameter specifying coupling strength between maps at different sites. We assume that each map couples to every other map with uniform coupling δ. The Jacobian of the map is $(-b)^N$ and, hence, for $|b| < 1$ the system is highly dissipative. In the numerical computations to be described, b is fixed at a value $b = 0.3$ and properties of the two-dimensional parameter space defined by a and δ are explored. Lyapunov exponents are computed [16] in order to quantify the type of attractors of Eqs. (1) and (2). Let λ_1 be the largest of the $2N$ Lyapunov exponents. Then $\lambda_1 > 0$ signifies the existence of a chaotic attractor and $\lambda_1 \leq 0$ indicates nonchaotic motion (quasiperiodic or periodic).

Figure 1 plots a chaotic parameter set in the (a,δ) space for $N = 10$. This parameter space was sampled over a two-dimensional 800×460 uniform grid in a parameter region defined by $1.0 \leq a \leq 1.4$ and $0 \leq \delta \leq 0.35$. The maximum Lyapunov exponent λ_1 was computed at each grid point. In Fig. 1, black dots denote parameter pairs for which $\lambda_1 > 0$, while white blank regions denote parameter regions of nonchaotic motion. Chaos occurs for $a > 1.06$. The most interesting feature is the presence of regions of interspersed black and blank dots. In these regions, near every chaotic parameter combination, there are parameter pairs that give rise to nonchaotic motion. Thus, chaotic parameter sets within these regions are riddled [2].

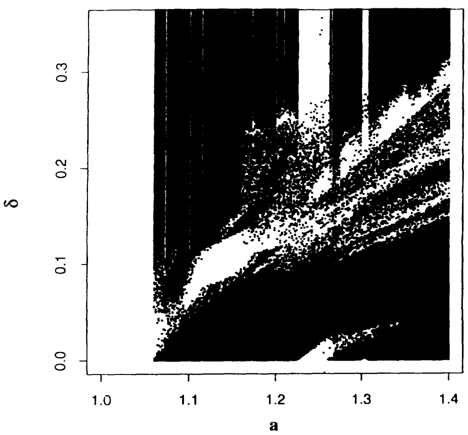

FIG. 1. Values of parameters a and δ that lead to chaotic attractor (black dots) and nonchaotic attractor (blank regions) for $1 \leq a \leq 1.4$ and $0 \leq \delta \leq 0.35$.

To examine "riddling" of chaotic parameter sets, we plot λ_1 and the number of positive Lyapunov exponents N_p versus δ for a fixed a value, as shown in Figs. 2(a) and 2(b) for $a = 1.4$, respectively. There are three distinct dynamic regimes as δ is increased from 0. For most δ values in regime 1 ($0 < \delta \leq 0.14$), $N_p = N$, indicating that maps on different sites behave independently [14,17]. For most values of δ within regime 3 ($\delta > 0.32$), there is only one positive Lyapunov exponent, indicating the existence of a strong coherence among maps at different

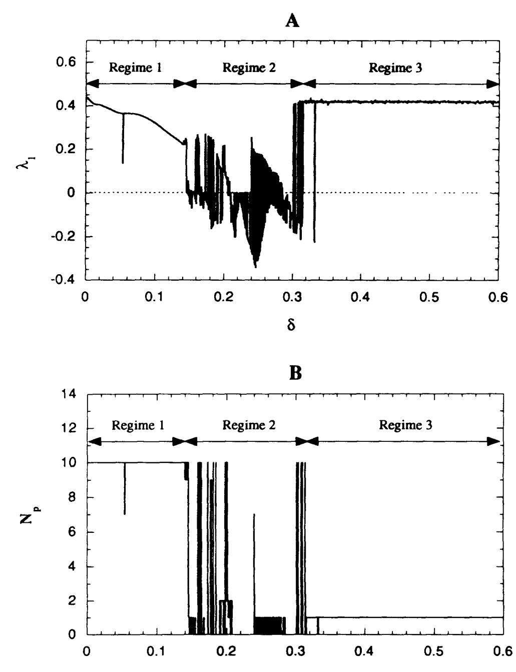

FIG. 2. (a) The largest Lyapunov exponent λ_1 and (b) the number of positive Lyapunov exponents N_p versus δ (coupling strength) for a system of $N = 10$ globally coupled Hénon maps.

sites [17]. Riddling occurs in regime 2 ($0.14 < \delta \leq 0.32$), where λ_1 fluctuates between positive and negative values, and N_p changes between 10 and 0. Figure 3(a) shows an expanded view of Fig. 2(a) for δ values in the range $0.248 < \delta < 0.249$. Sign changes in λ_1 persist even at this much finer scale. The data of Figs. 1-3 suggest that for any random choice of the coupling parameter δ in regime 2, an arbitrarily small perturbation about that value of δ can give rise to completely different asymptotic dynamical behavior (e.g., a transition from chaos to periodic motion, or vice versa). To verify this consequence of riddling of the parameter space in Figs. 1-3, we have computed the uncertainty exponent α [11]. The procedure is as follows. A coupling value δ is drawn from a uniform distribution defined over regime 2 for a fixed a value. Maximum Lyapunov exponents are then computed for both parameters δ and $\delta+\epsilon$ using the same initial conditions, where ϵ is a small fixed perturbation factor. If the two exponents have different sign, then δ is defined as an uncertain parameter value. This process is repeated for many different randomly selected δ values within regime 2 while holding the perturbation ϵ constant. Assume that among N_t values of δ chosen, there are N_u uncertain parameter values. The fraction of uncertain δ values is then $f(\epsilon) = N_u/N_t$. In our numerical computation we increase N_t until N_u reaches 200. This process is then repeated for different values of the perturbation ϵ.

Results of this calculation are shown in Fig. 3(b) for $a = 1.4$. The abscissa shows $\log_{10}(\epsilon)$ and the ordinate plots $\log_{10}[f(\epsilon)]$. These data show that $f(\epsilon)$ scales with ϵ as $\sim \epsilon^{\alpha}$ [1,11]. The slope of the straight line fit to the data points is an estimate of the uncertainty exponent α. This exponent is estimated to be 0.006 ± 0.008 at a 95% confidence level and, hence, the hypothesis that $\alpha = 0$ cannot be rejected [18]. This suggests strongly that the chaotic parameter set within regime 2 is riddled.

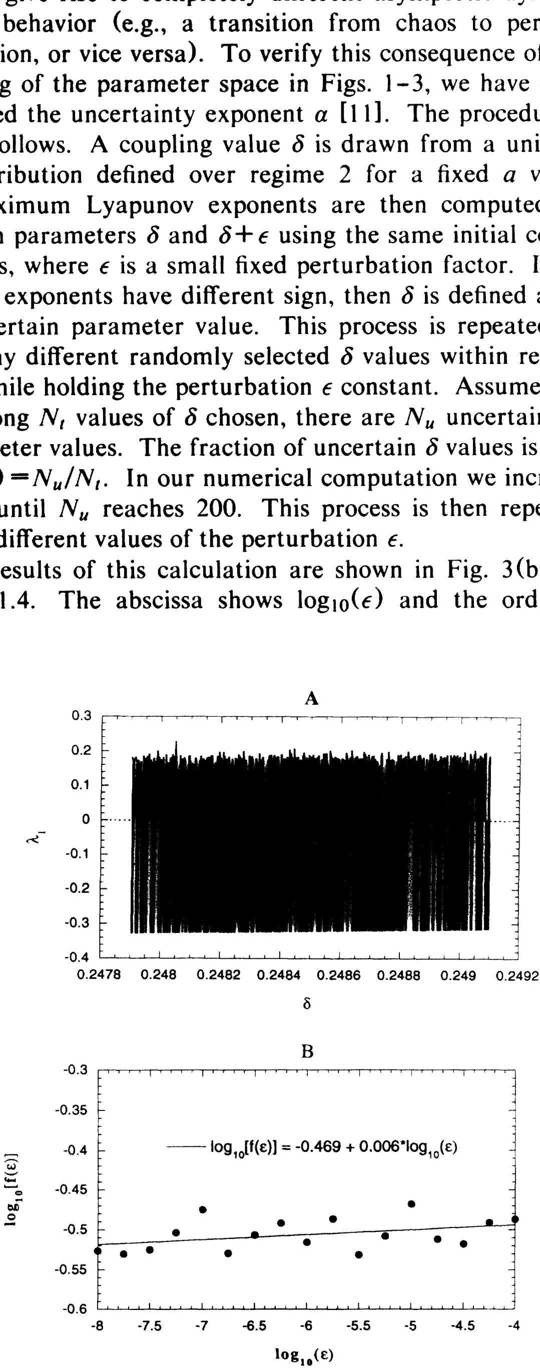

FIG. 3. (a) An expanded view of Fig. 2(a) for the parameter range $0.248 \leq \delta \leq 0.249$. (b) The uncertainty fraction $f(\epsilon)$ versus the uncertainty ϵ on a base-10 logarithmic scale.

A consequence of a riddled chaotic parameter set is an extremely sensitive dependence on parameter values of the asymptotic dynamical behavior of the system. To appreciate the implication of $\alpha \approx 0$, assume that α takes its upper bound value of 0.014 in Fig. 3. Assume the value of δ can be specified to within 10^{-16}; then there is a probability of $f(\epsilon) \sim 10^{0.014 \times (-16)} \approx 0.6$ that the final asymptotic state computed using δ is incorrect. Improving the precision with which δ is specified offers little improvement in the probability of computing the final state of the system correctly. For example, suppose computer precision is improved by 22 decades to 10^{-38}. Then the probability of incorrectly computing the asymptotic state is still $\sim 10^{0.014 \times (-38)} \approx 0.3$, a small improvement in uncertainty with respect to the magnitude of the improvement in computer precision. This indicates that computer calculation of the asymptotic state of the system in regime 2 cannot be reliable.

We have examined different values of N, the number of maps coupled in the system [19]. When $N = 28$, for example, the uncertainty exponent is estimated to be $\alpha = 0.00066 \pm 0.0068$ at a confidence level of 95%. Hence, the hypothesis that $\alpha = 0$ cannot be rejected.

We therefore conclude that chaotic parameter sets within some subregions of the parameter space of globally coupled Hénon maps are riddled. Since these systems are simplified models of spatiotemporal dynamical systems [12,13], it is highly likely that the same type of sensitive parameter dependence occurs in models of physical systems which are much more complicated than the model investigated in this paper (e.g., fluid turbulence).

The extreme sensitive parameter dependence described in the paper is not restricted only to systems of globally coupled Hénon maps. We have also investigated the following model spatiotemporal systems: (1) diffusively coupled logistic map lattices, the one most extensively studied in the literature, (2) globally coupled Zaslavsky map lattices, and (3) a system of diffusively coupled ordinary differential equations (coupled Duffing's oscillators) [14]. For all three systems, the same type of sensitive parameter dependence has been observed [17], although details of the nature of asymptotic attractors differ from system to system. For instance, in system (1), with 20 diffusively coupled logistic maps [$f(x) = 4x(1-x)$], when the coupling is between 0.7 and 0.9, there is one chaotic and one quasiperiodic attractor. Within this regime of coupling, the asymptotic attractors of the system exhibit an extreme sensitive dependence on the parameter (coupling strength) similar to Fig. 3(a). For systems (1) and (2), uncertainty exponents have been computed for many choices of the number of coupled maps; they are all near zero, similar to Fig. 3(b). For system (3), while no computation of the uncertainty exponent has been carried out due to the limitation of our computational source, extremely wild oscillations of the largest Lyapunov exponent in certain parameter regimes similar to that of Figs. 2(a) and 3(a) have been observed [17], which suggests a similar type of extreme sensitive parameter dependence. The system of coupled Hénon maps, including systems (1)-(3), are the only four spatiotemporal systems we have examined. Evidence of extreme sensitive parameter dependence in all these four systems suggests that the occurrence of riddled parameter space is a *robust* dynamical phenomenon in spatiotemporal systems.

We remark that chaos in low-dimensional dynamical systems (or temporal chaos) is characterized by a sensitive dependence of system dynamic variables on initial conditions in phase space. The work described herein, as well as that of previous investigators [4], demonstrates that dynamical systems may also exhibit sensitive dependence of asymptotic attractors on system parameters. For the quadratic map, this dependency is rather weak ($\alpha \approx 0.41$) [4,11]. In spatiotemporal chaotic systems such as the coupled Hénon maps studied in this paper, the dependence of asymptotic attractors on parameters is extremely sensitive because the uncertainty exponent is near zero or is in fact zero. Thus, for such systems, we cannot reliably predict the evolution of dynamic variables in phase space, nor can we predict statistical properties of the asymptotic attractors for particular parameter values.

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[17] Y. C. Lai and R. L. Winslow, "Extreme Sensitive Dependence on Parameters and Initial Conditions in Spatiotemporal Chaotic Dynamical Systems," Physica (Amsterdam) D (to be published). This paper includes a system of 10 diffusively coupled Duffing's oscillators. The computation required to identify riddled parameter space for this system is practically impossible on conventional workstations. We have used the massively parallel Connection Machines CM-5 to compute maximum Lyapunov exponents for many parameter values in a parallel fashion.

[18] The hypothesis that a fitted straight line has zero slope can be tested using linear regression theory. See, for example, B. W. Lindgren, *Statistical Theory* (Macmillan, New York, 1976). In particular, for a linear fit $y = \alpha x + b$, there is a confidence interval $[\alpha - k\sigma_y/\sqrt{(n-2)S_x^2}, \alpha + k\sigma_y/\sqrt{(n-2)S_x^2}]$ for the estimated slope α, where σ_y is the standard deviation of the fit, $S_x^2 = (1/n)\sum_i (x_i - \bar{x})^2$, n is the number of data points, $\bar{x}$ is the averaged value of x_i, and $k = \sqrt{F_{1-\gamma}(1, n-2)}$ (the F-distribution function, and $1-\gamma$ is the confidence level). Typically, k increases with $1-\gamma$. If the confidence interval contains 0, then the hypothesis that $\alpha = 0$ can be accepted with confidence level $1-\gamma$ [P. G. Hoel, S. C. Port, and C. J. Stone, *Introduction to Statistical Theory* (Houghton Mifflin, Boston, 1971)].

[19] We have examined systems of coupled maps with N up to 80. As N becomes larger, the computation of Lyapunov and uncertainty exponents becomes computationally intensive.