Crisis in Chaotic Scattering

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We show that in a chaotic scattering system the stable and unstable foliations of isolated chaotic invariant sets can become heteroclinically tangent to each other at an uncountably infinite number of parameter values. The first tangency, which is a crisis in chaotic scattering, provides the link between the chaotic sets. A striking consequence is that the fractal dimension of the set of singularities in the scattering function increases in the parameter range determined by the first and the last tangencies. This leads to a proliferation of singularities in the scattering function and, consequently, to an enhancement of chaotic scattering.

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Chaotic scattering occurs commonly in open Hamiltonian systems [1-3]. In such a case, the scattering function, which represents the dependence of some output variable characterizing the scattering trajectory after the scattering on some input variable characterizing the trajectory before the scattering, displays a Cantor set of singularities. Consequently, arbitrarily small changes in the input variable can result in large changes in the output variable. It has been established that chaotic scattering is due to the existence of nonattracting chaotic invariant sets in the phase space that contains an infinite number of unstable periodic orbits [1-3]. These unstable periodic orbits are intimately related to the set of singularities on a line intersecting the closure of the stable manifold of the periodic orbits embedded in the chaotic invariant set [4] and, hence, they are also closely related to the set of singularities in the scattering function. It has also been found that after the onset of chaotic scattering, further qualitative changes in the chaotic invariant set are possible as a system parameter changes [2]. In this Letter, we present a new phenomenon in chaotic scattering. It is akin to the merging crisis [5] in dissipative chaotic systems. We henceforth call this phenomenon *crisis in chaotic scattering.* Chaotic scattering occurs on both sides of the crisis. Before the crisis, there exist two *topologically and dynamically isolated* chaotic invariant sets in the phase space. As a system parameter changes, the closures of the stable and unstable manifolds of the two chaotic sets first touch each other at the crisis. Since both stable and unstable foliations have Cantor structures, as the parameter is varied further, both foliations pass through each other experiencing an uncountably infinite number of heteroclinic tangencies in the process. The initial tangency, then, corresponds to the crisis and provides the link between the two chaotic invariant sets.

There are two major consequences resulting from this crisis. The first one is that once the crisis has occurred, an uncountably infinite number of new periodic and chaotic trajectories are suddenly created. These trajectories live in the union of the two chaotic sets that existed before the crisis. However, as long as the stable and unstable foliations keep creating tangencies as the parameter varies, the number of periodic and chaotic trajectories keeps increasing. It means that there is an uncountably infinite number of new possibilities for scattering trajectories. The second major consequence is that the fractal dimension of the set of singularities in the scattering function increases during the crisis. We stress that a crisis in chaotic scattering is triggered by an infinite number of tangencies of stable and unstable foliations, which will occur when both foliations have a fractal structure. Such Cantor-like stable and unstable foliations are, however, typical in chaotic scattering systems [2].

In order to illustrate these findings, we consider a system in which particles are scattered from a two-dimensional array of nonoverlapping, elastic scatterers in the plane [3]. These scatterers are placed at constant intervals D along the y axis and each scatterer is represented by a circular attractive potential $V(r)$ that becomes negligibly small for $r > R$, where $R < D/2$. The effect of an individual scatterer on a scattering particle can be characterized by the elastic deflection angle $\Theta(l)$ as a function of the angular momentum l. Note that $\Theta(l)$ vanishes for $l > l_{\max} = uR$ (mass of the particle $=1$) due to the finite range of the potential, where u is the particle velocity in regions where the potential is negligible. Since the system is invariant under time reversal, we have $\Theta(-l) = -\Theta(l) \bmod (2\pi)$. The crisis studied in this paper occurs in an energy range where the scattering is hyperbolic [2,3,6,7]. For $V(r)$ we choose the Woods-Saxon potential which is often used in the context of nuclear physics [8], $V(r) = -V_0/\{1+\exp[(r-R_0)/a]\}$, where $V_0 > 0$, and R_0 and a are constants. At large distance r, $V(r)$ vanishes exponentially. For this system, it is convenient to choose the angular momentum l and angle β as

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dynamical variables [3], where β is the angle of a particle trajectory relative to the $-y$ axis when the particle is in regions where the potential is negligible. In this way, a mapping can be defined that relates (l,β), the dynamical variables of the particle trajectory with respect to a scatterer, to (l',β'), the dynamical variables of the particle with respect to the next scatterer after being scattered from the first scatterer. For $|l'| \le l_{\max}$, the mapping can be explicitly expressed as [3] $\beta' = [\beta + \Theta(l)] \mathrm{mod}(2\pi)$ and $l' = l - (Du)\mathrm{sgn}(\cos\beta')\sin\beta'$. If $|l'| > l_{\max}$, the particle exits the system. In our subsequent numerical experiments, we fix $V_0 = 10$, $R_0 = 0.5$, $\alpha = 0.1$, $D = 4$, and $R = 1.4$. Thus $V(r=R)/V_0 \sim 10^{-4}$ so that adjacent potentials do not appreciably overlap each other. When the particle energy is large $(E \gg 10)$, we observe that the phase space contains KAM surfaces and chaotic regions (nonhyperbolic chaotic scattering) [6]. For $E < 10$, we find numerically that all the KAM surfaces are destroyed and the phase space only contains hyperbolic chaotic invariant sets. Henceforth, we vary E and investigate the scattering behavior of the system for $E < 10$.

The chaotic invariant sets lie in the closure of the intersection of the stable and unstable manifolds. Note that the map has two unstable fixed points: (0,0) (corresponding to a straight trajectory along the $-y$ axis) and $(0,\pi)$ (corresponding to a straight trajectory along $+y$ axis). Numerically, we find that for $E > E_c \approx 4.4$, there exist two *topologically and dynamically isolated* chaotic invariant sets associated with the unstable fixed points. Note that the two chaotic sets must be identical due to the symmetry of the system with respect to $\beta = 0$ (or 2π) and $\beta = \pi$. At the energy value $E_f \approx 4.4$, the stable manifold of one chaotic set becomes heteroclinically tangent to the unstable manifold of the other chaotic set, as shown in Fig. 1(a). At this crisis point, both chaotic sets are dynamically linked and particles initiated near one chaotic set can reach and exit along the unstable manifold of the other chaotic set. As E decreases passing through E_f, the closures of the stable and unstable manifolds of the two chaotic sets heteroclinically cross each other forming additional chaotic sets at the intersection. Since both the stable and unstable foliations of the chaotic sets have Cantor structures before the crisis, there must be an uncountably infinite number of tangencies between E_f and $E_l \approx 4.1$ [corresponding to the last tangency, as shown in Fig. 1(b)].

Physically, the occurrence of crisis for this particular system can be understood as follows. For a fixed particle energy, $\Theta(l)$ is a function of the angular momentum l. At some $l = l_c$, $\Theta(l)$ attains its maximum value $\Theta_{\max}$. It is interesting to note that the locations where the tips of the stable manifolds cross the unstable manifolds in Figs. 1(a) and 1(b) correspond precisely to $\pm l_c$. This maximum deflection angle determines the extent to which a particle trajectory can turn over. The occurrence of crisis entails that a particle trajectory going upward can exit the system downward, and vice versa. This indicates that

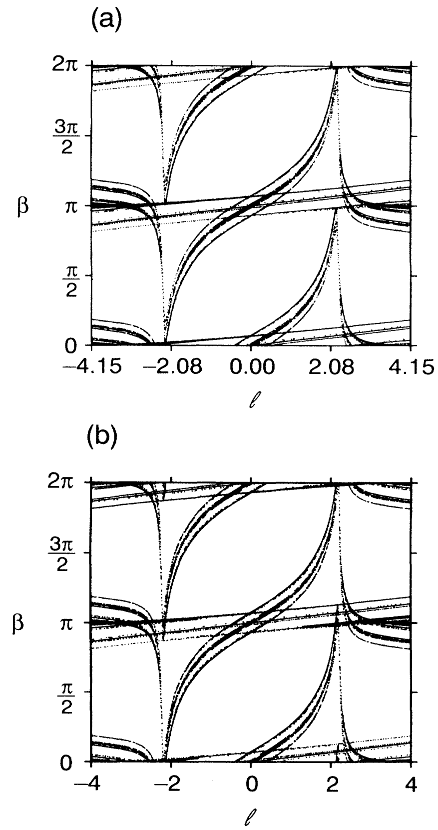

FIG. 1. Stable and unstable foliations for (a) $E = 4.4$ and (b) $E = 4.1$. The unstable foliation is nearly horizontal.

$|\Theta_{\max}|$ is close to π when the crisis occurs. Indeed, as E decreases, $|\Theta_{\max}|$ keeps increasing, as can be verified numerically. For $E \le E^* \approx 3.2$, $|\Theta_{\max}| = \infty$. The situation where $|\Theta_{\max}| = \infty$ is usually called "orbiting" [9].

Generally, the hallmark of chaotic scattering is the existence of a set consisting of an uncountably infinite number of singularities in the scattering function. This set of singularities can be conveniently characterized by its fractal dimension. [Recall that the number of boxes $N(\varepsilon)$ needed to cover a fractal set scales with the size of the box ε as $N(\varepsilon) \sim \varepsilon^{-d}$, where d is the box-counting dimension of the set.] To see the physical consequence of the crisis in chaotic scattering, we compute this fractal dimension d as the energy changes through the crisis. We use the uncertainty algorithm [10] to compute the fractal dimension. It can be shown that the uncertainty dimension is smaller than or equal to the box-counting dimension, and it has been proved that both dimensions are equal for nonattracting hyperbolic chaotic sets of typical dynamical systems [11]. Figure 2 shows the uncertainty dimension d for a Cantor set obtained by fixing $\beta_0 = 2.55$ [see Figs. 1(a) and 1(b)] versus the particle energy E curve. As can be seen from Fig. 2, $d \approx 0.486$ for $E > E_f$. It increases as E decreases from E_f to $E_l \approx 4.1$. For

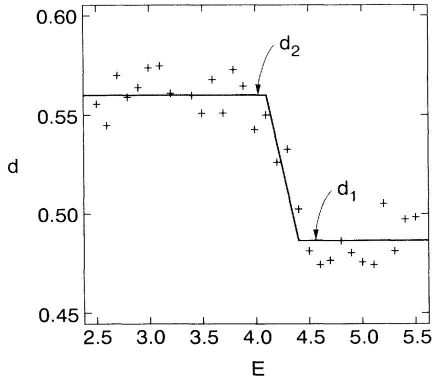

FIG. 2. Fractal dimension of the set of singularities of the scattering function as a function of E obtained by fixing $\beta_0 = 2.55$.

$E < 4.1$, $d \approx 0.560$. The increase in the fractal dimension indicates a proliferation of singularities in the scattering function. Consequently, chaotic scattering is enhanced after the crisis. Changing β_0 will not significantly change the dimension values in Fig. 2, as long as the line cuts through all components of the stable foliations. This has been verified by computing the dimension at various energy values with slightly different β_0. The overall features of Fig. 2 appear therefore to be robust.

The dynamical process observed in our scattering model, including the increase of the fractal dimension of the invariant set after the crisis, can be understood by considering the following piecewise linear one-dimensional model:

$$f_\lambda(x) = \begin{cases} -s|x+1|+\lambda, & x<0\,, \\ s|x-1|+\lambda, & x\geq 0\,, \end{cases} \tag{1}$$

where $s > \frac{1}{2}[3+\sqrt{9+4(\lambda-1)}]$ and $\lambda > -1$. For $-1 < \lambda < \lambda_f = s/(s+2)$, there are two isolated invariant sets. These are the "middle α" Cantor sets with $\alpha = (s-2)/s$, so the box-counting dimension of each one of these isolated invariant sets is $\ln 2/\ln s$. At $\lambda = \lambda_f$, the first tangency occurs, as shown in Fig. 3(a), which provides a link between both Cantor sets. Hence, for $\lambda > \lambda_f$, the invariant set of the map is the union of the two previously isolated Cantor sets plus the set created due to the linking of the two Cantor sets. At $\lambda_l = s/(s-2)$, the last tangency occurs, as shown in Fig. 3(b). For $\lambda > \lambda_l$, the large invariant Cantor set is self-similar and has box-counting dimension $2\ln 2/\ln s$. Hence, the graph of the fractal dimension of the invariant set has two constant pieces and the region $\lambda_f \leq \lambda \leq \lambda_l$ where the dimension increases from the lower constant value to the higher constant value. In fact, in this transition region the dimension can be shown to be nondecreasing.

Although the map for the scattering system is smooth and two dimensional, while our model system has discontinuities in the derivative and is one dimensional, our model captures the essential features of the scattering system. For instance, although the tangencies are quadratic for the scattering system, the stable manifold exhibits a very large curvature at the tangency. Both stable and unstable manifolds away from the tangency points are almost straight lines. Therefore our one-dimensional model is a good dynamic representation of the scattering system at the heteroclinic crossing. The factor of 2 increase in the dimension could be decreased by changing slightly the local slope of the map, and, thus, the analytic results would be even closer to the numerical results in Fig. 2. In our scattering system, Fig. 2 indicates that the d versus E curve during the crisis is nondecreasing. It is, however, hard to quantify this explicitly because of the difficulty in computing the uncertainty dimension, which involves numerical integration of deflection angles $\Theta(l)$.

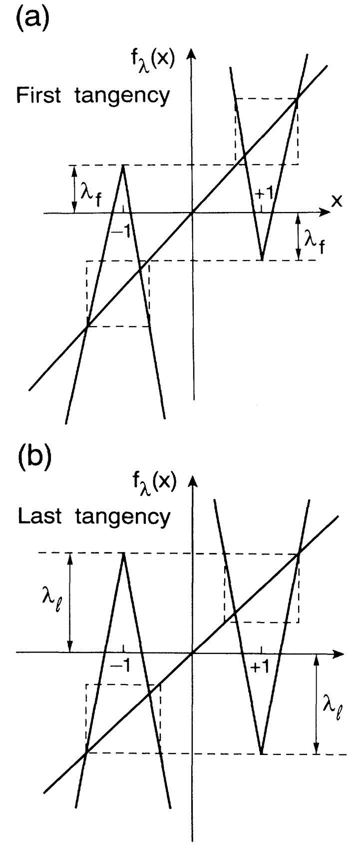

FIG. 3. One-dimensional model exhibiting heteroclinic crossing of two fractal invariant sets. (a) The first tangency at λ_f and (b) the last tangency at λ_l.

In summary, we use a scattering potential of the Woods-Saxon type which has the advantage of realistically capturing the energy dependence of quantum phase shifts [7] and classical scattering functions. Most importantly, the Woods-Saxon potential shows a new dynamical phenomenon (crisis), as shown above. This is due to the existence of nontrivial hyperbolic scattering in the "channeling" regime. By channeling we denote a scattering situation in which scattering trajectories going upwards (downwards) will never turn around and *scatter anew* with a scattering potential below (above) from which it came in the first place. Numerically, for $E \geq 5$, channeling exists, which can be easily verified by calculating the scattering function. Most importantly, the scattering is chaotic for the same energy value. The existence of both channeling and hyperbolic scattering (hyperbolic channeling) is a necessary condition for a crisis in chaotic scattering to occur because it provides two dynamically decoupled sets which merge at the critical point $E = E_f$.

We argue the generality of the phenomenon of crisis in chaotic scattering by using a simple one-dimensional model. While the stable and unstable manifolds associated with the chaotic invariant sets in the two-dimensional physical scattering system look rather special, namely, the stable manifolds have a sharp bend and the unstable manifolds are almost straight, the condition for a crisis to occur is independent of these specific geometrical shapes of the manifolds. As we have discussed in the simple one-dimensional model, crisis and the subsequent increase in the fractal dimension *depend only on the occurrence of an infinite number of heteroclinic tangencies of the stable and unstable foliations.* Since fractal stable and unstable foliations are a general feature of chaotic scattering systems [2], we expect crisis in chaotic scattering to be typical.

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[9] For a circular potential, orbiting means that at a certain angular momentum $l_{\rm crit}$, the particle trajectory exhibits near zero radial motion in the potential and hence, the particle circulates in the potential for an infinite number of times. The situation of orbiting occurs at specific values of energy and angular momentum which are determined by (i) $dr/dt|_{r=r_{\rm crit}} = \sqrt{2[E - V_{\rm eff}(r_{\rm crit})]} = 0$ and (ii) $dV_{\rm eff}(r)/dr|_{r=r_{\rm crit}} = 0$, where $V_{\rm eff}(r) \equiv V(r) + l^2/2r^2$ is the effective potential. The latter condition distinguishes "orbiting" from a turning point of a particle trajectory, which only requires $dr/dt = 0$. Orbiting is possible if $V_{\rm eff}(r)$ possesses a local quadratic maximum at some $r = r_{\rm crit}$. Therefore, $\Theta(l) \to -\infty$ for $l \to l_{\rm crit}$. For values of angular momenta in the neighborhood of $l_{\rm crit}$, deflection angles are no longer $-\infty$ but still assume large absolute values. Orbiting is an important phenomenon in scattering theory in general. For its importance in a special subdiscipline, for instance in nuclear physics, see, e.g, W. Nörenberg and H. A. Weidenmüller, *Introduction to the Theory of Heavy Ion Collisions* Lecture Notes in Physics Vol. 51 (Springer-Verlag, Berlin, 1976).

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