Lai *et al.* Reply We reported in [1] a mechanism for noise-induced variations of statistical (or trajectory) averages in chaotic systems. This is essentially the same mechanism responsible for noise-induced chaos [2]. We found that in a *physical* situation noise-induced variations of statistical averages obey a general algebraic scaling law. The basic observation is that, in any finite computational or observational time, dynamical invariant sets begin to connect only when noise exceeds a critical level, leading to observable variations of averages. Our scaling law characterizes how the averages vary with the noise level beyond the threshold. The universality arises because the underlying dynamical mechanism does not depend on the specific physical function for the average.

The author [3] presented a simple linear map to show that averages such as the second moment $\langle x^2 \rangle$ depend on the noise amplitude continuously without any threshold. Such a dependence is, however, not general as for a different physical function, the dependence is completely different. For instance, the average $\langle x \rangle$ is simply a constant and has no dependence on the noise amplitude. This type of nonuniversal variation with noise is thus unrelated to our scaling law. More worrisomely, the use of a linear map may be inappropriate because the dynamics we studied is exclusively nonlinear and it has no counterpart in linear systems. In particular, the scaling law arises because of the *nonlinear dynamics* involved in the noise-induced process, which is the dynamical connection between a periodic attractor and a chaotic saddle. This connection is possible only for a noise level above the threshold, where a trajectory spends most of the time near the periodic attractor but with intermittent excursions to the chaotic saddle. This nonlinear dynamical process is *independent* of the system details and should be valid for *any* periodic window. Thus the resulting scaling law is universal.

For Gaussian noise, if one is allowed an infinite amount of computational or experimental time, the two sets will connect for arbitrarily weak noise. This is similar to the situation where, under Gaussian noise in the infinite-time limit, no attractor in a finite phase-space region and its basin of attraction can be defined. The point is, for *finite time,* such a threshold can be defined in an *ad hoc* but physically meaningful manner. In fact, the existence of such a threshold has been well documented in both the physics and mathematics literature [2,4,5]. In essence, the controversy boils down to the issue of whether infinite amplitude events can occur for a Gaussian random process (mathematically yes, but physically no).

To define a threshold for a finite physical time, we note that under Gaussian noise of normalized amplitude D, the steady-state probability distribution for point $\mathbf{x}$ on the attractor can be written as [4] $W(\mathbf{x}) \sim Z(\mathbf{x}) e^{-\Phi(\mathbf{x})/D^2}$, where the prefactor $Z(\mathbf{x})$ is similar to the form describing fluctuations in thermal equilibrium and $\Phi(\mathbf{x})$ is analogous to the free energy. While the explicit form of $Z(\mathbf{x})$ and $\Phi(\mathbf{x})$ cannot be obtained from basic principles, the interesting feature is that they are *both independent of the noise strength.* This allows for a proper threshold to be defined [2]. For example, for parameter value p immediately above the saddle-node bifurcation point p_s initiating a period-m window, *for Gaussian noise* a threshold exists and scales with the parameter variation as $D_c \sim (p - p_s)^{3/4}$ [2]. We stress that D_c depends on the probability resolution, and therefore it is defined with respect to a given computational or experimental time. Our scaling law of trajectory averages and the associated scaling exponent should be interpreted with respect to this time.

That a noise threshold exists physically can be seen from a different angle. Consider the maximum Lyapunov exponent for flow, which is a trajectory average. In a periodic window, in the absence of noise its value is zero. Under arbitrarily small noise the attractor remains nonchaotic within any physical time. The exponent becomes positive only when noise exceeds a threshold. The author's points in (ii)–(iv) suggest that this exponent can in principle be positive, no matter how small the noise amplitude is. This implies that, in dynamical systems, under noise nonchaotic attractors would not exist, which is not reasonable. The key point here is again the finite observational time allowed in any physical situation.

Concerning the terminology issue with respect to the word "shadowing," we agree that its use may be improper in the context studied.

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