Superpersistent Chaotic Transients in Physical Space: Advective Dynamics of Inertial Particles in Open Chaotic Flows under Noise

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Superpersistent chaotic transients are characterized by an exponential-like scaling law for their lifetimes where the exponent in the exponential dependence diverges as a parameter approaches a critical value. So far this type of transient chaos has been illustrated exclusively in the phase space of dynamical systems. Here we report the phenomenon of noise-induced superpersistent transients in physical space and explain the associated scaling law based on the solutions to a class of stochastic differential equations. The context of our study is advective dynamics of inertial particles in open chaotic flows. Our finding makes direct experimental observation of superpersistent chaotic transients feasible. It also has implications to problems of current concern such as the transport and trapping of chemically or biologically active particles in large-scale flows.

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A superpersistent chaotic transient is characterized by the following scaling law for its lifetime:

$$\tau \sim \exp(\alpha|p - p_c|^{-\gamma}), \tag{1}$$

where p is a system parameter, $\alpha > 0$ and $\gamma > 0$ are constants. As p approaches the critical value p_c, the transient lifetime τ becomes superpersistent in the sense that the exponent in the exponential dependence diverges. This type of chaotic transient was first conceived to occur through the dynamical mechanism of unstable-unstable pair bifurcation, in which an unstable periodic orbit in the boundary of a chaotic invariant set coalesces with another unstable periodic orbit preexisted outside the set [1]. The same mechanism was shown to cause the riddling bifurcation [2]. The transients were also identified in a class of coupled-map lattices, leading to the speculation that asymptotic attractors may not be relevant for turbulence [3]. The mathematical models [1,2] used to analyze superpersistent chaotic transients were discrete-time maps. Recently, this type of transient chaos was demonstrated in systems described by differential equations [4], making *indirect* experimental observation of the transient possible [5]. In all the existing works, superpersistent chaotic transients occur in the *phase spaces* of dynamical systems.

In this Letter, we present evidence of *noise-induced* superpersistent chaotic transients in the *physical, or configuration space*. The context of our study is advection of particles with inertia in open chaotic flows. It has been known that ideal particles with zero mass and size simply follow the velocity of the flow and, as such, the advective dynamics can be described as Hamiltonian [6,7] in the physical space for which chaos can arise but not attractors. In an open Hamiltonian flow, ideal particles coming from the upper stream must necessarily go out of the region of interest in finite time. However, the inertia of the advective particles can alter the flow locally [8]. As a result, the underlying dynamical system becomes dissipative for which attractors can arise and, hence, particles can be trapped permanently in some region in the physical space [9]. This phenomenon has been demonstrated recently in a model of two-dimensional flow past a cylindrical obstacle [10]. As the authors of Ref. [10] pointed out, this result has implications in environmental science where forecasting aerosol and pollutant transport is a basic task, or even in homeland defense where the spill of a toxin or biological pathogen in large-scale flows is of critical concern. The possibility that toxin particles can be trapped in physical space is particularly worrisome. We are thus motivated to study the structural stability of such attractors [11]. In particular, we ask, can chaotic attractors so formed be persistent under small noise? We find that, in general, the attractor is destroyed by small noise and replaced by a chaotic transient, which is typically superpersistent as characterized by the scaling law (1), with the parameter variation $|p - p_c|$ replaced by the noise amplitude. For small noise, the extraordinarily long trapping time makes the transient particle motion practically equivalent to an attracting motion with similar physical or biological effects. Our finding suggests a way to directly observe superpersistent chaotic transients in laboratory experiments [12].

For an ideal, passive particle of zero inertia and zero size advected in a flow, the particle velocity $\mathbf{v}$ is the flow velocity $\mathbf{u}$ which, in a two-dimensional physical space, is determined by a stream function $\Psi(x, y, t)$: $u_x = \partial\Psi/\partial y$ and $u_y = -\partial\Psi/\partial x$. The dynamical system in the five-dimensional phase space is thus conservative (Hamiltonian). For particles of finite size, viscous friction arises and, as such, their velocities differ from those of the fluid. Consider a spherical particle of radius a and mass m_p, and fluid of dynamic viscosity μ and element

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mass m_f, the equation of motion of the advective particle is [8], $m_p d\mathbf{v}/dt = m_f d\mathbf{u}/dt - (m_f/2)(d\mathbf{v}/dt - d\mathbf{u}/dt) - 6\pi a\mu(\mathbf{v} - \mathbf{u})$, where on the right-hand side, the first term is the fluid force from the undisturbed flow field, the second term is the force due to the added mass effect, and the third represents the Stokes drag. While in principle, the fluid velocity $\mathbf{u}$ is disturbed by the particle motion, if the particle sizes are relatively small and their concentration is low, $\mathbf{u}$ can be considered as unchanged [10]. For convenience, one can introduce the mass ratio parameter $R = 2\rho_f/(\rho_f + 2\rho_p)$ and the inertial parameter $A = R/[\frac{2}{9}(a/L)^2 R_e]$, where ρ_p and ρ_f are the densities of the particle and the fluid, respectively, L is a typical large-scale mixing length, and R_e is the Reynolds number ($R_e \equiv UL/\nu$, where U is a typical large-scale velocity and ν is the kinematic viscosity of the fluid). The equation of motion can then be cast into a dimensionless form. To simulate random forcing due to the flow disturbance or other enviromental factors, we add terms $\epsilon\xi_x(t)$ and $\epsilon\xi_y(t)$ to the force components in the x and y directions, where $\xi_x(t)$ and $\xi_y(t)$ are independent Gaussian random variables of zero mean and unit variance, and ϵ is the noise amplitude. The final equation of motion under random perturbations is $d\mathbf{v}/dt - (3R/2)d\mathbf{u}/dt = -A(\mathbf{v} - \mathbf{u}) + \epsilon\boldsymbol{\xi}(t)$, where $\boldsymbol{\xi}(t) = [\xi_x(t), \xi_y(t)]^T$. Inertial particles are *aerosols* if $0 < R < 2/3$ and they are *bubbles* if $2/3 < R < 2$. The limit $A \to \infty$ corresponds to the situation of ideal particles (passive advection).

Following Ref. [10], we use the open flow model of the von Kármán vortex street in the wake of a cylinder of radius r, located at $(x, y) = (0, 0)$, where a time-periodic stream function $\Psi(x, y, t)$ (period $T_f = 1$ in a standard dimensionless form) governing the motions of vortices in a background flow of velocity u_0 can be constructed explicitly from the solutions of the two-dimensional viscous Navier-Stokes equations for the geometry of a circle of radius r in the middle of an infinite channel of width $w = 4r$ [16]. The Reynolds number is $R_e \approx 250$. The flow velocity $\mathbf{u}(x, y, t)$ can be obtained from $\Psi(x, y, t)$, allowing the particle motions to be computed [17].

It has been shown in Ref. [10] that attractors can be formed in the bubble regime. We then focus on this regime and fix $R = 1.47$ and $A = 30$. There are three attractors [10]: two chaotic and one at $x = \infty$. The chaotic attractors are located near the cylinder (but not stuck on it): one in $y > 0$ and another in $y < 0$. To gain insight as to what might happen to the attractors under noise, we examine the basins of attraction of these attractors. To do so we choose a 1000×1000 grid of initial conditions in the region $[-2.0 \le x(t_0) \le 1.5, -1.5 \le y(t_0) \le 1.2]$ covering the cylinder, and set the initial velocities to be $v_x(t_0) = u_x(x, y, t_0)$ and $v_y(t_0) = u_y(x, y, t_0)$, and then compute toward which attractor every initial particle is attracted. Figure 1 shows the basins of attraction of the two chaotic attractors (light blue and yellow, respectively), where the blank region denotes the basin of the attractor at infinity. Note that the phase space is five dimensional, so what is shown in Fig. 1 is in fact a two-dimensional slice of the basin structure in the full phase space, which corresponds to the physical space. Near the cylinder, the basin boundaries among the three attractors are apparently fractal [18]. Because of the explicit time dependence in the stream function and therefore in the flow velocities, the attractors and their basins move oscillatorily around the cylinder. The remarkable feature is that in the physical space, there are time intervals during which the attractors come close to the basin boundaries. Thus, under noise, we expect permanently trapped motion on any one of the two chaotic attractors to become impossible. In particular, particles can be trapped near the cylinder, switching intermittently on the two originally chaotic attractors, but this can last only for a finite amount of time: eventually all trajectories on these attractors escape and approach the $x = \infty$ attractor. That is, chaos becomes transient under noise [20].

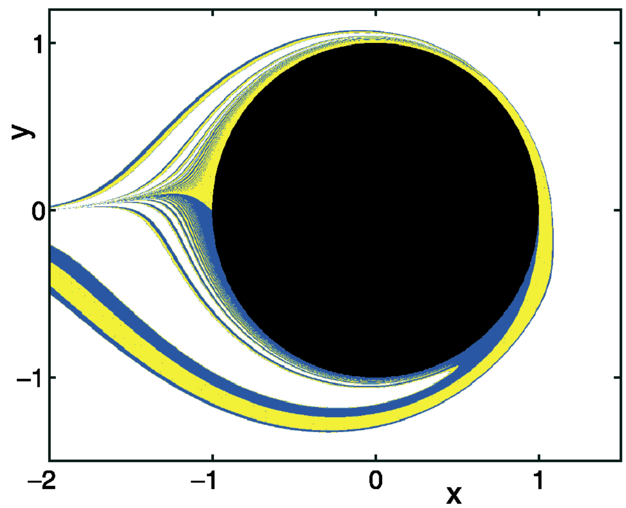

FIG. 1 (color). Basins of attraction of two chaotic attractors (light blue and yellow, respectively) in the absence of noise. The blank region denotes the basin of the attractor at $x = \infty$.

To understand the nature of the noise-induced transient chaos, we distribute a large number of particles in the original basins of the chaotic attractors, and examine the channel(s) through which they escape to the $x = \infty$ attractor under noise. Figures 2(a)–2(c) show, for three instants of time (t, $t + T_f/4$, and $t + T_f/2$, respectively), locations of an ensemble of particles in the physical space. Because of the symmetry of the flow [16], the particle trajectories at t and $t + T_f/2$ are symmetric to each other with respect to the x axis, as can be seen from Figs. 2(a) and 2(c). While there are particles still trapped in the original attractors, many others are already away from the cylinder. Since this is a two-dimensional projection of a five-dimensional dynamics, some fractal-like features overlap. The channels through which they escape are a set of thin openings surrounding the cylinder and extending to one of the vortices in the flow. After wandering near

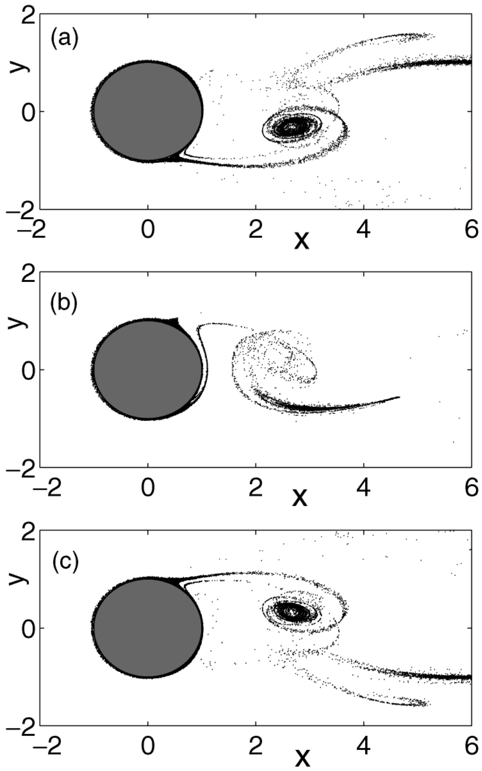

FIG. 2. (a)–(c) At three different instants of time, $T_f/4$ apart, locations of the temporally trapped and escaping particles in the physical space.

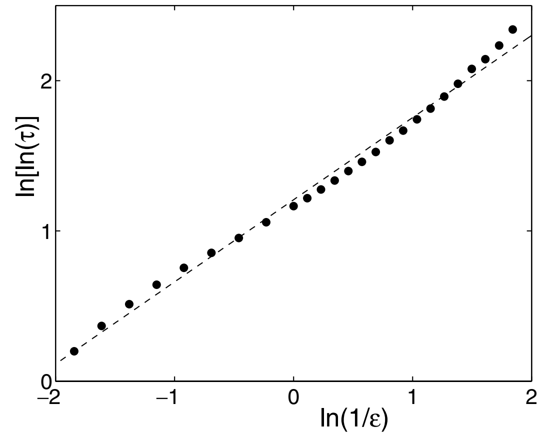

FIG. 3. Scaling of the average lifetime of the trapped chaotic particles versus the noise amplitude.

the vortex, particles go to the $x = \infty$ attractor. Because of the time-dependent nature of the flow, in the physical space the locations of these openings vary in time, but the feature that they are narrow is common. For a fixed noise amplitude, numerically we find that the lifetimes of the particles near the cylinder obey an extremely slow, exponentially decaying distribution, from which the average lifetime τ is obtained. Figure 3 shows τ versus the noise amplitude ϵ on a proper scale. A least-squares fit gives $\tau \approx \exp[3.3\epsilon^{-0.55}]$. Note that for $\epsilon = 0$, there is an attracting motion so that τ diverges. Figure 3 suggests, however, the way that τ diverges follows the superpersistent transient scaling law as ϵ is decreased (mathematically, $\tau \to e^{+\infty}$ as $\epsilon \to 0$).

We now provide a physical theory for the scaling of noise-induced superpersistent chaotic transients. Previous works suggest unstable-unstable pair bifurcation as the generic mechanism for the transients [1,2]. One can imagine two unstable periodic orbits of the same periods, one on the chaotic attractor and another on the basin boundary. In a noiseless situation, as a bifurcation parameter passes through a critical value, the two orbits *coalesce* and disappear simultaneously, leaving behind a narrow "channel" in the phase space through which trajectories on the chaotic attractor can escape. In our flow problem, transient chaos is induced by noise. The closeness of the attractor to the basin boundary implies that noise can induce an unstable-unstable pair bifurcation, creating a narrow, escaping channel. For small noise, the probability for the channel to open is small. Particles can move, however, to the location of the channel and remain there for a finite amount of time to escape through the channel while it is open. Suppose on average, it takes time T for a particle to travel through the channel. We expect T to increase as the noise amplitude ϵ is decreased, because the probability for channel to remain open is smaller for weaker noise. In fact, as we will argue below, we expect T to increase at least algebraically as ϵ is decreased and, $T \to \infty$ as $\epsilon \to 0$.

Suppose the largest Lyapunov exponent of the chaotic attractor is $\lambda > 0$. After an unstable-unstable pair bifurcation the opened channel is locally transverse to the attractor. In order for a trajectory to escape, it must spend at least time T at the location of the opening on the attractor. The trajectory must come to within distance of about $\exp[-\lambda T(\epsilon)]$ from the location of the channel. The probability for this to occur is proportional to $\exp[-\lambda T(\epsilon)]$. The average time for the trajectory to remain on the attractor, or the average transient lifetime, is thus $\tau \sim \exp[\lambda T(\epsilon)]$. To obtain the dependence of T on ϵ, we consider a small region about the "root" of the channel, or the location of the mediating periodic orbit. Let $\mathbf{q}$ and $\mathbf{p}$ be the local coordinates on the attractor and in the channel, respectively. We consider the following model: $d\mathbf{q}/dt = \mathbf{f}(\mathbf{q})$ and $d\mathbf{p}/dt = \epsilon\boldsymbol{\xi}(t) + g(\mathbf{q})\mathbf{p} + \mathbf{h}(\mathbf{p})$, where $\mathbf{f}(\mathbf{q})$ is a vector field that generates a chaotic attractor, $g(\mathbf{q}) < 0$ for $\mathbf{q} \neq \mathbf{0}$, $g(\mathbf{0}) = 0$, $\epsilon\boldsymbol{\xi}(t)$ is noise, and the lowest order of the function $\mathbf{h}(\mathbf{p})$ is $\mathbf{p}^2$. For $\epsilon = 0$, we have $\mathbf{p}(t) \to \mathbf{0}$ for $\mathbf{p}$ small so that the channel is closed and no trajectory can escape.

In order to construct a model that captures the essential transient dynamics, which at the same time is amenable to analysis, we assume that the escaping channel is approximately one dimensional and the length of the channel is $l \gg \epsilon$. This one-dimensional picture can be justified for typical cases where the periodic orbit at the opening of the channel in the original attractor is strongly unstable in the direction of channel so that the escaping dynamics in the channel is approximately one dimensional. Once a trajectory on the chaotic attractor falls into the opening of the channel, i.e., $\mathbf{q} = \mathbf{0}$, its motion is governed by a stochastic differential equation of the form $dz/dt = h(z) + \epsilon\xi(t)$. The probability density function $\phi(z, t)$ of the stochastic process obeys the Fokker-Planck equation $\partial\phi/\partial t = -\partial[h(z)\phi]/\partial z + (\epsilon^2/2)\partial^2\phi/\partial z^2$, where ϵ^2 is the diffusion coefficient. For z small we consider the lowest order of $h(z)$ and write $h(z) \approx az^{k-1}$, where $a > 0$ and $k \geq 3$. The average time required for a trajectory to travel through the channel is roughly the mean first passage time [21], $T = (2/\epsilon^2) \int_0^l dy \exp[-H(y)/\epsilon^2] \times \int_0^y \exp[H(z)/\epsilon^2]dz$, where $H(z) = \int 2h(z)dz = 2az^k/k$, $\int_0^y \exp[H(z)/\epsilon^2]dz = \sum_{n=0}^{\infty} b^n y^{kn+1}/[n!(kn+1)]$, and $b = (2a)/(k\epsilon^2)$. We otain $T = (2/\epsilon^2) \int_0^l \sum_{n=0}^{\infty} \{(by^k)^n/[n!(kn+1)]\} y \exp(-by^k)dy \lesssim \epsilon^{-2+4/k} \sum_{n=0}^{\infty} [n!(kn+1)k]^{-1}\Gamma(n+2/k)$. It can be shown [22] that the infinite series converges. We thus have $T \lesssim \epsilon^{-2+4/k}$ which, when substituted into $\tau \sim \exp[\lambda T(\epsilon)]$, gives the scaling law (1) that is characteristic of superpersistent chaotic transients. We see that the exponent is $\gamma = 2 - 4/k$. In general, we have $0 < \gamma < 2$.

In summary, we have reported superpersistent chaotic transients in physical space, where particles of finite inertia are advected in noisy flow with a cylindrical obstacle, behind which vortices form. We have presented numerical results and analysis for the noisy scaling law that defines the transient behavior. A similar flow setting was used to demonstrate chaotic scattering [15] in laboratory experiments, and we believe this could be a candidate for direct experimental observation of superpersistent chaotic transients in physical space. Another candidate is the experimental system studied in Ref. [23]. The possibility that realistic particles with inertia can be trapped in regions behind structures for long time can be of serious concern if the particles are chemically or biologically active.

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