Experimental Characterization of Transition to Chaos in the Presence of Noise

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Transition to chaos in the presence of noise is an important problem in nonlinear and statistical physics. Recently, a scaling law has been theoretically predicted which relates the Lyapunov exponent to the noise variation near the transition. Here we present experimental observation of noise-induced chaos in an electronic circuit and obtain the fundamental scaling law characterizing the transition. The experimentally obtained scaling exponent agrees very well with that predicted by theory.

An important problem in nonlinear and statistical physics is to understand the transition to chaos in dynamical systems under the influence of noise [1–15]. The problem is fundamental because it concerns the interplay between deterministic and stochastic dynamics. Some pioneering works in this direction are the following. The effect of noise on period-doubling transition to chaos was studied by Crutchfield *et al.* [1,2], where a renormalization-group approach was used to analyze the scaling behavior of the Lyapunov exponent near the transition [2]. The effect of noise on type-I intermittency was investigated by Hirsch *et al.* [3]. The influence of noise on periodic attractors for the Lorenz system was studied by Fedchenia *et al.* [11]. Noise-induced chaos in a system with homoclinic points was discussed by Anishchenko and Herzel [5], and the opposite phenomenon of noise stabilization of chaotic dynamics was studied by Herzel [6]. The problem of noise-induced chaos also has similarities with the problem of noise activation of excitable systems [12].

Transition to chaos in the presence of noise is important and relevant to problems in, for instance, laser physics [14] and biology [15]. Often, the following questions are asked: Suppose the system is originally in a nonchaotic state and it becomes chaotic under the influence of noise, what are the characteristic features of the transition to chaos? Are these features universal (in the sense that they can be observed in different systems regardless of the system details)?

Recently, the above questions have been addressed theoretically and numerically [16] in the general setting where a periodic attractor coexists with a nonattracting chaotic invariant set (chaotic saddle), as can be expected in any periodic window of a nonlinear dynamical system. Under sufficiently large noise, the attractor of the system becomes chaotic, which is characterized by the appearance of a positive Lyapunov exponent. It is argued [16] that, near the transition, the largest Lyapunov exponent λ_1 obeys the following scaling law with respect to noise variations:

$$\lambda_1 \sim (D - D_c)^\alpha, \tag{1}$$

where D is the noise amplitude, D_c is the critical amplitude at which the transition to chaos occurs (i.e., the attractor of the system is not chaotic for $D < D_c$ but chaotic for $D > D_c$), and $\alpha > 0$ is the scaling exponent determined by the properties of the coexisting chaotic saddle. In particular, for three-dimensional flows (or equivalently two-dimensional Poincaré maps), the exponent is given by

$$\alpha = 1 - \frac{1}{2\lambda_1^S \tau}, \tag{2}$$

where $\lambda_1^S > 0$ and τ are the positive Lyapunov exponent and the lifetime of the chaotic saddle, respectively, in the absence of noise. The scaling law (1) is subsequently argued to hold for *any* dimension, and explicit expressions for the scaling exponent have been obtained in all dimensions [17]. At a fundamental level, the dynamical mechanism underlying the transition can be related to the phenomenon of noise-induced unstable dimensional variability [16,17], a severe type of nonhyperbolicity that occurs commonly in multidimensional chaotic systems [18].

The purpose of this Letter is to provide direct experimental evidence for noise-induced chaos and the scaling law (1) associated with the transition. By utilizing a nonlinear electronic circuit, the Chua's circuit [19], we are able to obtain experimentally the algebraic scaling law over about 1.5 orders of the magnitude of the noise variation, with the scaling exponent in excellent agreement with the theoretical prediction.

We emphasize that, in principle, the nonlinear element (resistor) in the Chua's circuit has a piecewise-linear current-voltage relation [19]. The circuit is therefore a nonsmooth dynamical system, whose bifurcation scenarios are quite different from those in smooth systems. Nonetheless, in actual implementation, the theoretical discontinuity in the derivative of the current-voltage relation is typically smoothed out due to the finite-time

response behavior in the operational amplifiers that constitute the nonlinear resistor. Thus, the experimental circuit is suitable for testing the noise-scaling law observed in smooth dynamical systems.

We begin by describing briefly the theoretical argument that leads to the scaling law (1). Assuming that, in the absence of noise, there exist a periodic attractor and a coexisting chaotic saddle on a two-dimensional Poincaré plane. Under noise of amplitude D, the periodic attractor can be found in a disk of radius D around the original attractor. For small noise ($D < D_c$), there is no overlap between the disk and the stable manifold of the chaotic saddle, so the attractor of the system is nonchaotic. For $D > D_c$, a subset of the stable manifold of the chaotic saddle overlaps with the disk, giving rise to a nonzero probability that a trajectory near the periodic attractor is kicked out of the disk and moves toward the chaotic saddle along its stable manifold. Since the chaotic saddle is nonattracting, the trajectory will stay in its vicinity for only a finite amount of time before escaping along its unstable manifold. The trajectory will then enter the disk containing the original periodic attractor again, and so on. For $D \gtrsim D_c$, the probability for the trajectory to leave the disk is small, leading to an intermittent behavior where the trajectory spends long stretches of time near the original periodic attractor, with occasional bursts out of it, wandering near the chaotic saddle.

Now consider the Lyapunov spectrum of the noisy attractor of the three-dimensional flow. Let $\lambda_3^P \leq \lambda_2^P < \lambda_1^P = 0$ and $\lambda_3^S < \lambda_2^S = 0 < \lambda_1^S$ be the Lyapunov spectra of the periodic attractor and of the chaotic saddle, respectively, in the absence of noise. Let $\lambda_3 < \lambda_2 < \lambda_1$ be the Lyapunov spectrum of the noisy attractor. For $D < D_c$, the noisy attractor is only a fattened version of the original periodic attractor. Thus, we have $\lambda_i = \lambda_i^P$ ($i = 1, 2, 3$) and, in particular, $\lambda_1 = \lambda_1^P = 0$, indicating that the attractor is nonchaotic, in spite of the noise. For $D > D_c$, there is an intermittent hopping of the trajectory between regions that contain the original periodic attractor and the chaotic saddle. Let f_P and f_S be the fractions of time that the trajectory spends in the corresponding regions asymptotically. We have $\lambda_1 \approx f_P \lambda_1^P + f_S \lambda_1^S = f_S \lambda_1^S > 0$, indicating that the intermittent attractor is now chaotic. To obtain the scaling law (1), note that the probability f_S is proportional to the natural measure of the stable manifold of the chaotic saddle in the disk containing the original periodic attractor, which is determined by the dimension of the stable manifold. For a two-dimensional disk of size ϵ on a Poincaré plane, the natural measure of the stable manifold in it [20] is $\epsilon^{D_s} = (\epsilon^2)^{D_s/2}$, where ϵ^2 is proportional to the area of the disk, D_s is given by $D_s = 2 - 1/(\lambda_1^S \tau)$, and τ is the lifetime of the chaotic saddle of the Poincaré map (τ is thus in the unit of T, the average time that a typical trajectory crosses the Poincaré section). For $D \gtrsim D_c$, the area in which the stable manifold of the chaotic saddle overlaps with the disk is propor-

tional to $(D^2 - D_c^2)$. We thus have $f_S \sim (D^2 - D_c^2)^{D_s/2} \sim (D - D_c)^{1 - 1/(2\lambda_1^S \tau)}$, which is the scaling law (1).

The above consideration can be readily extended to all dimensions, yielding similar scaling laws of the largest Lyapunov exponent with respect to the noise [17]. Dimension formulas for chaotic saddles in high dimensions [21] can be utilized to give explicit expressions for the scaling exponents. It can also be argued [16,17] that an intermittent chaotic trajectory for $D \gtrsim D_c$ necessarily possesses unstable dimension variability, rendering the corresponding noisy chaotic attractor severely nonhyperbolic. Consequently, the chaotic attractor of the noisy flow possesses no neutral direction, in sharp contrast to attractors in deterministic flows or nonchaotic attractors in noisy flows [16,17].

Our experimental system is shown in Fig. 1, where 1(a) is the Chua's circuit [19] and 1(b) is the noise divider that we design to obtain controllable noisy signals with fine resolution in amplitude from a commercial noise generator. The differential equations describing the noiseless Chua's circuit are $C_1 dV_{C1}/dt = G(V_{C2} - V_{C1}) - g(V_{C1})$, $C_2 dV_{C2}/dt = G(V_{C1} - V_{C2}) + i_L$, and $L di_L/dt = -V_{C2}$, where $G = 1/R$ and $g(\cdot)$ is the following piecewise-linear function: $g(x) = m_0 x + (m_1 - m_0)[|x + B_p| - |x - B_p|]/2$, and m_0, m_1, and B_p are parameters. The
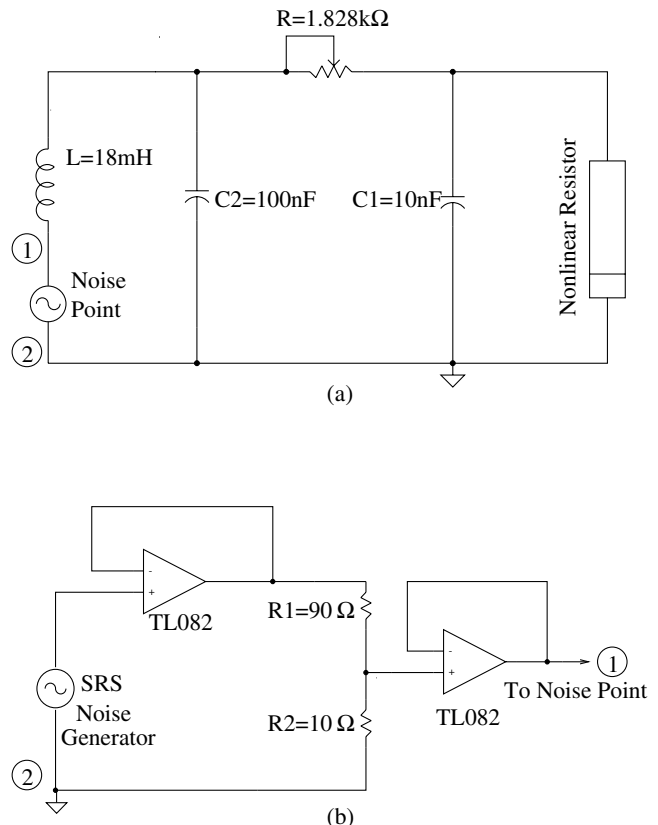

FIG. 1. Our experimental system: (a) Chua's circuit; (b) noise divider.

nonlinear resistor and the noise divider are implemented on two integrated circuit chips, each containing two operational amplifiers (JM08AK TL082 CP), with properly selected resistors. The whole circuit (including the noise divider) is soldered on a high-quality circuit board (Vector Electronic Co.). The TL082 chips are powered by a low-ripple dc power supply (Hewlett Packard E3631A) of ± 9 V. Gaussian white noise with variable amplitude is provided by a noise generator (SRS DS345 with output resistance of 50 Ω). We choose the resistance R to be the bifurcation parameter to set the system in a period-3 window of relatively large size in the parameter space so that, in the absence of the controllable noise, the system can be maintained stably in the window, in spite of the presence of the uncontrollable noises in the laboratory. Voltage signals from the capacitors C_1 and C_2 are recorded using a 12-bit data acquisition system (KPCI3110, Keithley) at the sampling rate of 200 kHz. The standard routine for computing the Lyapunov exponents from time series [22] is utilized. For a reliable computation of the exponents, roughly the amount of data required is on the order of $d(W/\delta)^{d_1}$, where d is the embedding dimension, d_1 is the information dimension of the attractor, W is the size of the attractor in the phase space, and δ is the minimal distance between two points in the reconstructed phase space. To satisfy this requirement while at the same time having a manageable data set, we choose the size of the data set to be 10^6 points at the given sampling rate. From the computation of the largest Lyapunov exponent, we find that the critical noise voltage for transition to chaos is $D_c \approx$ 6.0 mV. [The resolution of a commercial noise generator is typically about 10.0 mV. That is why we utilize two voltage buffers (TL082) and a voltage divider to construct the circuit in Fig. 1(b). This divider yields noise signals with finer resolution of about 1.0 mV.]

Note also that the Lyapunov exponent has the dimension of reciprocal time. Our estimated value of the largest Lyapunov exponent thus has the unit of inverse of the sampling time, which is 5 μs. The magnitude of the exponent also depends on the logarithm used in its evaluation. Here we use the standard natural logarithm. Thus, for instance, given an estimated value of $\lambda > 0$, the factor that an infinitesimal distance is magnified in time, say $t = 10$ (which corresponds to the actual physical time of 50 μs), is $\exp(\lambda t) = \exp(10\lambda)$.

The projection of the period-3 attractor on the $V_{C1} - V_{C2}$ plane in the absence of the controllable noise is shown in Fig. 2(a). For noise voltage $D < D_c \approx$ 6.0 mV, the attractor appears to be slightly smeared, but it remains nonchaotic as its largest Lyapunov exponent is estimated to be approximately zero. Such a nonchaotic attractor is shown in Fig. 2(b) for $D \approx$ 4.0 mV. The attractor near the transition is shown in Fig. 2(c) for $D \approx$ 6.0 mV, and an apparently chaotic attractor is shown in Fig. 2(d) for $D \approx$ 40.0 mV.
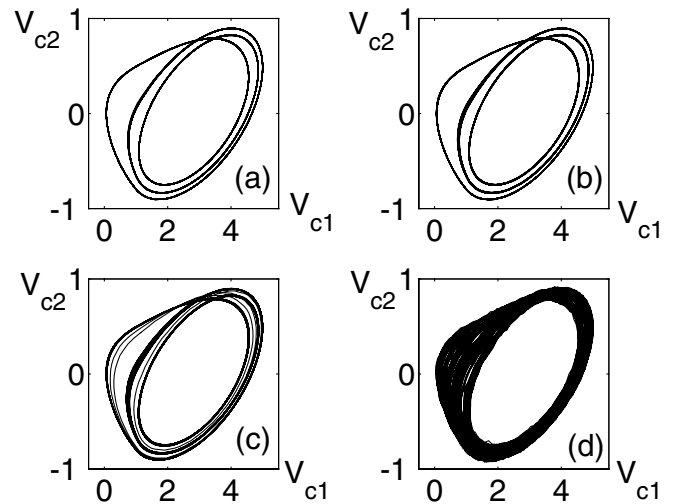

FIG. 2. For the period-3 window, the projection of the attractor on the $V_{C1} - V_{C2}$ plane for (a) $D \approx 0$, (b) $D \approx 4.0$ mV, (c) $D = D_c \approx 6.0$ mV, and (d) $D \approx 40.0$ mV. The attractors for (a) and (b) are nonchaotic, while that for (d) is apparently chaotic. V_{C1} and V_{C2} are measured in volts.

Figure 3(a) shows the estimated largest Lyapunov exponent λ_1 versus the noise voltage D. The precision of the estimated exponent is on the order of 10^{-2}, which is assessed from the value λ_1^0 obtained from the period-3 attractor, in the absence of controllable noise. As the noise is increased from about 6.0 mV, λ_1 starts to increase from zero. To obtain the scaling, we fine-tune the noise voltage in the range 6.0–80.0 mV and obtain the value of λ_1, using λ_1^0 as the reference point. The algebraic noisy-scaling law of λ_1 is shown in Fig. 3(b) on a logarithmic scale, where a robust algebraic relation between λ_1 and $(D - D_c)$ is observed. The experimental scaling exponent is estimated to be 0.98 ± 0.02.

To obtain the theoretical scaling exponent, it is necessary to estimate the quantities λ_1^S and τ of the chaotic saddle in the period-3 window, in the absence of noise. We use the following straightforward procedure: We turn off
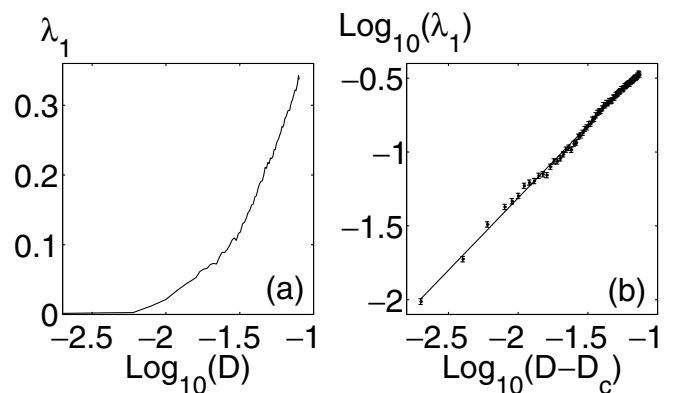

FIG. 3. (a) Estimated largest Lyapunov exponent λ_1 versus the noise voltage $\log_{10} D$. (b) Algebraic scaling of λ_1 with $D - D_c$.
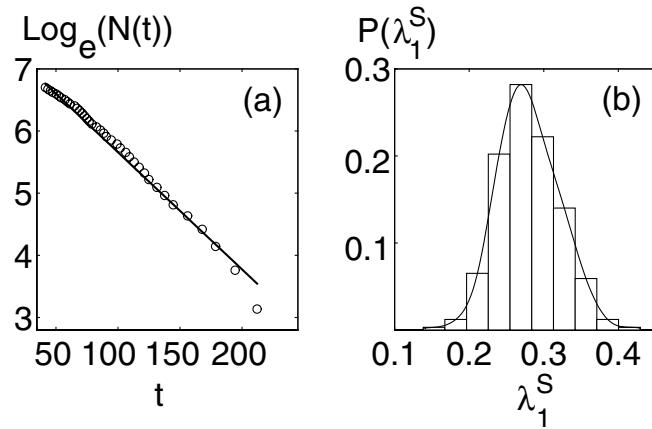
FIG. 4. (a) Exponential decay of $N(t)$ in the absence of noise. Time is measured in the unit of T, the average time interval for successive crossings of a typical trajectory through a Poincaré plane. (b) Histogram of λ_1^S.

noise and collect 1000 trajectories from the circuit, which are transiently chaotic. From the chaotic parts of the trajectories, we compute $N(t)$, the number of trajectories that remain chaotic at time t, which typically decays exponentially with time, as shown in Fig. 4(a). The inverse of the exponential decay rate is taken to be the estimated value of τ. We obtain $\tau \approx 54$, in the unit of T, the average time interval for successive crossings of a typical trajectory through a Poincaré plane. To measure λ_1^S, we compute the largest Lyapunov exponents associated with the chaotic parts of these trajectories, and construct a histogram of the calculated exponents, as shown in Fig. 4(b). The central value of the histogram is taken to be λ_1^S. We obtain $\lambda_1^S \approx 0.28$. These estimates give the theoretical scaling exponent [via Eq. (2)] $\alpha \approx 0.97$, which agrees very well with the experimental result.

In summary, we present experimental observation of transition to chaos as induced by the noise, and characterize the scaling behavior governing the transition. Our careful estimates of the relevant dynamical quantities provide a solid verification of the theoretically predicted scaling law. We believe this scaling law is universal and it can be observed in other experimental systems as well.

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