

## Shadowability of Statistical Averages in Chaotic Systems

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We ask whether statistical averages in chaotic systems can be computed or measured reliably under the influence of noise. Situations are identified where the invariance of such averages breaks down as the noise amplitude increases through a critical level. An algebraic scaling law is obtained which relates the change of the averages to the noise variation. This breakdown of shadowability of statistical averages, as characterized by the algebraic scaling law, can be expected in both low- and high-dimensional chaotic systems.

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An important problem in statistical and nonlinear physics is whether the statistical averages can be computed or measured reliably. The problem is particularly relevant when the underlying physical system exhibits chaos. For chaotic systems, it has been known that numerical trajectories are not always shadowable by true trajectories [1–6]. In particular, for typical low-dimensional chaotic systems, the existence of tangencies between stable and unstable manifolds can result in divergence between any numerical trajectory from a true one after a time that depends on the computational error [2–5]. Severe obstruction to shadowing can occur in high-dimensional chaotic systems when one of the Lyapunov exponents fluctuates about zero [6]. All these results concern the shadowability of individual numerical trajectories. For statistical averages, one tends to argue that because of the ergodicity of chaos, the effects of computational errors or noise are averaged out and, hence, the computed values of long-term statistical averages should be reliable. The purpose of this paper is to show that this intuition is not always correct. In fact there are common situations where the statistical averages change with noise and, hence, shadowing of computed statistical averages is not always guaranteed. A relevant situation is where measurements of average physical quantities take place. Our findings imply that if an identical experiment is to be performed in two different environments or at two different times where the noise levels are different, the measured averages of some physical quantities may vary. Since computing or measuring averages is an extremely common practice in physics and in many other scientific disciplines as well, we expect our finding to be of fundamental importance.

The general setting of our investigation is as follows. Consider a chaotic system described either by a discrete-time map:  $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, p) + D\xi_n$  or by a continuous-time flow:  $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x}, p) + D\xi(t)$ , where  $\mathbf{x} \in \mathcal{R}^N$  and  $p$  is a system parameter,  $D$  is the noise amplitude, and  $\xi_n$

or  $\xi(t)$  is a Gaussian white noise term. Assume that the parameter  $p$  is chosen such that the system is capable of generating chaos. Let  $G(\mathbf{x})$  be a continuous function that represents the physical quantity whose average is to be computed or measured. In discrete-time systems, the average is:  $\bar{G} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{i=0}^{T-1} G(\mathbf{x}_i, p)$ , where  $\{\mathbf{x}_i\}_0^{T-1}$  is a typical trajectory generated from a random initial condition. In continuous-time systems, the average is given by  $\bar{G} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G[\mathbf{x}(t), p] dt$ . For trajectories on an ergodic invariant set, the above averages are equal to the ensemble averages, which in practice, can be evaluated by utilizing a large number of short time series.

The principal result of this paper is that there are situations in chaotic dynamical systems where, if the noise amplitude  $D$  exceeds a critical value  $D_c$ , the statistical average  $\bar{G}$  can change with noise and scales with  $D$  in the following algebraic manner:

$$\Delta G(D) \equiv \bar{G}(D) - \bar{G}(D_c) \sim (D - D_c)^\alpha, \quad \text{for } D \geq D_c, \quad (1)$$

where  $\alpha > 0$  is a scaling exponent that depends on the details of the system such as the dimensionality, and  $\Delta G(D) = 0$  for  $D < D_c$ . The scaling behavior holds in both low and high dimensions, and it is expected to be observable because it occurs in parameter regions of positive Lebesgue measure. We are able to obtain explicit expressions for the scaling exponent in all dimensions. The implication is the following. In numerical computations, if different precisions or different computers are used, the ergodic average can have different values. In a laboratory experiment, measurements performed under nonidentical circumstances may yield different results.

To derive the scaling law (1), we consider situations where there are two coexisting dynamical invariant sets with distinct unstable dimensions and, noise can link the two sets and thereby induce unstable dimension variability [6–11] along a continuous trajectory. Such a

situation is by no means rare. For instance, it can occur in any periodic window where a periodic attractor and a nonattracting chaotic saddle coexist. The periodic attractor has no unstable direction (unstable dimension zero), while the chaotic saddle can have at least one unstable direction. In this case, the critical amplitude of the noise for which unstable dimension variability arises for a continuous trajectory is proportional to the phase-space distance between the attractor and the chaotic saddle in the absence of noise, which in turn is proportional to the size of the periodic window in the parameter space. For illustrative purpose, we focus on a periodic window. Let  $\overline{G^A}(0)$  and  $\overline{G^S}(0)$  be the statistical averages of the physical function  $G(\mathbf{x})$  associated with the periodic attractor and the chaotic saddle, respectively, in the absence of noise. In particular, if  $\{\mathbf{x}_i^A\}_{i=1}^p$  denotes the stable periodic orbit of period  $p$  on the Poincaré map, then we have  $\overline{G^A}(0) = \frac{1}{p} \sum_{i=1}^p G(\mathbf{x}_i^A)$ . For the chaotic saddle, let  $\{\mathbf{x}_i^S\}_{i=1}^\infty$  be a dense trajectory embedded in it which, numerically, can be obtained by the procedure in Ref. [12]. The average is given by  $\overline{G^S}(0) = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^L G(\mathbf{x}_i^S)$ . For  $D < D_c$ , the periodic attractor is isolated from the chaotic saddle so that asymptotic trajectories of system are confined in the neighborhood of the periodic attractor. This is true even when an initial condition is chosen in the vicinity of the chaotic saddle, in which case the resulting trajectory may wander near the saddle for a finite amount of time but, asymptotically, the trajectory approaches the periodic attractor. We thus have

$$\overline{G}(D) = \overline{G^A}(0), \quad \text{for } D < D_c, \quad (2)$$

because in the asymptotic time limit, the effect of noise vanishes on average. For  $D \geq D_c$ , the periodic attractor is dynamically connected with the chaotic saddle. Let  $P^A(D)$  and  $P^S(D)$  be the probabilities of a typical trajectory to visit the periodic attractor and the chaotic saddle, respectively, in the asymptotic time limit. The trajectory is intermittent in the sense that the switching time, i.e., the time for the trajectory to “hop” between the periodic attractor and the chaotic saddle, is negligible, compared with the times that the trajectory spends near these sets. (Of course this is expected to be true only for  $D \geq D_c$ .) Thus, for  $D \geq D_c$ , we have  $\overline{G}(D) \approx P^A(D)\overline{G^A}(D) + P^S(D)\overline{G^S}(D)$ . Because of the averaging effect of noise, the average quantities for trajectories restricted to the periodic attractor and the chaotic saddle are approximately invariant:  $\overline{G^A}(D) \approx \overline{G^A}(0)$  and  $\overline{G^S}(D) \approx \overline{G^S}(0)$ . Under these approximations and noting that, since  $D$  is only slightly above  $D_c$ , we have  $P^S(D) \ll P^A(D)$ , which means  $P^A(D) \approx 1$ , we obtain

$$\Delta G(D) = \overline{G}(D) - \overline{G^A}(0) \approx P^S(D)\overline{G^S}(0), \quad \text{for } D \geq D_c, \quad (3)$$

where the dependence of the probability  $P^S(D)$  on noise is the major factor that determines the scaling of  $\Delta G(D)$ .

This probability is proportional to the natural measure of the stable manifold of the chaotic saddle in the neighborhood of the periodic attractor due to noise, which is determined by the fractal dimension of the stable manifold.

For nonattracting chaotic saddles, Hunt *et al.* [13] obtain explicit formulas for the Lyapunov dimension  $d_s$  of the stable manifold. In particular, suppose the chaotic saddle in an  $N$ -dimensional phase space has  $K_u$  positive and  $K_s$  negative Lyapunov exponents ( $K_u + K_s = N$ ), which are ordered as follows:  $\lambda_{K_u}^+ \geq \lambda_{K_u-1}^+ \geq \dots \geq \lambda_1^+ > 0 > -\lambda_1^- \geq \dots \geq -\lambda_{K_s-1}^- \geq -\lambda_{K_s}^-$ . The forward entropy of the chaotic saddle is  $H = \sum_{i=1}^{K_u} \lambda_i^+ - 1/\tau$ , where  $\tau$  is the lifetime of the chaotic saddle. Then, the dimension of the stable manifold of the chaotic saddle is

$$d_s = K_s + J + \frac{H - (\lambda_1^+ + \dots + \lambda_J^+)}{\lambda_{J+1}^+}, \quad (4)$$

where  $J$  is determined by  $\lambda_1^+ + \dots + \lambda_J^+ + \lambda_{J+1}^+ \geq H \geq \lambda_1^+ + \dots + \lambda_J^+$ . Similar formulas exist for  $d_u$  and  $d$ , the Lyapunov dimensions of the unstable manifold and the chaotic saddle itself, respectively [13].

Utilizing the dimension formula Eq. (4), we can derive the scaling law (1) and the scaling exponent  $\alpha$  in both low and high dimensions. In particular, for chaotic saddles arising in one-dimensional maps, there is only one positive Lyapunov exponent  $\lambda > 0$ . We have  $K_s = 0$ ,  $K_u = 1$ , and  $d_u = 1$ . The Lyapunov dimensions of the chaotic saddle and its stable manifold are equal:  $d_s = d = H/\lambda = 1 - 1/(\lambda\tau)$ . For an interval of size  $\epsilon$ , the natural measure of the stable manifold is proportional to  $\epsilon^{d_s}$ . When the noise is slightly above the critical level:  $D \geq D_c$ , the length in which the stable manifold falls in the original basin of the periodic attractor is proportional to  $(D - D_c)$ . We thus have  $P^S(D) \sim (D - D_c)^{d_s} \sim (D - D_c)^{1 - 1/(\lambda\tau)}$ , which is the scaling law (1) with the following scaling exponent:

$$\alpha = 1 - \frac{1}{\lambda\tau}, \quad \text{for one-dimensional maps.} \quad (5)$$

For two-dimensional Poincaré maps (equivalently three-dimensional flows), consider a circle of size  $\delta$ . The natural measure of the stable manifold contained within the circle is proportional to  $\delta^{d_s} = (\delta^2)^{d_s/2}$ , where  $\delta^2$  is proportional to the area of the circle and  $\tau$  is the lifetime of the chaotic saddle of the Poincaré map ( $\tau$  is thus in the unit of  $T$ , the average time that a typical trajectory crosses the Poincaré section). Let  $\lambda_1 > 0 > -\lambda_2$  be the Lyapunov exponents of the chaotic saddle. For  $D \geq D_c$ , the area in which the stable manifold of the chaotic saddle penetrates the original basin of the periodic attractor is proportional to  $(D^2 - D_c^2)$ . In two dimensions, Eq. (4) gives  $d_s = 2 - 1/(\lambda_1\tau)$ . We thus have  $P^S(D) \sim (D^2 - D_c^2)^{d_s/2} \sim (D - D_c)^{1 - 1/(2\lambda_1\tau)}$ , which, when substituted into Eq. (3), gives the scaling law (1)

with the following scaling exponent:

$$\alpha = 1 - \frac{1}{2\lambda_1\tau},$$

for two-dimensional maps or three-dimensional flows. (6)

For an  $N$ -dimensional map [or an  $(N + 1)$ -dimensional flow], a similar derivation gives the scaling law (1) with the following scaling exponent:

$$\alpha = \frac{1}{N} \left[ K_s + J + \frac{H - (\lambda_1^+ + \dots + \lambda_J^+)}{\lambda_{J+1}^+} \right]. \quad (7)$$

A feature to notice in all these cases is that in common situations where the lifetime of the chaotic saddle in a periodic window is long:  $\tau \gg 1$ , the scaling exponent is expected to be close to unity.

We now provide numerical support for the scaling law (1) and the scaling-exponent formulas (5)–(7) in low and high dimensions. In one dimension, we consider the noisy logistic map:  $x_{n+1} = ax_n(1 - x_n) + D\xi_n$ . We choose  $a = 3.008$  for which there is a period-8 window. The physical function is chosen to be  $G(x) = \sin x$ . We observe that  $\overline{G}$  is constant for  $D < D_c$  and it starts to increase for  $D > D_c$ , where  $D_c \approx 10^{-5}$ . A plot of  $\Delta G$  versus  $D$  on a logarithmic scale indicates that it is algebraic. A least-squares fit gives the following estimate of the scaling exponent:  $0.94 \pm 0.06$ . For the logistic map in this period-8 window, the Lyapunov exponent and the chaotic-transient lifetime are estimated to be  $\lambda \approx 0.425$  and  $\tau \approx 645.4$ . The theoretical exponent is thus close to unity, which is consistent with the numerical value.

For a two-dimensional example (or a three-dimensional flow), we consider the following noisy Lorenz system [14]:  $dx/dt = 10(y - x) + D\xi_1(t)$ ,  $dy/dt = 71.45x - y - xz + D\xi_2(t)$ , and  $dz/dt = -(8/3)z + xy + D\xi_3(t)$ . At this parameter setting, there is a periodic window of period-4. Invariance of the statistical average holds only for small noise  $D < D_c$ , as shown in Fig. 1(a) for the physical function  $G(\mathbf{x}) = z^2(t)$ , where the critical noise amplitude is estimated to be  $D_c \approx 10^{-2.61}$ . Figure 1(b) shows the algebraic scaling between  $\Delta G(D) = z^2 - z_c^2$  and  $D - D_c$ , where  $z_c^2 \approx 4564.7$  and the numerical scaling exponent is estimated to be  $0.99 \pm 0.03$ , which agrees very well with the theoretical slope  $\alpha \approx 1.0$  ( $\tau \approx 588.0 \gg 1$  and  $\lambda_1^S \approx 0.63$ ).

For a high-dimensional example, we consider the following system of two coupled Rössler chaotic oscillators [15] under noise:  $dx_{1,2}/dt = -y_{1,2} - z_{1,2} + \epsilon(x_{2,1} - x_{1,2}) + D\xi_1(t)$ ,  $dy_{1,2}/dt = x_{1,2} + 0.2y_{1,2} + D\xi_2(t)$ ,  $dz_{1,2}/dt = 0.2 + z_{1,2}(x_{1,2} - 5.3) + D\xi_3(t)$ , where  $\epsilon$  is the coupling strength. For small coupling, the chaotic set of the system can have two positive Lyapunov exponents. There is a period-3 window for  $\epsilon = 0.01$  in which there is a periodic attractor and a chaotic saddle with two positive exponents. Figure 2(a) shows the statistical aver-

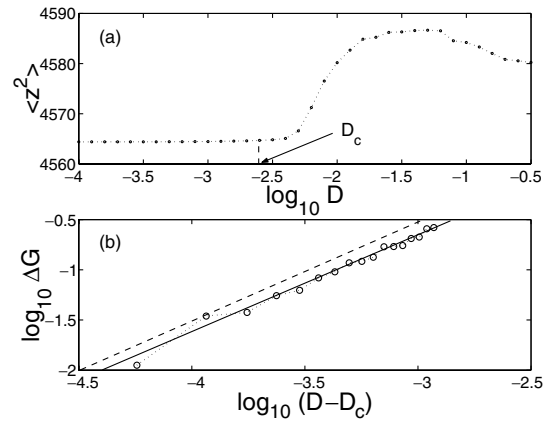


FIG. 1. For the Lorenz system, (a) Statistical average of the function  $G(\mathbf{x}) = z^2(t)$  versus the noise amplitude, where the average is constant for  $D < D_c$ , increases for  $D > D_c$ , and  $D_c \approx 10^{-2.61}$ . (b) Algebraic scaling between  $\Delta G(D)$  and  $D - D_c$ , where the dashed line represents the theoretical slope.

age of the function  $G(x) = \sin x$  as a function of the noise amplitude  $D$ , where we obtain  $D_c \approx 10^{-2.25}$ . Figure 2(b) shows the algebraic scaling of  $\Delta G = \langle \sin x \rangle - \langle \sin x \rangle_c$  with  $D - D_c$ , where the numerical scaling exponent is approximately  $1.01 \pm 0.08$ . The theoretical exponent is about 0.99 [ $N = 5$ ,  $K_s = 3$ ,  $J = 1$ ,  $\lambda_1^+ \approx 0.34$ ,  $\lambda_2^+ \approx 0.29$ ,  $\tau \approx 113.2$ , and  $H = \lambda_1^+ + \lambda_2^+ - 1/\tau \approx 0.62$ , so  $\alpha = (1/5)[3 + 1 + (\lambda_2^+ - \tau^{-1})/\lambda_2^+ \approx 0.99]$ ], which agrees very well with the numerical one.

In summary, we have identified common circumstances in chaotic dynamical systems under which statistical averages of dynamical variables and their functions depend on the noise amplitude in an algebraic manner when it exceeds a critical level [16]. Our results suggest that, even in low-dimensional chaotic systems, computations or measurements of statistical or ergodic averages may not necessarily be invariant under noise. For instance, if a physical experiment is to be carried out in different times

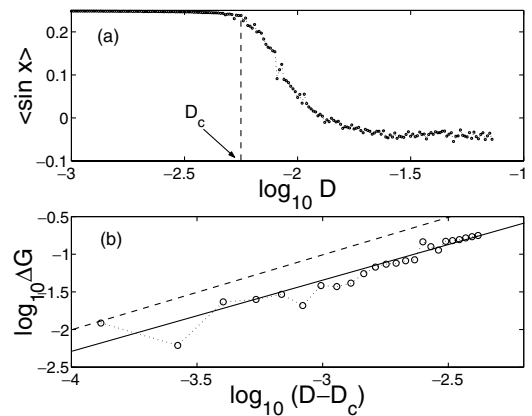


FIG. 2. For the coupled Rössler system, (a) Statistical average of the function  $G(x) = \sin x$  versus the noise amplitude. (b) Algebraic scaling between  $\Delta G(D)$  and  $D - D_c$ , where the dashed line represents the theoretical slope.

or in different laboratories, the measurement of average quantities may not yield the same result because of the possible variations in the noise level. The algebraic scaling established in this paper suggests that this “failure of shadowability” of statistical averages is, however, quite *mild* as compared with the breakdown of shadowing of individual numerical trajectories, particularly in situations of unstable dimension variability where numerical trajectories diverge exponentially from the true trajectories and this can occur frequently in time [6,8–10]. The interplay between chaos and stochasticity is of fundamental importance for both nonlinear dynamics and statistical physics, we believe that our scaling results provide new insights into the problem.

We stress that our scaling result is general because it is expected to be valid for chaotic systems of all dimensions. While the scaling law is derived under some assumptions, it is relevant to and important for real experiments because its characteristic feature is independent of the details of the system and of the type of the noise [19] as well, as numerical computations using different model chaotic systems yield qualitatively the same scaling result. This means that, although there exists no precise model for any experimental system and the internal and/or environmental noise level may be high, the algebraic scaling behavior of statistical averages with noise is expected to be observable. While the required critical noise level seen in some of our numerical examples is relatively small (e.g.,  $10^{-5}$  for the Lorenz system), its value is determined by the size of the periodic window in the parameter space. In a situation where an experimental system operates in a relatively large periodic window, the critical noise level may well exceed the total intrinsic and environmental noise strength, rendering the algebraic scaling law of statistical averages experimentally testable.

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