Coherence Resonance in Coupled Chaotic Oscillators

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Existing works on coherence resonance, i.e., the phenomenon of noise-enhanced temporal regularity, focus on excitable dynamical systems such as those described by the FitzHugh-Nagumo equations. We extend the scope of coherence resonance to an important class of dynamical systems: coupled chaotic oscillators. In particular, we show that, when a system of coupled chaotic oscillators is under the influence of noise, the degree of temporal regularity of dynamical variables characterizing the difference among the oscillators can increase and reach a maximum value at some optimal noise level. We present numerical results illustrating the phenomenon and give a physical theory to explain it.

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The phenomenon of noise-induced enhancement of the temporal regularity of a physical signal was first noticed by Sigeti *et al.* [1]. Recently, this phenomenon was rediscovered and renamed as *coherence resonance* in excitable nonlinear dynamical systems [2–4] and in certain one-dimensional stochastic systems [5]. Typically, in such systems, the time traces of dynamical variables of physical interest consist of an infinite sequence of bursts occurring at random times. Coherence resonance is referred to the fact that noise can actually be utilized to improve the temporal regularity of the bursting time series [2–5]. In particular, at both small and large noise levels, the time series appears random in the sense that its Fourier spectrum is broadband and apparently exhibits no pronounced peaks. At some intermediate noise levels, the bursting time series appears more regular, which is characterized by the appearance of a finite set of peaks at certain frequencies. If one defines a measure, say, the ratio of the height of the most pronounced peak in the Fourier spectrum to its half width, to quantify the temporal regularity of the bursting time series, then one finds that the measure tends to increase as the noise level is raised and reaches a maximum value at some optimal noise level. Coherence resonance is different from the extensively studied phenomenon of stochastic resonance [6], as the former concerns the temporal aspect of the signal while the latter deals with quantities related to the amplitude such as the signal-to-noise ratio. Another difference is that coherence resonance usually does not require an external periodic driving [2,7] versus stochastic resonances that do.

Most existing works on coherence resonance address excitable systems [7] such as those described by the FitzHugh-Nagumo equations [8] in which the dynamics typically consists of a slow motion near some fixed point and rapid excursions away from it. In such systems, the measured time series usually consists of a silent phase and a bursting one, corresponding, respectively, to motions near the fixed point and the excursions. The temporal regularity of the bursting time series and, consequently, coherence resonance are of great physical or biological importance. The purpose of this Letter is to point out that coherence resonance can actually be expected in another important class of dynamical systems: coupled nonlinear oscillators, which are relevant to a variety of physical and biological situations [9,10]. In particular, we argue that, when identical or slightly nonidentical chaotic oscillators are coupled together, the temporal regularity of some measured signal characterizing the degree of the synchronization among the oscillators can be modulated by external noise in the sense of coherence resonance. Such signals, for example, can simply be the difference among, or the function of, the corresponding dynamical variables of the oscillators. We give numerical examples and a quantitative analysis elucidating the dynamical mechanism for the coherence resonance. Because of the ubiquity of the occurrence of coupled nonlinear oscillators in nature and in engineering systems, the correct identification of coherence resonance will be both theoretically interesting and practically useful for applications such as signal processing.

We begin by presenting numerical results from the following system of two coupled Lorenz oscillators:

$$\begin{aligned} \dot{x}_{1,2} &= \sigma_{1,2}(y_{1,2} - x_{1,2}) + K(x_{2,1} - x_{1,2}) + D\xi_x(t), \\ \dot{y}_{1,2} &= \gamma_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2} + D\xi_y(t), \\ \dot{z}_{1,2} &= -b_{1,2}z_{1,2} + x_{1,2}y_{1,2} + D\xi_z(t), \end{aligned} \tag{1}$$

where $\sigma_{1,2}$, $\gamma_{1,2}$, and $b_{1,2}$ are the parameters of the Lorenz oscillator [11], K is the coupling parameter, $\xi_{x,y,z}(t)$ are independent Gaussian random processes that simulate the external noise, and D quantifies the noise strength. We first consider the case where the two Lorenz oscillators are identically chaotic: we set $\sigma_{1,2} = 10.0$, $\gamma_{1,2} = 28.0$, and $b_{1,2} = 8/3$ so that each Lorenz oscillator, when uncoupled, exhibits a chaotic attractor. This identity stipulates that the asymptotic synchronization state $\mathbf{x}_1(t) = \mathbf{x}_2(t)$, where $\mathbf{x} = \{x, y, z\}$, is a solution of Eq. (1). A stability analysis through the calculation of the transverse Lyapunov exponent [10] indicates that, in the noiseless situation, the synchronization state is unstable for $K < K_c$ and stable for $K > K_c$, where $K_c \approx 3.92$. Our point is that, for K near K_c under the influence of noise, the coupled system exhibits dynamical characteristics required for coherence resonance. In particular, for $K \lesssim K_c$, the synchronization state is weakly unstable so that the difference between the dynamical variables $\Delta\mathbf{x}(t) \equiv \mathbf{x}_1(t) - \mathbf{x}_2(t)$, exhibits on-off intermittency [12], regardless of whether noise is absent or present. For $K \gtrsim K_c$, the synchronization is stable so that $\Delta\mathbf{x}(t) \rightarrow \mathbf{0}$ asymptotically when $D = 0$ (noiseless situation), but, for $D \neq 0$, $\Delta\mathbf{x}(t)$ exhibits, again, on-off intermittency. The dynamical mechanism for on-off intermittency for both $K \lesssim K_c$ (with or without noise) and $K \gtrsim K_c$ (with noise) can be understood by analyzing the transverse stabilities of the infinite set of unstable periodic orbits embedded in the attractor [13]. The characteristic feature of on-off intermittency is the existence of two distinct states: the "off" state, in which $\Delta\mathbf{x}(t) \approx \mathbf{0}$, and the "on" state, where $\Delta\mathbf{x}(t)$ deviates significantly from the off state. Typically, the system tends to reside in the off state for a certain amount of random time that is exponentially distributed [14], with intermittent bursts away from the off state (the on state). Roughly, the off and on states here correspond to the motion near the fixed point and the excursion away from it, respectively, in an excitable system. Thus, qualitatively, under the influence of noise, we expect coherence resonance to occur in coupled chaotic systems.

To characterize the degree of the temporal regularity of the bursting signals at different noise levels, we compute the Fourier power spectra. Figures 1(a)–1(c) show the power spectra of $\Delta y(t)$ for $D = 10^{-3}$, $D = 3 \times 10^{-2}$, and $D = 0.3$, respectively. For small noise [Fig. 1(a)], the spectrum exhibits no peak, except for the one at $\omega = 0$, indicating a lack of temporal regularity in the bursting time series. The situation is similar for large noise [Fig. 1(c)]. A pronounced peak at $\omega \neq 0$ does exist at the intermediate noise level [Fig. 1(b)], indicating the existence of a strong time-periodic component in the time series. The apparent temporal regularity seen in Fig. 1(b) can be quantified by the characteristics of the peak at a nonzero frequency ω_p in the spectrum. In particular, we utilize the quantity $\beta_S \equiv HQ_s$, where H is the height of the spectral peak, $Q_s = \omega_p/\Delta\omega$, and $\Delta\omega$ is the half width of the peak [2–5]. By its definition, a high value of β_S indicates a strong temporal regularity in the bursting time series. Figures 2(a) and 2(b) show, for Eq. (1) at $K = 4.0 > K_c$ and $K = 3.5 < K_c$, respectively, β_S versus the noise amplitude D. We see that β_S is small at small noise levels, increases as the noise is increased, reaches a maximum at an optimal noise level, and decreases as the noise is increased further. These are features that are associated with stochastic resonance, where, typically, a signal-to-noise ratio is plotted against the noise level [6], but here the relevant quantity concerns the time regularity.

When the oscillators are not identical, the synchronization state is no longer invariant. If the mismatch between the oscillators is small, on-off intermittency persists, even in a wider parameter regime in a noiseless situation [14]. With the intermittency, noise will effectively regulate the temporal characteristics of the bursting time series and, consequently, coherence resonance will arise. We have obtained numerical results that are similar to these in Fig. 2. This is of practical importance: in laboratory experiments small mismatches among the coupled oscillators are inevitable, but coherence resonance in such systems appears to be a robust phenomenon against the mismatches.

We now give a physical theory for coherence resonance in coupled chaotic systems. Consider the following

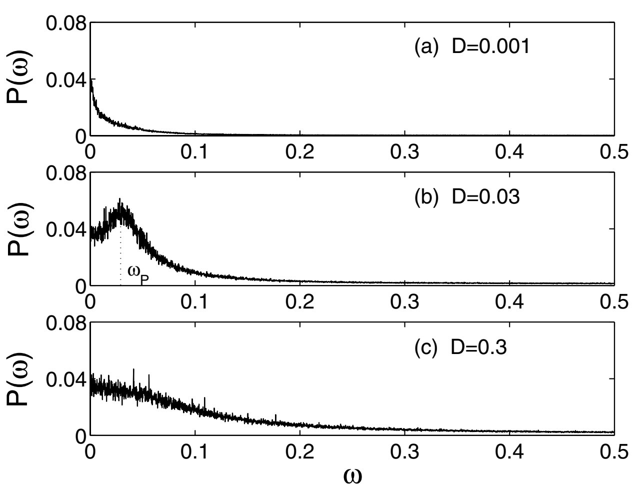

FIG. 1. For a pair of two coupled identical Lorenz chaotic oscillators at coupling $K = 4.0 > K_c$. Fourier power spectra of $\Delta y(t)$ at the following noise levels: (a) $D = 10^{-3}$, (b) $D = 3 \times 10^{-2}$, and (c) $D = 0.3$.

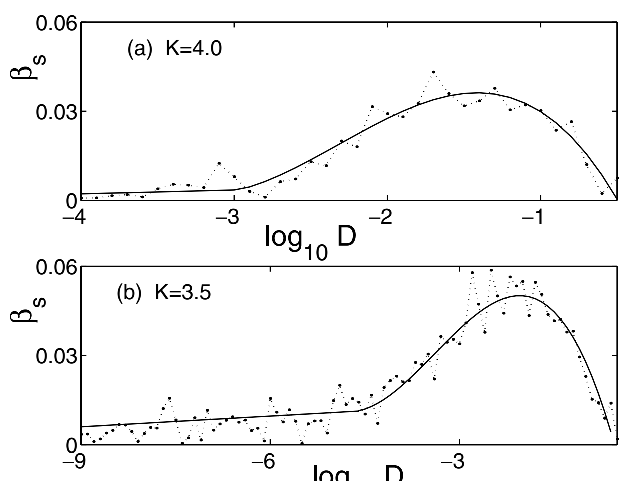

FIG. 2. Coherence-resonance measure β_S vs the intensity of noise for the pair of coupled identical Lorenz chaotic oscillators with coupling strengths (a) $K = 4.0 > K_c$ and (b) $K = 3.5 < K_c$, where the solid lines are the polyfit curves.

system of two coupled chaotic oscillators: $\dot{\mathbf{x}}_{1,2} = \mathbf{f}_{1,2}(\mathbf{x}_{1,2}) + \mathbf{K} \cdot (\mathbf{x}_{2,1} - \mathbf{x}_{1,2})$, where $\mathbf{x}_{1,2} \in \mathcal{R}^N$, $\mathbf{f}_1$ and $\mathbf{f}_2$ are the velocity fields of the chaotic oscillators when uncoupled, and $\mathbf{K}$ is the coupling matrix. If $\mathbf{f}_1 = \mathbf{f}_2$, the oscillators are identical so that the synchronization state $\mathbf{x}_1 = \mathbf{x}_2$ is invariant [10]. For simplicity, we consider the situation where the two oscillators are slightly nonidentical, $\mathbf{f}_1 \approx \mathbf{f}_2$, and explore the dynamics near the approximate synchronization state: $\mathbf{x}_1 \approx \mathbf{x}_2$. By introducing two new variables, $\mathbf{u} = (\mathbf{x}_1 + \mathbf{x}_2)/2$ and $\mathbf{v} = (\mathbf{x}_1 - \mathbf{x}_2)/2$, we obtain, near the approximate synchronization state $\mathbf{v} \approx \mathbf{0}$, the following equations:

$$\dot{\mathbf{u}} \approx \frac{1}{2}[\mathbf{f}_1(\mathbf{u}) + \mathbf{f}_2(\mathbf{u})] + \frac{1}{2}\left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}_2}{\partial \mathbf{u}}\right) \cdot \mathbf{v}, \quad (2)$$

$$\dot{\mathbf{v}} \approx \left[\frac{1}{2}\left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{u}} + \frac{\partial \mathbf{f}_2}{\partial \mathbf{u}}\right) - 2\mathbf{K}\right] \cdot \mathbf{v} + \frac{1}{2}[\mathbf{f}_1(\mathbf{u}) - \mathbf{f}_2(\mathbf{u})],$$
$$\equiv [-\boldsymbol{\lambda} + \mathbf{N}(t)] \cdot \mathbf{v} + \boldsymbol{\xi}_i(t), \quad (3)$$

where $\partial \mathbf{f}_{1,2}/\partial \mathbf{u}$ are the Jacobian matrices of the velocity fields $\mathbf{f}_{1,2}$, $\boldsymbol{\lambda}$ is a matrix whose elements are the average values of the corresponding elements of the matrix in front of $\mathbf{v}$ in Eq. (3), $\mathbf{N}(t)$ is a zero-mean random matrix, and $\boldsymbol{\xi}_i(t)$ stands for the small chaotic modulation term in Eq. (3) which vanishes if the oscillators are identical. Since both oscillators are chaotic, we see from Eq. (2) that the variables $\mathbf{u}$ can typically be chaotic because they are the approximate average of $\mathbf{x}_1$ and $\mathbf{x}_2$ under a small chaotic modulation term proportional to $\mathbf{v}$. The variable $\mathbf{v}$, on the other hand, obeys an equation that describes on-off intermittency under "noise" because $\mathbf{u}$ is chaotic [14].

To make the analysis feasible, we consider one scalar variable that exhibits on-off intermittency. That is, we consider the one-dimensional version of Eq. (3). Under the influence of external noise $\xi_e(t)$, we have

$$\dot{v} = [-\lambda + N(t)]v + \xi_i(t) + \xi_e(t)$$
$$\equiv [-\lambda + N(t)]v + \xi(t), \quad (4)$$

where $\xi(t)$ now stands for the combination of internal chaotic modulation and external noise. Note that Eq. (4) is similar to the paradigmatic model for analyzing on-off intermittency under the influence of noise [15]. To quantify how the temporal regularity of $v(t)$ is modulated by noise, we use the following measure, introduced in Ref. [2], for convenience: $\beta_T = \langle T\rangle/\sqrt{\mathrm{Var}(T)}$, where T is the interval between the bursts, and $\langle T\rangle$ and $\mathrm{Var}(T)$ are the average value and variance of $T(t)$, respectively. We note that the measure β_T is in fact equivalent to the β_S that we use in numerical experiments [16]. To obtain $\langle T\rangle$ and $\mathrm{Var}(T)$, we consider the following Fokker-Planck equation associated with Eq. (4), which describes, *approximately,* the evolution of the probability distribution function $P(v,t)$ of the random variable $v(t)$:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial v}\left[\left(-\lambda v + \frac{1}{2}\epsilon v\right)P\right] + \frac{1}{2}\frac{\partial^2}{\partial v^2}[(\epsilon v^2 + D)P], \quad (5)$$

where D is the noise strength and ϵ is the strength of $N(t)$. Noting that the intermittent interval T is in fact the first-passage time, we solve Eq. (5) for quantities that are required for characterizing the time regularity of $v(t)$ under the conditions that there is an absorbing boundary at $v = a$ and a reflecting boundary at $v = b$. We obtain [17], for the first and second moments of T, the following:

$$\langle T\rangle = 2\int_{v_0}^{a} dy\,(\epsilon y^2 + D)^{\lambda/\epsilon - 1/2} \times \int_{b}^{y} (\epsilon z^2 + D)^{-1/2-\lambda/\epsilon}\,dz,$$
$$\langle T^2\rangle = 4\int_{v_0}^{a} dy\,(\epsilon y^2 + D)^{\lambda/\epsilon - 1/2} \times \int_{b}^{y} (\epsilon z^2 + D)^{-1/2-\lambda/\epsilon}\langle T(z)\rangle\,dz, \quad (6)$$

where v_0 is the initial value of $v(t)$. The quantity β_T can then be obtained from Eq. (6). Figure 3 shows a typical behavior of β_T as a function of D that we obtain numerically by evaluating the integrals in Eq. (6) under the following parameters (arbitrary): $v_0 = -5$, $a = 1$, $b = -20$, and $\lambda = \epsilon = 10^{-4}$. The signature of coherence resonance can be seen clearly from Fig. 3, where β_T attains a maximum value at some optimal noise strength. The theoretical prediction [Fig. 3] thus agrees, qualitatively, with the numerical results from the system of two coupled Lorenz chaotic oscillators [Fig. 2].

We summarize by listing the set of necessary conditions for coherence resonance: (1) there exists a reference state near which a trajectory can spend long time spans;

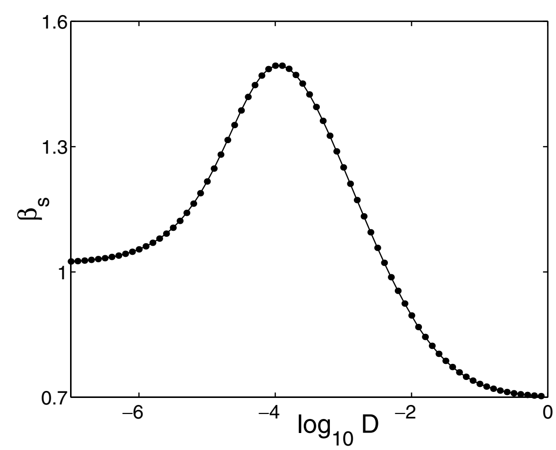

FIG. 3. Theoretical prediction of the measure of coherence resonance β_T versus noise strength for a general system of coupled chaotic oscillators.

(2) the system has the potential to temporally burst out of the reference state; (3) the system is nonlinear. Under these conditions, it is possible for a signal characterizing the bursting behavior to become temporally more regular under the influence of noise. Excitable systems apparently satisfy these conditions [2–4]. The analysis and numerical computations presented in this Letter indicate that coupled chaotic oscillators, a class of dynamical systems of intense recent interest, also satisfy these conditions and, hence, they can generically exhibit coherence resonance. Such systems can be readily constructed in the laboratory, say, by using electronic circuits, for experimentally verifying the theoretical prediction of this Letter (in fact we are currently pursuing this). Since coupled chaotic oscillators occur in many different contexts of natural sciences [9], we expect our finding to be important [18].

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[16] A physical process can be described either in the time domain $f(t)$ or in the frequency domain by its Fourier transform $F(\omega)$. When $f(t)$ is approximately periodic, its Fourier spectrum exhibits a peak at $\omega_p = 1/\langle T \rangle$ with width $\Delta\omega$. Since $T \sim 1/\omega$, we have $\langle T \rangle + \Delta T \sim 1/(\omega_p + \Delta\omega) \approx 1/\omega_p - \Delta\omega/\omega_p^2$. Thus, $\Delta T \sim \Delta\omega/\omega_p^2$ and, hence, $\beta_T = \langle T \rangle/\Delta T \sim \omega_p/\Delta\omega \sim \beta_S$.

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[18] For example, imagine a biological system consisting of two coupled, chaotically behaving neurons. Knowing that noise can enhance temporal regularity in some outputs of the system is clearly of importance if regular behavior is desirable, as a tuning of an internal parameter of the system is practically impossible. Similar applications can be anticipated in a system of two coupled chaotic lasers, where a regular output signal is desirable.