Experimental Observation of Superpersistent Chaotic Transients

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We present the first experimental observation of *superpersistent chaotic transients*. In particular, we investigate the effect of noise on phase synchronization in coupled chaotic electronic circuits and obtain the scaling relation that is characteristic of those extremely long chaotic transients.

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Transient chaos is ubiquitous in nonlinear dynamical systems [1,2]. In such a case, dynamical variables of the system behave chaotically for a finite amount of time before settling into a final state that is usually not chaotic. A common situation for transient chaos to arise is where the system undergoes a *crisis* at which a chaotic attractor collides with the basin boundary separating it and another coexisting attractor [1]. After the crisis, the chaotic attractor is destroyed and converted into a nonattracting chaotic saddle. Dynamically, a trajectory then wanders in the vicinity of the chaotic saddle for a period of time before asymptoting to the other attractor. Chaotic transients of this sort are not superpersistent in the sense that their average lifetimes scale with the system parameter only algebraically. Specifically, let p be a system parameter and assume that as p is increased a crisis occurs at the critical parameter value p_c. There is thus transient chaos for $p > p_c$. It is well established both theoretically [3] and experimentally [4] that the average lifetime τ of the chaotic transients scales with the parameter variation, as follows: $\tau \sim (p - p_c)^{-\gamma}$, $p > p_c$, where $\gamma > 0$ is the algebraic scaling exponent.

There exists, however, a distinct class of chaotic transients that are superpersistent in the following sense of scaling [5]:

$$\tau \sim \exp[A(p - p_c)^{-\beta}], \qquad p > p_c\,, \tag{1}$$

where $A > 0$ is a constant, $\beta > 0$ is the scaling exponent, p_c is a critical parameter value, and transient chaos occurs for $p > p_c$. We see that as $p \to p_c$, the lifetime of the transient behaves like $e^{+\infty}$; henceforth it is called *superpersistent*. Physically, the scaling relation (1) means that as p approaches p_c, the transient lifetime is significantly longer than that associated with "regular" chaotic transient characterized by an algebraic scaling law. Because of the scaling (1), the asymptotic attractor of the system is practically unobservable for $p \gtrsim p_c$. While "regular" chaotic transients have been observed in experiments, so far there has been no direct experimental verification of superpersistent chaotic transients. Because of the extremely long nature of these transients, it is highly nontrivial to observe and quantify them in laboratory experiments.

In this Letter, we present the first direct experimental evidence and characterization of these transients. Our experimental study is motivated by a recent theoretical discovery [6] that such transients can occur in the context of phase synchronization in coupled chaotic oscillators [7]. Generally, when two chaotic oscillators are coupled together, synchronization in their dynamical variables (complete synchronization) can occur, but phase synchronization usually occurs at coupling strength much smaller than that required for complete synchronization. Briefly, if trajectories in each chaotic oscillator can be regarded as a rotation, then the phase angle of the rotation increases steadily with time: $\phi(t) = \omega t + \theta(t)$, where ω is the average rotation frequency and $\theta(t)$ is a term characterizing chaotic fluctuations. In the absence of coupling, the phase angles of the two oscillators $\phi_1(t)$ and $\phi_2(t)$ are uncorrelated. That is, if one measures the difference $\Delta\phi(t) \equiv |\phi_1(t) - \phi_2(t)|$, one finds that $\Delta\phi(t)$ increases steadily with time. However, when a small amount of coupling is present, $\Delta\phi(t)$ can be confined within 2π, while the amplitudes of the rotations are still completely uncorrelated. The bifurcation that leads to phase synchronization is subsequently investigated [8].

Our experimental study is built upon the following intuition [6]: under the influence of noise, phase synchronization between two weakly coupled chaotic oscillators cannot be sustained forever, but the time that temporal phase synchronization lasts can be extremely long. In particular, it is predicted [6] that additive white noise can induce phase slips in units of 2π between the coupled oscillators which would otherwise be synchronized in phase in the absence of noise. The average time duration between successive phase slips appears to obey the following scaling law with the noise amplitude ϵ, which is typical of the superpersistent chaotic transients:

$$\tau \sim \exp(K\epsilon^{-\alpha})\,, \tag{2}$$

where $\alpha > 0$ is the scaling exponent that depends on system parameters such as the coupling strength, and $K > 0$ is a constant. An implication is that in the presence of only small noise, the average time duration to observe phase synchronization can be extremely long. Phase synchronization is robust in this sense. Since phase

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synchronization occurs in general for continuous-time dynamical systems (flows) only [7], it provides a natural setting for observing and quantifying superpersistent chaotic transients in laboratory experiments, as we will describe in this Letter.

Before we detail our experimental results, we briefly describe the theoretical background for superpersistent chaotic transients. It is argued in Ref. [5] that such a transient is dynamically due to the unstable-unstable pair bifurcations. Briefly, suppose there is an unstable periodic orbit on the chaotic attractor and there is another orbit of the same period on the basin boundary. As p is increased towards p_c, the two orbits *coalesce* and are destroyed simultaneously, leaving behind a "channel" in the phase space through which trajectories on the chaotic attractor can escape. Because of the opening of the channel, the chaotic attractor is converted into a chaotic transient, but the channel created by the mechanism of unstable-unstable pair bifurcation is typically supernarrow, which can be seen as follows. Let T be the time required for a trajectory to pass through the channel. For $p \gtrsim p_c$, T is large and scales with $(p - p_c)$ as $T \sim (p - p_c)^{-\beta}$. In order for such a tunneling to occur, a trajectory on the chaotic set has to stay near the location of the channel for a time T. Because of ergodicity, the probability for a trajectory to stay near the particular location of the unstable periodic orbit for time T is proportional to $e^{-\lambda T}$, where $\lambda > 0$ is the Lyapunov exponent of the chaotic set. The average lifetime of the transient is the inverse of this probability, which is thus proportional to $e^{\lambda T}$. Substituting the scaling of T with $(p - p_c)$ gives the scaling relation (1). Thus, a trajectory initiated in the basin of the original attractor can spend a tremendous amount of time in the region where the original attractor lives, leading to a superpersistent chaotic transient. Another context where these transients occur is riddling in chaotic systems [9]. In Ref. [6], it is argued that the same mechanism is responsible for the extremely long duration of the temporary phase synchronization between weakly coupled chaotic oscillators under the influence of noise, with the noise amplitude's playing the same role as the parameter difference $(p - p_c)$.

Our experimental setup consists of a pair of unidirectionally coupled Chua's circuits [10], as shown schematically in Fig. 1. The differential equations that describe the circuits are

$$\frac{dx_1}{dt} = \frac{1}{C_b}[G(y_1 - x_1) - f(x_1)], \qquad \frac{dx_2}{dt} = \frac{1}{C_b}[G(y_2 - x_2) - f(x_2)] + \frac{x_1 - x_2}{R_c},$$
$$\frac{dy_{1,2}}{dt} = \frac{1}{C_a}[G(x_{1,2} - y_{1,2}) + z_{1,2}], \qquad \frac{dz_{1,2}}{dt} = -\frac{1}{L}(y_{1,2} + R^0_{1,2}z_{1,2}), \tag{3}$$

where x, y, and z are proportional to the voltages across the capacitors C_b and C_a, and the current through the inductor L, respectively, $G = 1/R$, $R^0_1 = R^n_1 R^i/(R^n_1 + R^i)$, $R^0_2 = R^n_2 + R_L$, and $f(x) = G_b x + (G_a - G_b)[|x + E| - |x - E|]/2$ is the piecewise linear function that describes the current-voltage relation of the nonlinear diodes in the circuit. The one-way coupling from circuits 1 to 2 is realized by an operational amplifier (LM741, Harris) and a resistor R_c, where the value of R_c controls the coupling strength. External white noise is injected to the driving circuit. All parameters in the two circuits, except R^n_1 and R^n_2, are set at approximately equal values. Nonidentity between the two circuits is stipulated by setting $R^n_1 = 10\ \Omega$ and $R^n_2 = 6\ \Omega$, which corresponds to about 8.2% of difference in the parameters $R^0_{1,2}$ in Eq. (3). Both circuits are assembled on a high-quality printed-circuit board, and the whole system is enclosed in an electromagnetic shielding box to minimize the influence of uncontrollable external disturbances. The entire system is powered by a low-ripple and low-noise power supply (HPE3631A, HP), and a synthesized functional generator (DS345, SRS) is used as the white noise source whose amplitude can be controlled digitally. The oscillating frequencies of the circuits are in the audiorange, and the signals [dynamical variables in Eq. (3)] are measured by using a 12-bit data acquisition board (KPCI-3110, Keithley) with sampling frequency 2 orders of magnitude higher than those of the circuits.

In order to be able to define and observe the phase variables from both circuits, we choose the parameters such that both circuits generate chaotic attractors with a well defined center of rotation (single scroll) in the three-dimensional phase space [7,11]. Figures 2(a) and 2(b) show the projections in the (x, y) plane of the two attractors from the driving and driven circuits, respectively, in the absence of coupling. The phase angles $\phi_{1,2}(t)$ are obtained by measuring time series of a single dynamical variable, say $y(t)$, from both circuits and constructing the corresponding analytic signals [12]: $\Psi_{1,2}(t) = y_{1,2}(t) + iH[y_{1,2}(t)] \equiv A_{1,2}(t)\exp[i\phi_{1,2}(t)]$, where $H[y_{1,2}(t)]$ are the Hilbert transforms of $y_{1,2}(t)$ and $A_{1,2}(t)$ are the amplitudes of the chaotic rotations in the two circuits, respectively. There exist also other methods for defining the phase [7], but due to the finite sampling rate of the data acquisition device, we find it desirable to choose the one based on the analytic signal, as only two time series, one from each circuit, are necessary.

To see the effect of noise on phase synchronization, we choose the coupling resistence R_C as the bifurcation parameter and first locate, in the absence of noise, an approximate range of its values in which the phase difference $\Delta\phi \equiv \phi_2(t) - \phi_1(t)$ is bounded within 2π. An example of phase synchronization is shown by the lower trace in Fig. 3, where time is in units of T_s, the sampling interval

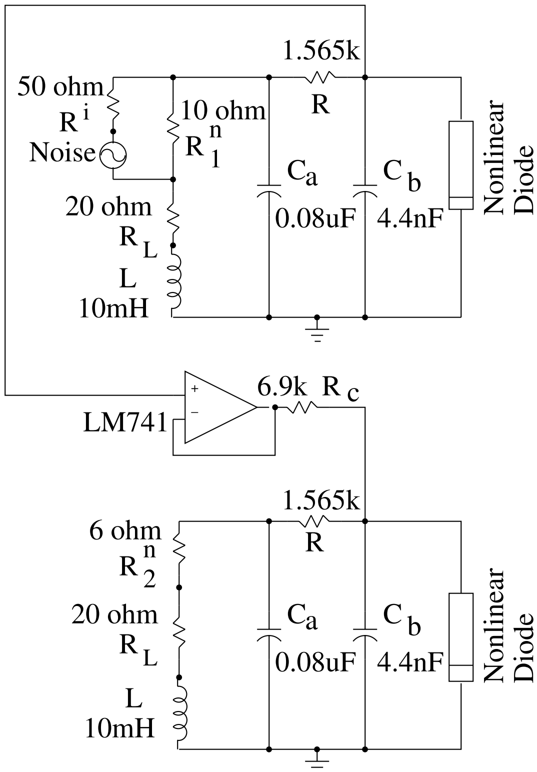

FIG. 1. Schematic diagram of the system of pair of unidirectioanally coupled Chua's circuits and parameters of the circuit components.

of the data acquisition device (approximately 6.67 μs [13]). Additive white noise clearly induces phase jumps in units of 2π, as shown by the upper trace in Fig. 3, where the noise voltage is $\epsilon \approx 0.8$ V. We see that phase synchronization can now be maintained only temporarily in time intervals between successive phase jumps. The durations of these time intervals are apparently random. The temporal phase synchronization observed can thus be regarded as a *transient* chaotic phenomenon, as the dynamics from both circuits are chaotic, despite the phase coherence. The phase jumps are typically rare and become extremely infrequent at smaller noise voltages. A relevant question is then how the transient time intervals are distributed. Figure 4 shows such a distribution at $\epsilon \approx 0.8$ V. We find that the distribution is apparently exponential, indicating that an average transient time τ can be defined. In Fig. 4,

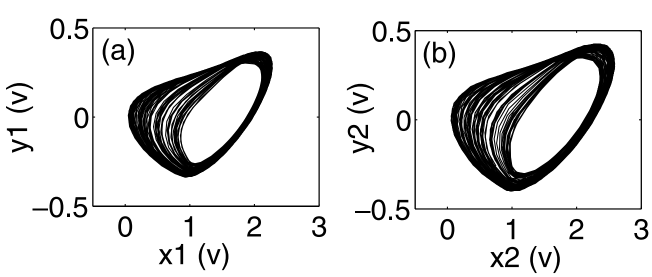

FIG. 2. Chaotic attractors from the driving (a) and driven (b) Chua's circuits.

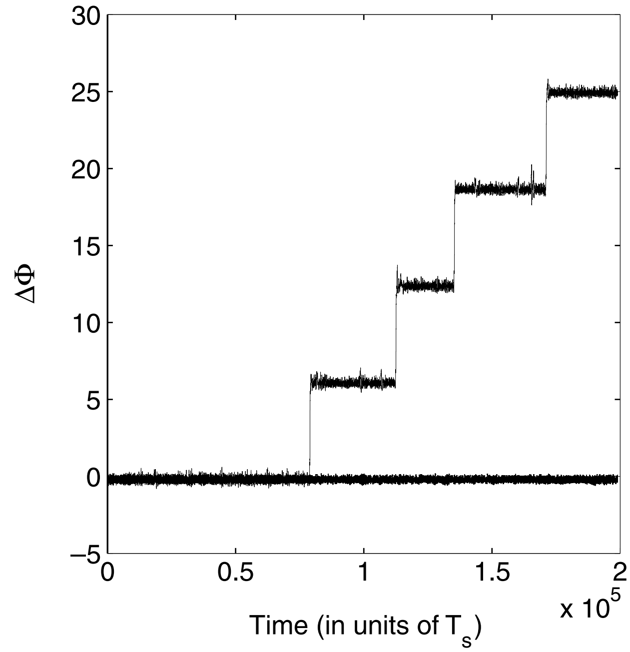

FIG. 3. Phase synchronization in the absence of noise (lower trace) and phase jumps in units of 2π (upper trace) under the influence of noise of voltage $\epsilon \approx 0.8$ V. The chaotic phase synchronization is transient only under noise.

we obtain $\tau \approx 1.67 \times 10^4$ in units of T_s. Physically, it is more convenient to use the average period of rotations on the chaotic attractors. We find that roughly, the average period is about 13.7 sampling time intervals. The transient lifetime for $\epsilon = 0.8$ V is thus about 1217 in units of the average number of the chaotic rotations.

We now report the experimental measurement of the scaling law between the transient lifetime and the noise

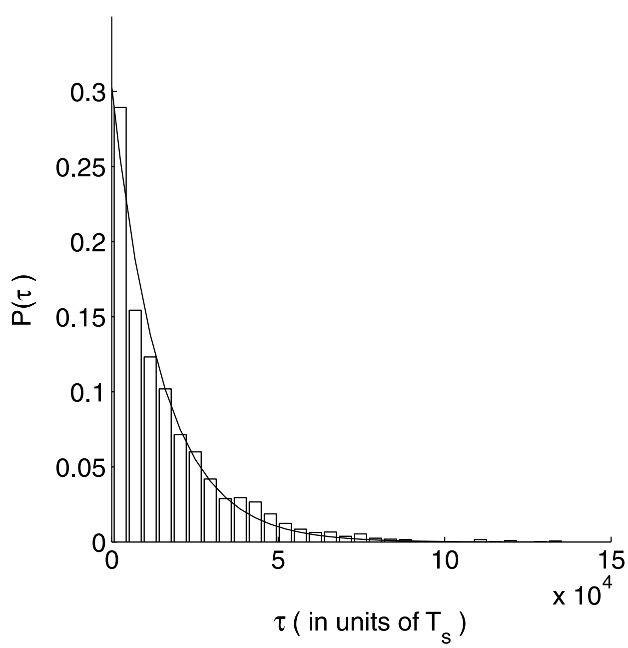

FIG. 4. Probability distribution of the lifetime of chaotic transients at noise voltage $\epsilon \approx 0.8$ V.

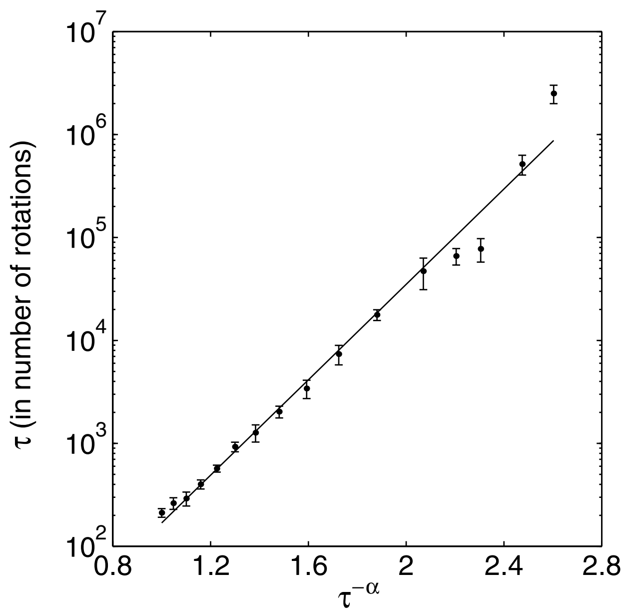

FIG. 5. Experimental scaling of superpersistent chaotic transients.

amplitude. As the noise voltage is reduced, the lifetime increases extremely significantly. We find that for $\epsilon \approx$ 0.3 V, the average lifetime approaches 10^7 cycles [14] of the chaotic rotation, which corresponds to about 40 min of real time. As such, it is practically impossible to obtain a consistent measurement of the lifetimes for lower noise levels. On the other hand, the size of the basin of attraction of the chaotic attractor imposes an upper bound for the noise level. We find that applying noise voltage greater than, say, 2.0 V, causes frequent collapse of the chaotic rotations. In fact, in order to obtain a reliable value of the lifetime at the larger end of the noise level via repetitive measurements, it is necessary to restrict the noise voltage to less than 2.0 V. We are thus forced to work with a range of noise level that is slightly less than 1 order of magnitude. Changing the circuit configuration does not seem to improve the situation. Nonetheless, the scaling obtained suggests a clear signature of superpersistent chaotic transients, as shown in Fig. 5, where the average lifetime (in units of rotation) versus $\epsilon^{-\alpha}$ is plotted on a semilogarithmic scale. To obtain the scaling, we choose 16 levels of noise in the range 0.3 V $< \epsilon <$ 2.0 V, and for each noise level, we measure time series with 5 to 100 phase-synchronized time intervals to obtain τ. The confidence intervals at each measurement of τ are obtained by repeating the whole process for 10 times. The scaling exponent is $\alpha \approx 0.91$ for Fig. 5. The relatively large fluctuations of the experimental data about the theoretical fitting for $\epsilon^{-\alpha} > 2.0$ (equivalently $\epsilon < 0.47$) come from the fact that many independent, long runs are necessary to accumulate enough numbers of 2π-phase jumps at small noise levels versus cases of $\epsilon^{-\alpha} < 2.0$ where typically only a few, shorter runs are enough to yield sufficient numbers of phase jumps for statistical analysis. During the long runs, small drifts in the parameters of the circuits can occur. In addition, the influence of uncontrollable changes in the experimental environment becomes relatively significant for small noise levels. We believe that these factors contribute to the fluctuations. Nonetheless, Fig. 5 suggests a robust scaling Eq. (2), despite the extremely challenging task of maintaining stable operation of the circuit system in a controlled noisy environment for long periods of time and manipulating huge data files from the measurement. We are thus confident that we have observed, for the first time, superpersistent chaotic transients in laboratory experiments.

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[13] The Keithley data acquisition board utilized in our experiments can produce a sampling rate of 150×10^3, which corresponds to the sampling time interval of $1/150 \times 10^3 \approx 6.67$ μs. The board collects data exactly at this sampling rate, with no external triggerings being necessary.

[14] A lifetime on the order of 10^7 cycles, typically encountered at small noise levels, corresponds to about 40 min of recording in real time. To obtain τ, we require a minimum of five phase-synchronized intervals from each time series. Thus, it is necessary to acquire the data for about 200 min without any interruption. Extreme cares were exercised in our experiments to ensure that. For one time series, the typical size of the file is about 4 Gbytes.