Antiphase synchronism in chaotic systems

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We report our finding and analysis of a type of synchronism that occurs in chaotic systems with symmetry. Specifically, we find that the amplitudes of the dynamical variables of such a system can be synchronized with those of its replica, but that the variables can have different signs with respect to each other. This type of antiphase chaotic synchronism is observable in wide parameter regimes even for hyperchaotic systems. The mechanism of the synchronism suggests a systematic and *a priori* way to construct synchronizable chaotic systems. Application to nonlinear digital communication is pointed out. [S1063-651X(98)02607-5]

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I. INTRODUCTION

One of the most striking discoveries in the study of chaos is that chaotic systems can be made to synchronize with each other [1]. This discovery by Pecora and Carroll in 1990 was both theoretically surprising and practically significant. Theoretically, chaos stipulates that nearby trajectories diverge exponentially in time and, thus, synchronization of chaotic systems seems unlikely in the presence of inevitable small differences in parameters of the systems, and noise. It was shown by Pecora and Carroll [1], however, that when an appropriately chosen state variable of a chaotic system is used to drive a subsystem (the ''slave''), the subsystem synchronizes with its replica if its Lyapunov exponents are all negative. Practically, synchronization of chaos provides a way to transmit information via a chaotic carrier and, therefore, synchronous chaotic systems can be utilized for communication [2]. Due to these appealing features, synchronism in chaotic systems has become a direction of intense recent research [3].

In this paper we report our finding of a class of synchronism that exists in chaotic systems with symmetry. Specifically, consider a chaotic system described by either an N-dimensional continuous flow $d\mathbf{z}/dt=\mathbf{F}(\mathbf{z},p)$ or an N-dimensional discrete map $\mathbf{z}_{n+1}=\mathbf{F}(\mathbf{z}_n,p)$, where $\mathbf{z}$ is the state variable, $\mathbf{F}$ is a nonlinear vector function that has a simple type of symmetry, and p is a system parameter. When the variable $\mathbf{z}$ is decomposed in the manner $\mathbf{z}=(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$ is the driving system and $\mathbf{y}$ is the slaving subsystem, we find that the subsystem $\mathbf{y}$ can synchronize with its replica in amplitude but with opposite sign for initial conditions chosen from large regions in the phase space. That is, for a replica $\mathbf{y}'$ of the slaving subsystem, the following can occur:

$$\mathbf{y}(t)=-\mathbf{y}'(t) \quad \text{as} \quad t\to\infty. \tag{1}$$

We call this type of behavior *antiphase chaotic synchronism*. There also exist regions of the phase space, from which initial conditions chosen yield full synchronization, that is, $\mathbf{y}(t)=\mathbf{y}'(t)$ as $t\to\infty$.

There are several features associated with the above synchronization scheme. (i) In most studies reported in the literature [1–3], in order to search for synchronizable chaotic systems, one usually tests various combinations of a subset of state variables to look for a subsystem that possesses only negative Lyapunov exponents. Our scheme provides a systematic and *a priori* way to design synchronizable chaotic systems. (ii) In our scheme, synchronization can be readily achieved even when the system has more than one positive Lyapunov exponent (hyperchaotic). (iii) The combination of in-phase and antiphase synchronization provides a way to encode messages into an array of synchronous chaotic systems for massive communication of digital information (see Sec. V).

The rest of the paper is organized as follows. In Sec. II, we present a theory for the antiphase synchronism. In Sec. III, we give a numerical example with a two-dimensional map. In Sec. IV, we demonstrate, by utilizing a six-dimensional hyperchaotic flow, that antiphase synchronism can also occur in continuous chaotic systems. In Sec. V, we present discussions and conclusions.

II. THEORY

Consider an N-dimensional map $\mathbf{z}_{n+1}=\mathbf{F}(\mathbf{z}_n,p)$ with the decomposition of the system into a driving subsystem $\mathbf{x}$ (dimension N_x) and a subsystem $\mathbf{y}$ to be synchronized (dimension N_y, where $N_x+N_y=N$). We write the following equations for $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n),$$
$$\mathbf{y}_{n+1}=h(\mathbf{x}_n,p)\mathbf{G}(\mathbf{y}_n), \tag{2}$$

where $\mathbf{f}(\mathbf{x}_n)$ is a nonlinear map that generates a chaotic attractor, $h(\mathbf{x},p)$ is a scalar driving function, and $\mathbf{G}(\mathbf{y}_n)$ is a vector function that possesses symmetry. For simplicity, we consider the reflecting symmetry in $\mathbf{G}(\mathbf{y}_n)$: $\mathbf{G}(-\mathbf{y}_n)=-\mathbf{G}(\mathbf{y}_n)$. There is then an invariant subspace defined by $\mathbf{y}=\mathbf{0}$, in which there is a chaotic attractor generated by the

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map $\mathbf{f}(\mathbf{x}_n)$. The subspace $\mathbf{y}=\mathbf{0}$ is invariant because a trajectory starting with $\mathbf{y}=\mathbf{0}$ is confined to $\mathbf{y}=\mathbf{0}$ at all subsequent times. The replica of the subsystem to be synchronized is

$$\mathbf{y}'_{n+1}=h(\mathbf{x}_n,p)\mathbf{G}(\mathbf{y}'_n). \tag{3}$$

In order to achieve synchronization between $\mathbf{y}$ and $\mathbf{y}'$, the largest Lyapunov exponent of the $\mathbf{y}$ subsystem must be negative [1]. For our system [Eq. (2)], this exponent can be written as $\Lambda_{\mathbf{y}}=\lim_{T\to\infty}(1/T)\Sigma_{n=1}^{T}\ln|h(\mathbf{x}_n,p)\mathbf{DG}(\mathbf{y}_n)\cdot\mathbf{u}|$, where $\mathbf{u}$ is a unit vector in the $\mathbf{y}$ subspace, and $\mathbf{DG}(\mathbf{y}_n)\equiv\partial\mathbf{G}/\partial\mathbf{y}|_{\mathbf{y}_n}$ is the $N_y\times N_y$ Jacobian matrix of the function $\mathbf{G}(\mathbf{y})$ evaluated along a typical trajectory in the phase space. To search for synchronizable subsystems that satisfy $\Lambda_{\mathbf{y}}<0$, we express $\mathbf{DG}(\mathbf{y}_n)$ by using a Taylor expansion, $\mathbf{DG}(\mathbf{y}_n)=\mathbf{DG}(\mathbf{0})+\mathbf{A}(\mathbf{y}_n)$, where $\mathbf{DG}(\mathbf{0})\equiv\mathbf{DG}(\mathbf{y})|_{\mathbf{y}_n=\mathbf{0}}$ is the Jacobian matrix evaluated at $\mathbf{y}=\mathbf{0}$, and $\mathbf{A}(\mathbf{y}_n)$ represents all the high-order terms in the expansion, which is an $N_y\times N_y$ matrix that depends on $\mathbf{y}_n$. We thus obtain

$$\Lambda_{\mathbf{y}}=\Lambda_T+\lambda, \tag{4}$$

where

$$\Lambda_T=\lim_{T\to\infty}\frac{1}{T}\sum_{n=1}^{T}\ln|h(\mathbf{x}_n,p)\mathbf{DG}(\mathbf{0})\cdot\mathbf{u}|,$$

and

$$\lambda=\lim_{T\to\infty}\frac{1}{T}\sum_{n=1}^{T}\ln|h(\mathbf{x}_n,p)\mathbf{A}(\mathbf{y}_n)\cdot\mathbf{u}|. \tag{5}$$

Notice that Λ_T is the transverse Lyapunov exponent defined locally with respect to the invariant subspace $\mathbf{y}=\mathbf{0}$ [4]. When $\Lambda_T<0$, trajectories in the vicinity of the $\mathbf{y}=\mathbf{0}$ approach it asymptotically. The chaotic attractor in the invariant subspace thus attracts initial conditions in the entire phase space if there are no other attractors. This leads to the asymptotic solution $\mathbf{y}=\mathbf{0}$, which is not interesting from the standpoint of synchronization. To achieve nontrivial chaotic synchronization, we must have $\Lambda_T>0$. In this case, trajectories in the vicinity of $\mathbf{y}=\mathbf{0}$ can be repelled away from it and the dynamics near the invariant subspace is chaotic. From Eq. (4), we see that in order to have $\Lambda_{\mathbf{y}}<0$, we can choose $\Lambda_T\gtrsim 0$ and $\lambda<0$. As we shall see in numerical examples, it is in fact quite straightforward to choose the functions $h(\mathbf{x}_n,p)$ and $\mathbf{G}(\mathbf{y})$ to satisfy this condition.

The key observation that antiphase synchronism can occur is that the system can have symmetry-broken attractors when $\Lambda_T\gtrsim 0$ [5]. Specifically, it was shown in Ref. [5] that the transition from $\Lambda_T<0$ to $\Lambda_T>0$, as the parameter p changes through a critical value p_c [6], can in general be a symmetry-breaking bifurcation. For $p<p_c$ ($\Lambda_T<0$), the chaotic attractor in the invariant subspace is the only attractor of the system. For $p>p_c$ ($\Lambda_T>0$), the attractor in the invariant subspace is no longer an attractor of the entire phase space. Instead, two isolated attractors, perfectly symmetric with respect to each other, are born at $p=p_c$, one lying in the upper half space $\mathbf{y}>\mathbf{0}$ and another in the lower half space $\mathbf{y}<\mathbf{0}$. The boundary between basins of attraction of these two attractors is $\mathbf{y}=\mathbf{0}$. These attractors are symmetry broken because they are confined only within half of the phase space and, hence, they do not possess the reflecting symmetry in the equations of the system Eq. (2). Due to symmetry in the system equations, all the statistical properties such as averages and the Lyapunov exponents are identical for both attractors. In this case, if $\Lambda_{\mathbf{y}}<\mathbf{0}$, trajectories starting from two random initial conditions, one in $\mathbf{y}>\mathbf{0}$ and another in $\mathbf{y}<\mathbf{0}$, tend to evolve as ''mirror image'' of each other. Thus, depending on the choice of initial conditions, both in-phase and antiphase synchronism can occur. In particular, when initial conditions for both the subsystem and its replica are chosen in the $\mathbf{y}>\mathbf{0}$ (or $\mathbf{y}<\mathbf{0}$) space, we have an in-phase synchronism: $\lim_{n\to\infty}\mathbf{y}_n=\mathbf{y}'_n$. However, if the initial condition of the subsystem is chosen in $\mathbf{y}>\mathbf{0}$ but that of its replica is chosen in $\mathbf{y}<\mathbf{0}$, or vice versa, then antiphase synchronism, $\lim_{n\to\infty}\mathbf{y}_n=-\mathbf{y}'_n$, can occur.

III. NUMERICAL EXAMPLE: A TWO-DIMENSIONAL MAP

We first give a simple numerical example to illustrate antiphase synchronism. We consider the following two-dimensional version of Eq. (2):

$$x_{n+1}=rx_n(1-x_n),$$

$$y_{n+1}=\frac{1}{2\pi}px_n\sin(2\pi y_n), \tag{6}$$

where the invariant subspace is $y=0$, in which the dynamics is described by the one-dimensional logistic map in x, and p is a parameter. We choose the parameter r such that the logistic map generates a chaotic attractor. We concentrate on the phase space region $(0\leqslant x\leqslant 1,-0.5\leqslant y\leqslant 0.5)$ because of the range of the logistic map and the periodicity in the y equation. The replica of the y subsystem to be synchronized is $y'_{n+1}=(1/2\pi)px_n\sin(2\pi y'_n)$. The transverse Lyapunov exponent of Eq. (6) is $\Lambda_T=\int_0^1\ln|px|\rho(x)dx$, where $\rho(x)$ is the invariant density of x for the logistic map. Thus, we have $p_c=\exp[-\int_0^1\ln|x|\rho(x)dx]$, where $\Lambda_T\geqslant 0$ for $p\geqslant p_c$ and $\Lambda_T<0$ for $p<p_c$. It was shown in Ref. [5] that the bifurcation at p_c is a symmetry-breaking bifurcation. For $p<p_c$, $y=0$ is the only attractor of Eq. (6). For $p\gtrsim p_c$, there are two attractors, completely symmetric to each other with respect to $y=0$, one in the upper half plane $y>0$ and another in the lower half plane $y<0$. The boundary between basins of attraction of the two attractors is the line segment $y=0$ and $0\leqslant x\leqslant 1$. For Eq. (6), a symmetry-increasing bifurcation occurs at parameter value $p_s=\pi$ [5] after which the two symmetry-broken attractors merge into a single chaotic attractor with two positive Lyapunov exponents through a crisis. Thus we expect antiphase synchronism to occur in the parameter range $p_c<p\lesssim p_s$.

The Lyapunov exponent of the y subsystem is given by $\Lambda_y=\Lambda_T+\int\ln|\cos(2\pi y_n)|\rho_y(y)dy$, where $\rho_y(y)$ is the probability distribution of y. The integral in y is always negative and, hence, it is possible to have $\Lambda_y<0$ while $\Lambda_T>0$. We note that for $p<p_c$, we have $\Lambda_y=\Lambda_T$ because $y_n=0$ asymptotically and, therefore, $\int\ln|\cos(2\pi y_n)|\rho_y(y)dy=0$. Figure 1

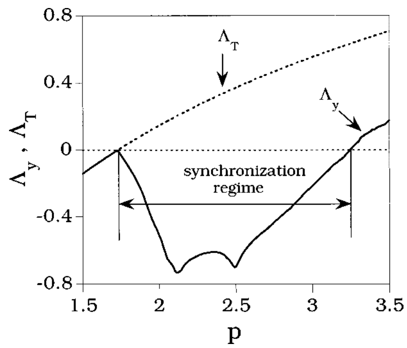

FIG. 1. Λ_y and Λ_T vs the parameter p in the model system Eq. (6).

shows Λ_y and Λ_T versus p for $1.5\leqslant p\leqslant 3.5$ ($r=3.8$ in the logistic map). We see that Λ_y remains negative for $p_c<p<p_s$ except when p is very close to p_s. Figure 2(a) shows the time series y_n and y_n' for a case of antiphase synchronism, where the initial conditions are $y_0>0$ and $y_0'<0$. We see that the two trajectories rapidly become symmetric to each other with respect to $y=0$. Figure 2(b) shows, on a logarithmic scale, the quantity $|y_n^2-y_n'^2|$ versus time. Clearly, the amplitudes of the y subsystem and its replica become synchronized as $n\rightarrow\infty$, but the phases of the chaotic time series are just opposite.

IV. NUMERICAL EXAMPLE: A SIX-DIMENSIONAL FLOW

The antiphase synchronism can also occur in continuous chaotic systems. To demonstrate this, we now study a six-dimensional hyperchaotic flow

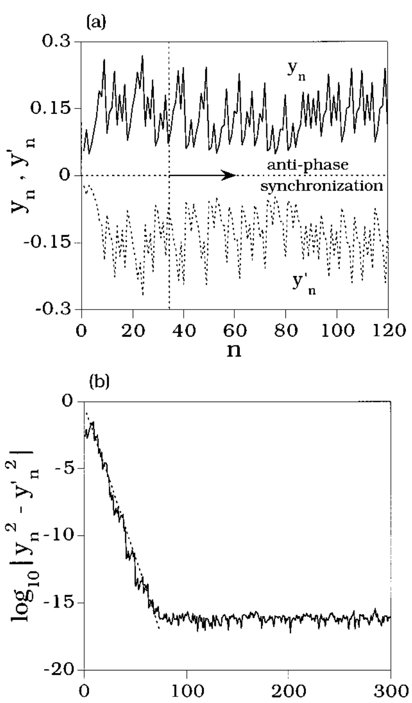

FIG. 2. Antiphase synchronism at $p=1.85$. (a) Time series y_n and y_n'. (b) $|y_n^2-y_n'^2|^{1/2}$ vs n on a logarithmic scale.

$$\begin{aligned}
\frac{dx_1}{dt}&=-x_2-x_3+ay,\\
\frac{dx_2}{dt}&=x_1+0.25x_2+x_4+bz^2,\\
\frac{dx_3}{dt}&=3.0+x_1x_3,\\
\frac{dx_4}{dt}&=-0.5x_3+0.05x_4,\\
\frac{dy}{dt}&=z,\\
\frac{dz}{dt}&=-\alpha z-\gamma y^3+(\beta+f_1\sin x_1+f_2\sin x_2)\sin(2\pi y),
\end{aligned}\tag{7}$$

where the invariant subspace is four-dimensional (x_1,x_2,x_3,x_4) defined by $y=0$ and $z=0$, the transverse subsystem to be synchronized is two-dimensional (y,z), and a, b, α, γ, β, f_1, and f_2 are parameters. Note that the cases where $a\neq 0$, $b\neq 0$, and $a=b=0$ correspond to bidirectional and unidirectional couplings from the invariant subspace to the transverse subspace, respectively. The choices of the coupling terms such as ay and bz^2 are chosen rather arbitrarily. In Eq. (7), the variables (x_1,x_2,x_3,x_4) constitute the hyperchaotic Rössler chaotic system with two positive Lyapunov exponents [7]. The replica of the transverse subsystem to be synchronized is

$$\begin{aligned}
\frac{dy'}{dt}&=z',\\
\frac{dz'}{dt}&=-\alpha z'-\gamma(y')^3+(\beta+f_1\sin x_1\\
&\quad+f_2\sin x_2)\sin(2\pi y').
\end{aligned}\tag{8}$$

For concreteness, we fix $a=1.0$, $b=2.0$, $\gamma=2.0$, $f_1=3.5$, and $f_2=5.0$, and change α and β to identify synchronizable parameter regimes with $\Lambda_T>0$ and $\Lambda_{\mathbf{y}}<0$. We find that there are large parameter regions for which antiphase synchronism can be achieved. Figure 3(a) shows such a case for $\alpha=10$ and $\beta=1.6$, where $z(t)$ versus t and $z'(t)$ versus t are plotted. Figure 3(b) shows, on a semilogarithmic scale, the quantity $\Delta(t)\equiv\sqrt{[|y(t)|-|y'(t)|]^2+[|z(t)|-|z'(t)|]^2}$ versus t. We see that $[y(t),z(t)]$ approaches $[-y'(t),-z'(t)]$ rapidly. We note that at this parameter setting, the full six-dimensional system [Eq. (7)] possesses the following Lyapunov spectrum (approximately): (0.109, 0.021, 0, -1.891, -7.749, -24.450) and, hence, the synchronism illustrated in Fig. 3 occurs in a hyperchaotic system with two positive Lyapunov exponents.

V. DISCUSSIONS AND CONCLUSIONS

In summary, we find that both in-phase and antiphase synchronism can occur in chaotic systems with symmetry in parameter regimes where there is symmetry breaking. The

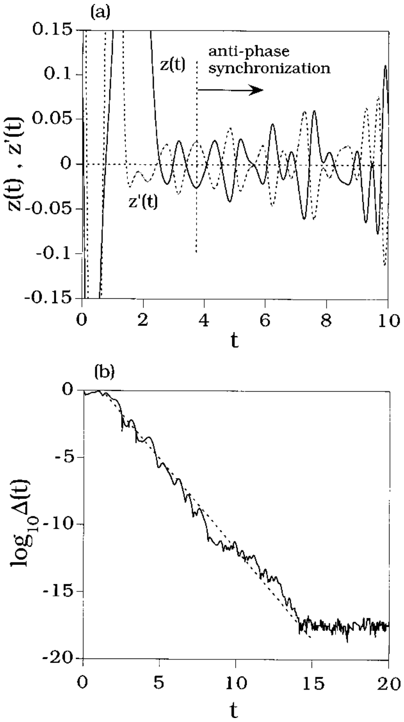

FIG. 3. Antiphase synchronism for the six-dimensional hyperchaotic flow [Eq. (7)]. (a) Time series $z(t)$ and $z'(t)$. (b) $\Delta(t)$ vs t on a semilogarithmic scale. See text for details.

synchronization mechanism elucidated in this paper suggests a general and *a priori* approach to construct synchronizable chaotic systems. The synchronism can be readily realized even for hyperchaotic systems. Due to these advantages, we expect the chaotic synchronism reported here to be practically useful.

The antiphase synchronism reported in this paper relies on the system possessing a simple symmetry. That is, in order to realize $\mathbf{y}(t)\rightarrow-\mathbf{y}'(t)$, where $\mathbf{y}$ and $\mathbf{y}'$ are the two subsystem's to be synchronized, it is necessary that both subsystems have an identical symmetry. Antiphase synchronization occurs simply because the trajectories of one subsystem live on a chaotic attractor, while the trajectories of the second subsystem wander on a chaotic attractor which is completely symmetric to the first attractor. Mathematically, this demands that the functions $h(\mathbf{x},p)$ and $\mathbf{G}(\mathbf{y})$ in Eq. (2) be identical for both subsystems. A slight mismatch between these functions for both subsystems may be allowed, but in such a case the quality of the synchronization, measured by $[|\mathbf{y}(t)|-|\mathbf{y}'(t)|]$, will be proportional to the amount of the mismatch. Antiphase synchronism may fail if the mismatch is too large.

A potential usage of the phenomenon of antiphase synchronism lies in nonlinear digital communication, which has become a field of recent interest. So far there have been two different approaches to the problem. One is to use the principle of synchronous chaos [1–3] to embed and transmit digital information. Call this method 1. Another is to extend the principle of controlling chaos [8] to dynamical systems with well-defined symbolic dynamics to encode information [9,10]. This approach makes explicit use of the fundamental principle that chaotic systems are natural information sources. By manipulating the symbolic dynamics of a chaotic system in an intelligent way, the system produces trajectories in which digital information is embedded in the symbolic dynamics. Call this method 2. Here we wish to point out that the coexistence of in-phase and antiphase synchronization may be quite useful in nonlinear digital communication. *The idea is to combine the principles of both method 1 and method 2, by utilizing antiphase and in-phase synchronization, to massively encode a large amount of digital information into an array of chaotic systems*. Say we construct an array of M synchronizable subsystems (or oscillators) $\mathbf{y}_i$ ($i=1,...,M$), all driven by the same chaotic signal $\mathbf{x}$. Initial conditions are chosen so that some of the oscillators are out of phase with the remaining oscillators. Due to the existence of the two distinct phases, one can now assign binary symbols to the array of oscillators. For instance, oscillators with $\mathbf{y}>\mathbf{0}$ are assigned symbol **1**, and those with $\mathbf{y}<\mathbf{0}$ are assigned symbol **0**. A digital message, represented by a finite sequence of binary symbols, can now be encoded into the array of oscillators, with each oscillator bearing one information bit. The whole message is thus encoded *simultaneously*. To encode a new message, one waits until $|\mathbf{y}|$ is close to the symmetric axis $\mathbf{y}=\mathbf{0}$, at which time small perturbations to the oscillator's dynamical variables are applied to change the state of the oscillator from $\mathbf{y}>\mathbf{0}$ to $\mathbf{y}<\mathbf{0}$, or vice versa, depending on details of the binary representation of the new message. By symmetry, *all oscillators come close to the symmetric axis simultaneously*. Thus small perturbations are applied at the same time. An advantage of this type of encoding is that the amount of information that can be encoded can be made large by simply increasing the number of driven oscillators. This is thus essentially a multichannel digital communication scheme, and the aspect of the synchronization utilized offers many advantages such as a good timing for decoding messages.

Finally, we remark that recently, the phenomena of phase and lag synchronization have been discovered and studied [11]. In such a case, an anglelike phase function of a chaotic oscillator, defined with respect to some rotation of chaotic trajectories in the phase space, can be made to stay close to the phase function of another chaotic oscillator (not necessarily identical to the first oscillator) when the two oscillators are weakly coupled to each other. The antiphase synchronism discussed in this paper is different from both the phase and lag synchronization. This can be seen by noting that, dynamically, antiphase synchronism occurs when the largest Lyapunov exponents of both subsystems become negative. Thus antiphase synchronism is essentially the same type of synchronism first studied by Pecora and Carroll [1]. The difference is that we allow for the coexistence of chaotic attractors which are symmetric to each other.

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