Bifurcation to strange nonchaotic attractors

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Strange nonchaotic attractors are attractors that are geometrically strange, but have nonpositive Lyapunov exponents. These attractors occur in regimes of nonzero Lebesgue measure in the parameter space of quasiperiodically driven dissipative dynamical systems. We investigate a route to strange nonchaotic attractors in systems with a symmetric invariant subspace. Assuming there is a quasiperiodic torus in the invariant subspace, we show that the loss of the transverse stability of the torus can lead to the birth of a strange nonchaotic attractor. A physical phenomenon accompanying this route to strange nonchaotic attractors is an extreme type of intermittency. We expect this route to be physically observable, and we present theoretical arguments and numerical examples with both quasiperiodically driven maps and quasiperiodically driven flows. The transition to chaos from the strange nonchaotic behavior is also studied. [S1063-651X(97)10908-4]

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I. INTRODUCTION

A central problem in the study of deterministic dynamical systems is to identify different types of asymptotic behaviors of the system and to understand how the behavior changes as a system parameter changes. The asymptotic behaviors can be, for instance, a steady state, a periodic oscillation, a quasiperiodic motion, and a random or a chaotic motion. There has been a lot of work in the past addressing how dynamical systems develop chaos from periodic or quasiperiodic motions. It is known so far that there are four major routes to chaotic attractors [1–5]: (i) the period-doubling cascade route [2]; (ii) the intermittency transition route [3]; (iii) the crisis route [4]; and (iv) the route to chaos in quasiperiodically driven systems [5].

This paper concerns bifurcations to a type of motion in deterministic systems that is neither regular (periodic or quasiperiodic) nor chaotic. The motion occurs on *strange nonchaotic attractors*, which are attractors that are geometrically complicated, but *asymptotically*, typical trajectories on the attractors exhibit no sensitive dependence on initial conditions [5–16]. Here, the word *strange* refers to the complicated geometry of the attractor: a strange attractor is not a finite set of points and it is not piecewise differentiable. The word *chaotic* refers to a sensitive dependence on initial conditions: trajectories originating from nearby initial conditions diverge exponentially in time. Mathematically, strange nonchaotic attractors occur in all dissipative dynamical systems that exhibit the period-doubling route to chaos: the attractor at the accumulation point of the period-doubling cascade is a fractal set, but its largest Lyapunov exponent is not positive. However, such a strange attractor is not observable in reality because the set of parameter values for the accumulation of the period-doubling cascade has a Lebesgue measure zero in the parameter space. Strange nonchaotic attractors are, however, observable in dissipative systems driven by several *incommensurate* frequencies (quasiperiodically driven systems) [5–16]. For example, it was demonstrated that in systems driven by two incommensurate frequencies, there exist regions of positive Lebesgue measure in the parameter space for which strange nonchaotic attractors exist [5,8]. More recent work demonstrated that typical trajectories on a strange nonchaotic attractor actually possess positive Lyapunov exponents in finite time intervals, although asymptotically, the exponent is negative [12]. These attractors also exhibit unusual spectral and correlation properties [11]. Strange nonchaotic attractors can arise in physically relevant situations such as quasiperiodically forced damped pendulums and localization of quantum particles in quasiperiodic potentials [7], and also in biological oscillators [9]. These exotic attractors have been observed in physical experiments [13,14].

While the existence of strange nonchaotic attractors was firmly established, a question that remains interesting is how these attractors are created as a system parameter changes through a critical value, i.e., what are the possible routes to strange nonchaotic attractors? One route was investigated by Heagy and Hammel [10] who discovered that, in quasiperiodically driven maps, the transition from two-frequency quasiperiodicity to strange nonchaotic attractors occurs when a period-doubled torus collides with its unstable parent torus. Near the collision, the period-doubled torus becomes extremely wrinkled and develops into a fractal set at the collision, while the Lyapunov exponent remains negative throughout the collision process. Recently, Feudel, Kurths, and Pikovsky found that the collision between a stable torus and an unstable one at a dense set of points leads to a strange nonchaotic attractor [15]. A renormalization-group analysis was also devised for the transition to strange nonchaotic attractors in a particular class of quasiperiodically driven maps [16].

In this paper, we present a route to strange nonchaotic attractors in dynamical systems with a symmetric low-dimensional invariant subspace S in the phase space [17]. Since S is invariant, initial conditions in S result in trajectories which remain in S forever. We consider the case where *there is a quasiperiodic torus in* S, as shown schematically in Fig. 1. Whether the torus attracts or repels initial condi-

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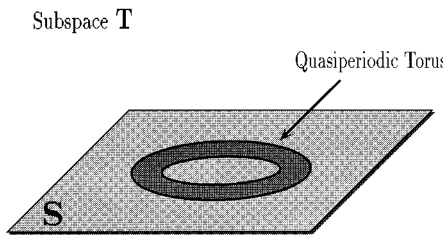

FIG. 1. Schematic representation of the invariant subspace **S** in which the quasiperiodic torus lies and the transverse subspace **T**.

tions in the vicinity of S is determined by the sign of the largest Lyapunov exponent Λ_T computed for trajectories in S with respect to perturbations in the subspace T which is *transverse* to S. When Λ_T is negative, S attracts trajectories transversely in the phase space and, the quasiperiodic torus in S is an attractor of the full phase space. When Λ_T is positive, trajectories in the vicinity of S are repelled away from it, and, consequently, the torus is transversely unstable and it is hence not an attractor of the full phase space. Assume that as a system parameter changes through a critical value a_c, Λ_T passes through zero from the negative side. This bifurcation is referred to as the ''blowout bifurcation'' [18]. Our main result is that the blowout bifurcation can lead to the birth of a strange nonchaotic attractor. A physical phenomenon accompanying this route to strange nonchaotic attractors is that the dynamical variables in the transverse subspace T exhibit an extreme type of temporally intermittent bursting behavior: on-off intermittency [19,20]. Thus, as a by-product, our work also demonstrates that on-off intermittency can occur in quasiperiodically driven dynamical systems, whereas to our knowledge, these intermittencies have been reported only for systems that are driven either randomly or chaotically. A short account of this work has been published recently [21].

The rest of the paper is organized as follows. In Sec. II, we study the blowout bifurcation route to strange nonchaotic attractors in discrete dynamical systems. In particular, we study a class of quasiperiodically driven maps for which the bifurcation can be understood fairly completely. We also investigate on-off intermittency after the birth of the strange nonchaotic attractor and the transition to chaos. In Sec. III, we demonstrate that all results obtained from the map can be observed in a quasiperiodically driven physical system, mathematically described by a continuous flow. Discussions are present in Sec. IV.

II. QUASIPERIODICALLY DRIVEN MAPS

A. Blowout bifurcation to strange nonchaotic attractors

We consider the following general class of N-dimensional maps,

$$\mathbf{x}_{n+1}=\mathbf{f}(\mathbf{x}_n),$$
$$\mathbf{y}_{n+1}=F(\mathbf{x}_n,p)\mathbf{G}(\mathbf{y}_n), \tag{1}$$

where $x\in\mathbb{R}^{N_x}$ $(N_x\geqslant 1)$, $y\in\mathbb{R}^{N_y}$ $(N_y\geqslant 1)$, and $N_x+N_y=N$. The vector function $\mathbf{G}(\mathbf{y})$ satisfies $\mathbf{G}(\mathbf{0})=\mathbf{0}$ so that $\mathbf{y}=\mathbf{0}$ defines the invariant subspace [22]. We assume that both the $\mathbf{x}$ and $\mathbf{y}$ dynamics are bounded. The N_x-dimensional vector function $\mathbf{f}(\mathbf{x})$ is a map that has a quasiperiodic torus so that the largest Lyapunov exponent of the $\mathbf{x}$ dynamics is $\Lambda_x=0$. The scalar function $F(\mathbf{x},p)$ is thus the quasiperiodic driving to the transverse $\mathbf{y}$ subsystem, and p is the bifurcation parameter. The largest transverse Lyapunov exponent is given by

$$\Lambda_T=\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L}\ln|F(x_n,p)\mathbf{DG}(\mathbf{y}_n)|_{y_n=0}\cdot\mathbf{u}|, \tag{2}$$

where $\mathbf{u}$ is a unit vector in $\mathbb{R}^{N_y}$. The largest Lyapunov exponent Λ_y of the y subsystem is given by

$$\Lambda_y=\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{L}\ln|F(x_n,p)\mathbf{DG}(\mathbf{y}_n)\cdot\mathbf{u}|, \tag{3}$$

where now $\mathbf{y}_n$ is *not* set to be $\mathbf{0}$ when the Jacobian matrices $\mathbf{DG}(\mathbf{y}_n)$'s are evaluated. Since the $\mathbf{x}$ dynamics represents the quasiperiodic driving to the $\mathbf{y}$ dynamics, and the largest Lyapunov exponent in $\mathbf{x}$ is zero, we see that Λ_y is in fact the largest nontrivial Lyapunov exponent of the system which determines whether the system is chaotic or nonchaotic. In particular, if $\Lambda_y>0$ $(\leqslant 0)$, the system is chaotic (nonchaotic).

We now argue that a blowout bifurcation can lead to the birth of a strange nonchaotic attractor. Let p_c be the bifurcation point, i.e., as the parameter p passes through p_c, the transverse Lyapunov exponent Λ_T passes through zero from the negative side. Thus, for $p<p_c$ $(\Lambda_T<0)$, the quasiperiodic torus in the invariant subspace $\mathbf{y}=\mathbf{0}$ is transversely stable so that typical trajectories are eventually attracted transversely towards $\mathbf{y}=\mathbf{0}$ and asymptote to the quasiperiodic torus there. For $p\gtrsim p_c$ $(\Lambda_T\gtrsim 0)$, the quasiperiodic torus in $\mathbf{y}=\mathbf{0}$ is transversely unstable and, hence, typical trajectories are chaotic locally near $\mathbf{y}=\mathbf{0}$. There are now some time intervals during which a trajectory in the vicinity of $\mathbf{y}=\mathbf{0}$ can be repelled from it. In this case, if there are no other attractors in the phase space, the trajectory comes back to the neighborhood of $\mathbf{y}=\mathbf{0}$ in an intermittent fashion. Since the trajectory is bounded in both $\mathbf{x}$ and $\mathbf{y}$, the asymptotic attractor in the full phase space $(\mathbf{x},\mathbf{y})$ exhibits a complicated geometric shape due to the local chaoticity in the vicinity of the invariant subspace. However, if the nontrivial Lyapunov exponent Λ_y is negative, which indeed occurs if the magnitude of the eigenvalues of the Jacobian matrices $\mathbf{DG}(\mathbf{y}_n)$'s evaluated along the trajectory is less than one, then the attractor, though geometrically complex, is not chaotic because both Lyapunov exponents Λ_x and Λ_y are not positive. Consequently, a strange nonchaotic attractor is born. In the sequel, we present a model that is partially analyzable, together with numerical results, to confirm the blowout bifurcation route to strange nonchaotic attractors.

B. A two-dimensional map

We study the following two-dimensional version of Eq. (1),

$$x_{n+1}=(x_n+2\pi\omega)\bmod(2\pi), \tag{4}$$

$$y_{n+1}=\frac{1}{2\pi}(a\cos x_n+b)\sin(2\pi y_n),$$

where a and b are parameters, and $\omega\in(0,1)$ is an irrational number so that the x dynamics is the circle map that generates a quasiperiodic torus with uniform invariant density $\rho(x)=1/(2\pi)$ in $x\in[0,2\pi]$ (two-frequency quasiperiodicity). The one-dimensional invariant subspace is $y=0$. The transverse Lyapunov exponent is

$$\Lambda_T=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\ln|a\cos x_j+b|$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\ln|a\cos x+b|dx. \tag{5}$$

We obtain,

$$\Lambda_T=\begin{cases}\ln|b|-\ln 2/[1+\{1-(a/b)^2\}], & \text{if } a\leqslant b\\ \ln|b|+\ln a/(2b), & \text{if } a>b.\end{cases} \tag{6}$$

We have, for example, $a_c=2$ for the case $a>b>0$, where $\Lambda_T\leqslant 0$ for $a\leqslant a_c$ and $\Lambda_T>0$ for $a>a_c$. The nontrivial y-Lyapunov exponent is

$$\Lambda_y=\Lambda_T+\lambda_y\approx\Lambda_T+\int\ln|\cos(2\pi y)|\rho(y)dy, \tag{7}$$

where $\rho(y)$ is the invariant density of y for $a>a_c$. Note that for $a<a_c$ we have $y=0$ (asymptotically) and, hence, $\lambda_y=0$. In this case, $\Lambda_y=\Lambda_T<0$ so that the asymptotic attractor is the quasiperiodic torus in the invariant line $y=0$. For $a>a_c$, we have $\Lambda_T\gtrsim 0$. From Eq. (7), we see that $\ln|\cos(2\pi y)|<0$ and, hence, $\lambda_y<0$. Thus, it is possible to have $\Lambda_T\gtrsim 0$ but $\Lambda_y<0$. Since, (i) the y dynamics is bounded, and (ii) the y map apparently does not have other stable attractors for $x\in[0,2\pi]$, although a typical trajectory can no longer stay in the vicinity of $y=0$ for $a\gtrsim a_c$, it must come to the neighborhood of $y=0$ intermittently. Thus, geometrically, the trajectory traces out a complicated structure in the phase space. But since $\Lambda_y<0$, the map possesses no positive Lyapunov exponent. Consequently, we expect the attractor to be strange but not chaotic for $a\gtrsim a_c$. The key observation is thus that the positiveness of Λ_T renders strange the asymptotic attractor, but the negativeness of the nontrivial Lyapunov exponent warrants that the attractor be nonchaotic.

We now present numerical results to test the nature of the attractor after the blowout bifurcation. Figure 2(a) shows a trajectory of 10 000 points on such an attractor recorded after 10^6 preiterations for $\omega=(\sqrt{5}-1)/2$ (the inverse golden mean), $a=2.1>a_c$ and $b=1$. The transverse Lyapunov exponent is $\Lambda_T\approx 0.049$. The attractor is geometrically strange, but it is nonchaotic because at this set of parameter values, the Lyapunov spectrum is $\Lambda_x=0$ and $\Lambda_y\approx -0.104$. The power spectrum for a time series $\{y_n\}$ of $M=2^{16}$ points using a standard fast fourier transform algorithm is shown in Fig. 2(b). Although the power spectrum is broadband, it appears to contain a discrete set of "spikes," which are typical features of the power spectrum of strange nonchaotic attractors [6–8,5,10,12]. Figures 2(a) and 2(b) thus suggest that immediately after the blowout bifurcation, the quasiperiodic torus in $y=0$ becomes a repeller in the transverse direction and a strange nonchaotic attractor is born in the full two-dimensional phase space.

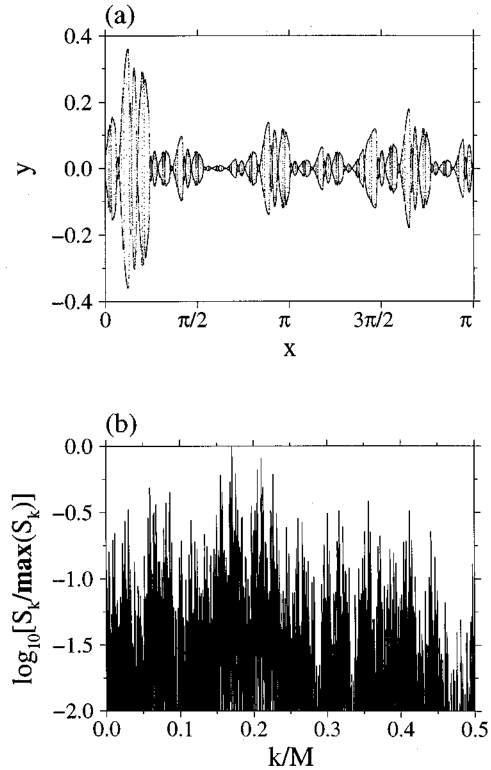

FIG. 2. For map (4), (a) the strange nonchaotic attractor at $a=2.1$. The Lyapunov spectrum is $\Lambda_x=0$ and $\Lambda_y\approx -0.104$. The transverse exponent is $\Lambda_T\approx 0.049$ which causes the attractor to have a strange geometry. (b) Power spectrum of the time series $\{y_n\}$ at $a=2.1$.

C. Singular-continuous spectrum analysis

To lend more credence to our assertion that the attractor for $a\gtrsim a_c$ is strange nonchaotic, we perform a singular continuous spectrum analysis that was first proposed in the investigation of models of quasiperiodic lattices and quasiperiodically forced quantum systems [23]. In general, power spectra of dissipative dynamical systems can be either discrete, or continuous, or a combination of both. Discrete spectra are usually generated by regular motions such as periodic or quasiperiodic motions, whereas continuous spectra correspond to irregular motions such as chaotic or random motions. A singular-continuous spectrum is a mixture of both discrete and continuous spectra [24]. This spectrum is particularly relevant to our study because strange nonchaotic behavior exhibits a mixture of properties of regular and irregular behaviors, as pointed out by Pikovsky and Feudel [11] [see also Fig. 2(b)]. To characterize a singular-continuous spectrum, we define, from a time series $\{x_n\}$ of a trajectory on a strange nonchaotic attractor, the following partial Fourier sum,

$$X(\Omega,T)=\sum_{n=1}^{T}x_n e^{i2\pi n\Omega}, \tag{8}$$

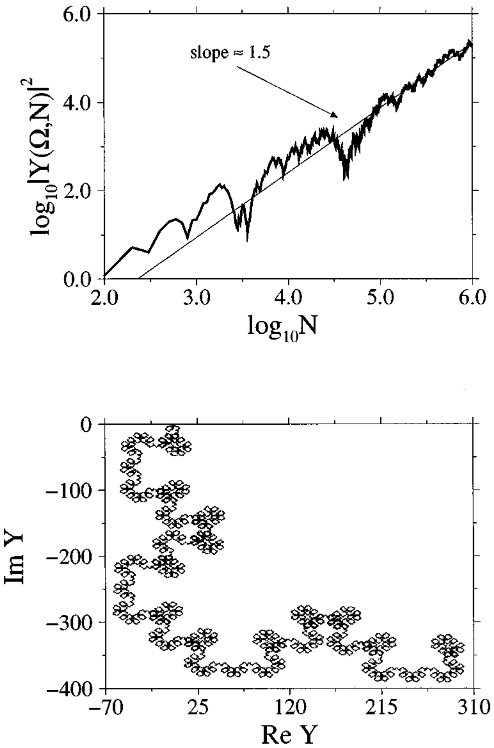

FIG. 3. For map (4), (a) singular-continuous spectrum analysis of the time series $\{y_n\}$. Shown is $Y(\Omega,T)$ vs T on a logarithmic scale. We have, $|Y(\Omega,T)|^2 \sim T^{1.5}$. (b) The fractal path in the complex plane (ReY,ImY).

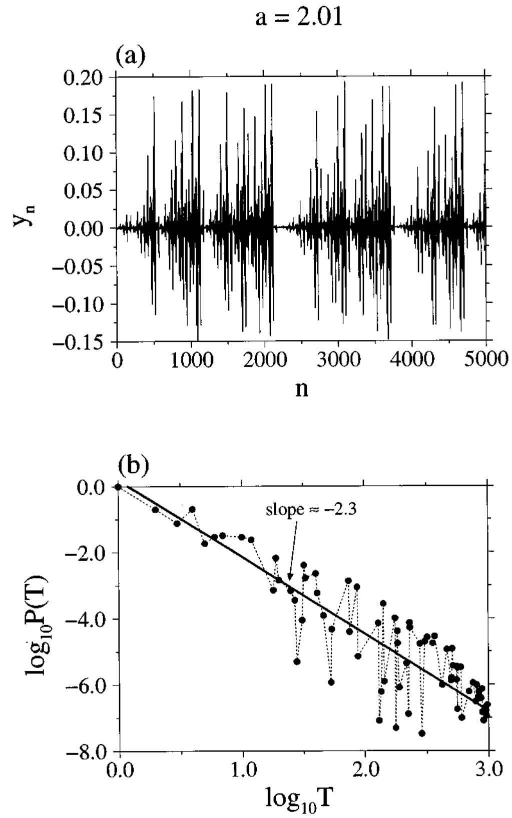

FIG. 4. For map (4), (a) on-off intermittency in y_n at $a=2.01$. The transverse Lyapunov exponent is $\Lambda_T \approx 0.005$. (b) Probability distribution of the laminar phases. Shown is $\log_{10}(T)$ vs $\log_{10} P(T)$.

where Ω is proportional to the irrational driving frequency ω in Eq. (4). When T is regarded as time, $|X(\Omega,T)|^2$ grows with time T as [24],

$$|X(\Omega,T)|^2 \sim T^{\alpha} \tag{9}$$

where α is the scaling exponent. Since $X(\Omega,T)$ is a complex variable, one can regard the real and imaginary parts of $X(\Omega,T)$ as two independent axes–the time evolution of $X(\Omega,T)$ can thus be represented by an orbit, or a ''walker'' in the complex plane (Re[$X(\Omega,T)$],Im[$X(\Omega,T)$]). When the motion is regular so that the power spectrum is discrete, we expect the average distance of the walker from the origin to increase linearly as T increases. In this case, we have $|X(\Omega,T)|^2 \sim T^2$, or $\alpha=2$. If the motion is random or chaotic, the behavior of the orbit in the complex plane is similar to that of a random walker so that the average distance of the walker from the origin increases like $\sqrt{T}$ as T increases. Thus, in this case, we have $|X(\Omega,T)|^2 \sim T$, or $\alpha=1$. The spectrum associated with trajectories on strange nonchaotic attractors falls somewhere in between these two categories. It was demonstrated [11] that for strange nonchaotic attractors, the quantity $X(\Omega,T)$ generally has the following features: (1) $1<\alpha<2$ and (2) the path (Re[X],Im[X]) in the complex plane is fractal.

Figure 3(a) shows, for $\Omega=\omega/4$, $\log_{10}|Y(\Omega,T)|^2$ versus $\log_{10}T$, where $Y(\Omega,T)$ is the partial Fourier sum from the time series $\{y_n\}$: $Y(\Omega,T)=\Sigma_{n=1}^{T} y_n e^{i2\pi n\Omega}$. The path (Re[$Y$],Im[$Y$]) in the complex plane is shown in Fig. 3(b). We have $|Y(\Omega,T)|^2 \sim T^{1.5}$ and, the path (Re[Y],Im[Y]) apparently exhibits a fractal self-similar structure. These results strongly suggest that the attractor in Fig. 2(a) is indeed strange nonchaotic.

D. On-off intermittency in quasiperiodically driven systems

A feature associated with the birth of the strange nonchaotic attractor is the occurrence of on-off intermittency [19,20] in y for $a \gtrsim a_c$ ($\Lambda_T \gtrsim 0$). This is shown in Fig. 4(a), where y_n versus the time n is plotted for $a=2.01$ ($\Lambda_T \approx 0.005$ and $\Lambda_y \approx -0.011$). We see that there are time intervals during which y_n stays near $y=0$ (the ''off'' state), but there are also intermittent bursts of y_n (the ''on'' state) away from the off state. This is a typical consequence of the blowout bifurcation [18], the origin of which can be understood by considering the fluctuations of finite time transverse Lyapunov exponent. Imagine we choose an ensemble of initial conditions in x, compute Λ_T for each initial condition at a finite time, and then construct a histogram of these exponents. Since the asymptotic Λ_T is only slightly positive, there is a spread of the histogram into the negative side, indicating that a trajectory can spend long stretches of time near $y=0$ in finite times. But since Λ_T is positive, the trajectory is repelled away from $y=0$ intermittently. Thus on-off intermittency occurs.

We stress that the on-off intermittent time series in Fig. 4(a) is in fact produced by a quasiperiodic driving to the

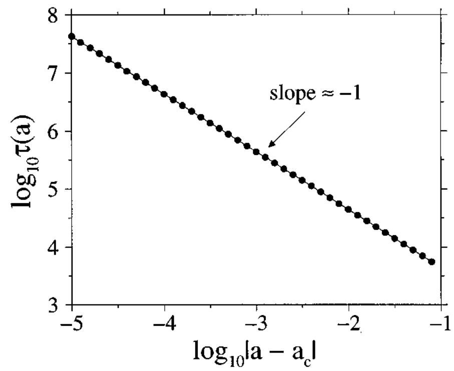

FIG. 5. For map (4), $\log_{10}\tau(a)$ vs $\log_{10}(a)$, where $\tau(a)$ is the average lifetime of the transient on-off intermittent behavior for $a < a_c$. We have $\tau(a) \sim |a-a_c|^{-1}$.

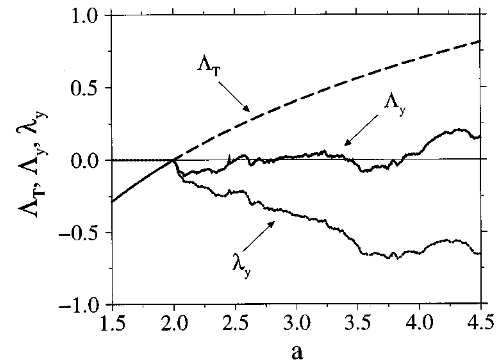

FIG. 6. For map (4), Λ_T, Λ_y, and λ_y vs parameter a, where $\Lambda_y = \Lambda_T - |\lambda_y|$.

transverse dynamics, whereas systems that generate on-off intermittency reported so far in the literature [19,20] are driven either randomly or chaotically. This can be understood by examining large time scales. In such scales, a quasiperiodic driving can be regarded roughly as a random or a chaotic driving. Figure 4(b) shows the probability distribution $P(T)$ of the laminar phases T plotted on a logarithmic scale, where T is the time interval for which $y_n \leqslant \epsilon = 10^{-5}$ and 10^8 such time intervals have been computed to construct $P(T)$. Roughly, $P(T)$ decays algebraically for small T ($T \gtrsim 1000$), a feature typical of the conventional on-off intermittent time series produced by random or chaotic driving [20]. For large T ($T \gtrsim 1000$), we find that $P(T)$ decays exponentially. However, only a distinct set of T values is observed and, hence, the plot in Fig. 4(b) exhibits large fluctuations. For the 10^8 laminar phases examined with a threshold $\epsilon = 10^{-5}$, there are only about 80 distinct ones, indicating that most ''off'' time intervals have the same length. In principle, there can be an infinite number of distinct laminar phases, but our numerical computation indicates that $P(T)$ tends to concentrate on a limited set of values of the laminar phases due to quasiperiodic driving. This is qualitatively different from the conventional on-off intermittency produced by random or chaotic driving where all possible values of T can be observed and statistical fluctuations in the plot of $\ln P(T)$ versus $\ln T$ are considerably smaller [20].

E. Transient on-off intermittent behavior preceding the bifurcation

Before the birth of strange nonchaotic attractor, a typical trajectory exhibits transient on-off intermittent behavior before finally approaching $y=0$. For a given parameter value a, the average transient lifetime τ depends on the parameter difference $|a-a_c|$. Figure 5 shows $\log_{10}\tau$ versus $\log_{10}|a-a_c|$ for $10^{-5} \leqslant |a-a_c| \leqslant 10^{-1}$, where for each value of a, 50 trajectories are used to compute the average τ. A trajectory is regarded as having $y=0$ if it stays within 10^{-100} of $y=0$ for 10 000 successive iterations. Clearly, we have $\tau \sim |a-a_c|^{-1}$ from Fig. 5. This can be understood by noting that $\tau \sim 1/\Lambda_T$, and for a near a_c, we have $\Lambda_T \sim |a-a_c|$.

F. Transition to chaotic attractors

We address the following question: *how does a strange nonchaotic attractor become a chaotic attractor as a increases from a_c*? To answer this question, notice that $\Lambda_y = \Lambda_T + \lambda_y$, where $\lambda_y < 0$. Thus, Λ_y becomes positive when $\Lambda_T > |\lambda_y|$. As a increases, Λ_T increases monotonically near a_c, but λ_y exhibits fluctuations. This is shown in Fig. 6, where Λ_T, Λ_y, and λ_y versus a are plotted for 600 values of a in [1.5,4.5]. For each value of a, Λ_y and λ_y are computed with 10^6 iterations and 2×10^5 preiterations. Despite the large number of iterations used in the computation, λ_y and hence Λ_y exhibit fluctuations. When Λ_T and $|\lambda_y|$ have comparable magnitudes, Λ_y can change from negative to positive and vice versa. Consequently, there exist a limited number of parameter intervals for strange nonchaotic attractors ($\Lambda_y \leqslant 0$) which are interspersed with parameter intervals for chaotic attractors ($\Lambda_y > 0$). This behavior has actually been observed in physical systems such as the quasiperiodically forced pendulum [8] and the continuous time model in Sec. III. Figure 6 suggests that the existence of two competing exponents such as Λ_T (local) and λ_y (global) is responsible for the alternation of strange nonchaotic and chaotic behaviors when the system parameter increases away from the bifurcation point. When a is increased further through some critical value, say a_g, where for $a > a_g$, Λ_T is sufficiently large that $\Lambda_T < |\lambda_y|$ does not occur, the system possesses a positive Lyapunov exponent Λ_y and, consequently, strange nonchaotic attractors are no longer possible.

III. QUASIPERIODICALLY DRIVEN FLOWS

We now demonstrate that all results in Sec. II for discrete maps also occur in more realistic physical systems modeled by continuous flows. In particular, we consider the following class of N-dimensional flow:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(x,z,p), \qquad \frac{d\mathbf{z}}{dt} = \boldsymbol{\omega}, \tag{10}$$

where $\mathbf{x}$ is N_x dimensional, $\mathbf{z}$ is N_z dimensional, $N_x + N_z = N$, $\boldsymbol{\omega} \equiv (\omega_1, \omega_2, \ldots, \omega_{N_z})$ is a frequency vector, and p is a bifurcation parameter. The function $\mathbf{F}$ satisfies $\mathbf{F}(\mathbf{0},\mathbf{z},p) = \mathbf{0}$ so that $\mathbf{x} = \mathbf{0}$ defines the invariant subspace S. The frequencies $(\omega_1, \omega_2, \ldots, \omega_{N_z})$ are incommensurate so that the

z-dynamics gives a quasiperiodic torus. The largest transverse and nontrivial Lyapunov exponents of the systems are given by

$$\Lambda_T = \lim_{l\to\infty} \frac{1}{t} \ln \frac{|\delta \mathbf{x}(t)|}{|\delta \mathbf{x}(0)|} \tag{11}$$

and

$$\Lambda = \lim_{l\to\infty} \frac{1}{t} \ln \frac{|\delta \mathbf{X}(t)|}{|\delta \mathbf{X}(0)|}, \tag{12}$$

respectively, where the infinitesimal vectors $\delta\mathbf{x}(t)$ and $\delta\mathbf{X}(t)$ are evolved according to

$$\begin{aligned} d\,\delta\mathbf{x}(t)/dt &= (\partial\mathbf{F}/\partial\mathbf{x})|_{x=0}\cdot\delta\mathbf{x}, \\ d\,\delta\mathbf{X}(t)/dt &= (\partial\mathbf{F}/\partial\mathbf{x})\cdot\delta\mathbf{X}, \end{aligned} \tag{13}$$

for random initial vectors $\delta\mathbf{x}(0)$ and $\delta\mathbf{X}(0)$ [18]. To be concrete, we consider a physical example, mathematically described by the following version of Eq. (10),

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = -\kappa y - \gamma x^3 + (\beta + f_1 \sin z_1 + f_2 \sin z_2)\sin(2\pi x), \tag{14}$$

$$\frac{dz_1}{dt} = \omega_1, \quad \frac{dz_2}{dt} = \omega_2$$

where the invariant subspace S is two-dimensional and it is given by $(x,y)=(0,0)$, ω_1 and ω_2 are two incommensurate frequencies so that there is a two-frequency quasiperiodic torus in S generated by the dynamics in z_1 and z_2. The largest Lyapunov exponent for trajectories restricted to the torus is zero. In Eq. (14), κ (dissipation), γ, β, f_1, and f_2 are parameters. Equation (14) is a slightly modified version of the experimental model used by Zhou, Moss, and Bulsara, which is relevant to the radio-frequency-driven superconducting quantum interference device [14]. In our numerical experiments, we arbitrarily choose κ as the bifurcation parameter and fix other parameters at $\gamma=2.0$, $\beta=-1.1$, $\omega_1=2.25$, and $\omega_1/\omega_2=\frac{1}{2}(\sqrt{5}+1)$ (the golden mean), $f_1=3.5$ and $f_2=5.0$. We observe that a blowout bifurcation occurs at $\kappa_c\approx 4.17$, where $\Lambda_T>0$ (<0) for $\kappa<\kappa_c$ $(>\kappa_c)$.

Figure 7(a) shows the (x,y) projection of a trajectory of 50 000 iterations (after 10 000 preiterations) on the stroboscopic surface of section defined by $z_1(t_n)=2n\pi$ $(n=1,2,\ldots)$ for $\kappa=4.1$ $(\Lambda_T\approx 0.019)$. The largest nontrivial Lyapunov exponent of the system is $\Lambda\approx -0.134$. Thus, there is no positive Lyapunov exponent for this parameter setting. The geometric shape of the attractor, however, appears strange, as can be seen from Fig. 7(a). The mechanism for the strangeness of the attractor is similar to that in map (4) [Fig. 2(a)]. In particular, a typical trajectory on the torus in S is transversely unstable for $\kappa\lesssim\kappa_c$ $(\Lambda_T\gtrsim 0)$. We verify numerically that there are apparently no other attractors in the phase space for $\kappa\lesssim\kappa_c$ [25]. Thus, as κ decreases through the critical value κ_c, a strange nonchaotic attractor is born:

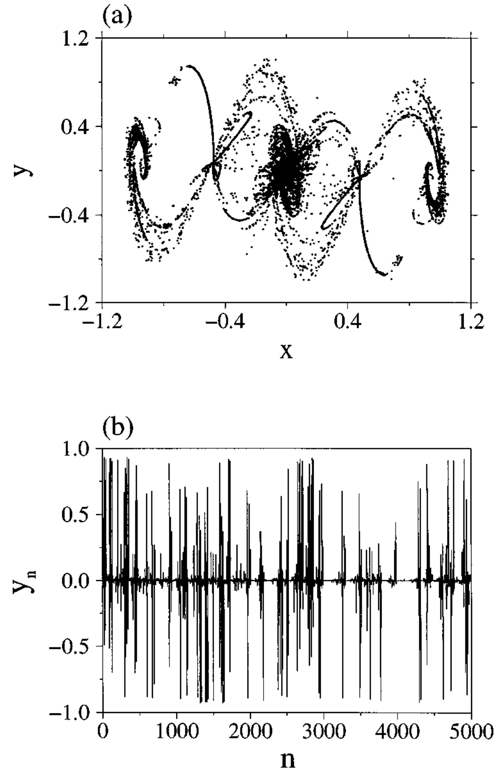

FIG. 7. For the flow Eq. (14), (a) a trajectory of 50 000 points on the stroboscopic section at $\kappa=4.1$ (see text for other parameters). Apparently, the attractor is geometrically strange. (b) On-off intermittency in the time series $y(t)$ obtained from (a).

the positiveness of the transverse Lyapunov exponent Λ_T gives rise to the strangeness of the attractor, and the negativeness of the largest nontrivial Lyapunov exponent Λ guarantees attractor's being nonchaotic. Figure 7(b) shows on-off intermittency after the bifurcation, where the time series $\{y_n\}$ is plotted for $\kappa=4.1$. We see that there are time intervals when y_n stays near $y=0$ (the off state), but there are also intermittent bursts of y_n (the on state) away from the off state. This is similar to Fig. 4(a). Again, here on-off intermittency is produced by a quasiperiodic driving.

We have also performed the singular-continuous spectrum analysis to check the nature of the attractor in Fig. 7(a). Figures 8(a) and 8(b) show, for $\Omega=(\sqrt{5}+1)/8$, respectively, $\log_{10}|X(\Omega,T)|^2$ versus $\log_{10} T$ and the path (ReX,ImX) computed from the time series $\{x_n\}$ on the surface of section in Fig. 7(a). We have $\alpha\approx 1.2$ and, the path (ReX,ImX) apparently exhibits a fractal self-similar structure. These results thus lend strong credence to our conclusion that the attractor in Fig. 7(a) is indeed strange nonchaotic.

The nontrivial Lyapunov exponent exhibits variation similar to Fig. 6 in map (4) after the blowout bifurcation, as shown in Fig. 9, where Λ_T, Λ and $\lambda=\Lambda-\Lambda_T$ versus κ are plotted for 2000 values of κ in [3.2,4.8]. For each value of κ, Λ_T, Λ, and λ are computed with 50 000 iterations and 10 000 preiterations on the surface of section. When Λ_T and $|\lambda|$ have comparable magnitudes, Λ can change from negative to positive and vice versa. Consequently, there exist interspersed parameter intervals for chaotic attractors $(\Lambda>0)$ and for strange nonchaotic attractors $(\Lambda<0)$. We have ob-

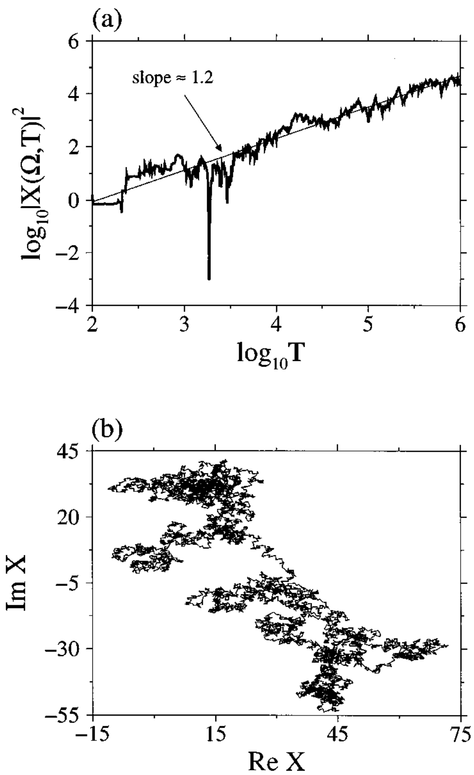

FIG. 8. For the flow Eq. (14), (a) singular-continuous spectrum analysis of the time series $\{x_n\}$: $\log_{10}|X(\Omega,T)|^2$ vs $\log_{10} T$. We have $|X(\Omega,T)|^2 \sim T^{1.2}$. (b) The corresponding path (ReX,ImX).

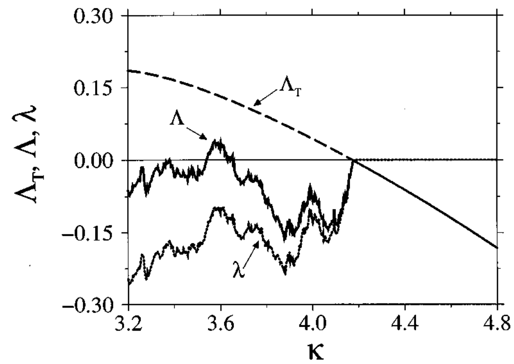

FIG. 9. For the flow Eq. (14), Λ_T, Λ_y, and λ_y vs the parameter κ (dissipation).

served that when κ is decreased further through some critical value, Λ_T is sufficiently large so that $\Lambda_T < |\lambda|$ does not occur, the system possesses a positive Lyapunov exponent Λ and, consequently, strange nonchaotic attractors are no longer possible and the asymptotic attractor is chaotic. Similar to the case of map (4), the average time of the transient on-off intermittent behavior preceding the blowout bifurcation follows the scaling law $\tau \sim |\kappa - \kappa_c|^{-1}$.

IV. DISCUSSIONS

In the study of nonlinear dynamical systems, the notion *strangeness* is often associated with a chaotic process. Chaotic attractors and chaotic saddles typically possess a fractal structure which is geometrically strange. There are also processes that are chaotic but not strange. For example, in Hamiltonian systems, the motion after disappearance of all Kolmogorov-Arnold-Moser tori is chaotic, but a typical trajectory wanders in a two-dimensional region that is not geometrically strange (chaotic sea) [26]. Strange nonchaotic attractors were found to be observable first in 1984 [6]. Since then, there is a continuous interest in the subject of strange nonchaotic attractors [5–16]. An important question then is how these exotic attractors arise in dynamical systems. The main contribution of this paper is the investigation of one route to strange nonchaotic attractors for systems with a symmetric invariant subspace. We present arguments and numerical verifications to show that a strange nonchaotic attractor can be created when a quasiperiodic torus embedded in the invariant subspace becomes transversely unstable. The numerical examples utilized to illustrate our findings consist of both discrete maps and continuous flows, the latter represents more realistic physical systems. Since symmetry is quite common in dynamical systems, we expect this route to strange nonchaotic attractors to be observable.

An interesting finding is that the two distinct dynamical phenomena, strange nonchaotic behavior and on-off intermittency, commonly thought of as arising in very different contexts in the study of nonlinear systems, can actually be closely related. The link is the blowout bifurcation that destabilizes, transversely, the quasiperiodic torus in the invariant subspace. Our study thus demonstrates that blowout bifurcation can occur even if the driving is not chaotic or random but quasiperiodic. As a consequence, on-off intermittency can arise in quasiperiodically driven dynamical systems. Since both strange nonchaotic attractors [13,14] and on-off intermittency [27] have been experimentally observed in physical systems, we believe that the findings reported in this paper can be tested in laboratory experiments.

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[1] In the period-doubling route (i), a chaotic attractor appears in a parameter region immediately following the accumulation of an infinite number of period doublings [2]. In the intermittency route (ii), as a parameter passes through a critical value, a simple periodic orbit is replaced by a chaotic attractor in such a way that the chaotic behavior is interspersed with a periodic-like behavior in an intermittent fashion [3]. In the crisis route (iii), a chaotic attractor is suddenly created to replace a nonat-

tracting chaotic saddle as the parameter passes through the crisis value [4]. In systems such as the two-frequency quasiperiodically forced systems, chaos can arise through the following route (iv): (three-frequency quasiperiodicity)→(strange nonchaotic behavior)→(chaos) [5].

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